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On the transversal vibrations of a conveyor belt with a low and time-varying velocity. Part I: the string-like case

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Abstract

In this paper initial-boundary-value problems for a linear wave (string) equation are considered. These problems can be used as simple models to describe the vertical vibrations of a conveyor belt, for which the velocity is small with respect to the wave speed and is assumed to move with a time-varying speed. Formal asymptotic approximations of the solutions are constructed to show the complicated dynamical behavior of the conveyor belt. It will also be shown that the truncation method cannot be applied to this problem in order to obtain approximations valid on long time scales.

1. Introduction

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see Refs. [1–4] for an overview), and is still of interest today.

The main purpose of studying the dynamic behavior of a belt system is to determine the *natural frequencies* of the vibrations. By knowing these natural frequencies, the so-called resonance-free belt system can be designed (see Ref. [3]). Resonances that can cause severe vibrations may be initiated by some parts of the belt system, such as the varying belt speed, the roll eccentricities, and other belt imperfections. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support component, and the increase of energy consumption of the system.

Belt vibrations can be classified into two types, i.e., whether it is of a string-like or of a beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be

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neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibrations of the belt system may be described as

string-like by

$$v_{tt} + 2Vv_{xt} + V_t v_x + (\kappa V^2 - c^2)v_{xx} = 0, \tag{1}$$

and beam-like (*with a string effect*) *by*

$$v_{tt} + 2Vv_{xt} + V_t v_x + (\kappa V^2 - c^2)v_{xx} + (EI/\rho A)v_{xxxx} = 0, \tag{2}$$

where $v(x, t)$ is the displacement of the belt in the y (vertical) direction, V is the time-varying belt speed, c is the wave speed, E is Young's modulus, I is the moment of inertia with respect to the x-(horizontal) axis, ρ is the mass density of the belt, A is the area of the cross-section of the belt, κ is a constant representing the relative stiffness of the belt (its value is in [0, 1]), x is the co-ordinate in the horizontal direction, and t is the time.

The beam-like system with a low time-varying speed will be considered in a forthcoming paper. In this paper, the string-like case will be studied in which the belt velocity $V(t)$ is given by

$$V(t) = \varepsilon(V_0 + \alpha \sin(\Omega t)), \tag{3}$$

where ε is a small parameter with $0 < \varepsilon \ll 1$, and V_0 and α are constants with $V_0 > 0$ and $V_0 > |\alpha|$. The velocity variation frequency of the belt is given by Ω. In fact the small parameter ε indicates that the belt speed $V(t)$ is small compared to the wave speed c. The condition $V_0 > |\alpha|$ guarantees that the belt will always move forward in one direction. It will be shown that certain values of Ω can lead to complicated internal resonances of the belt system.

While for more accurate results, a non-linear model is required, it is initially helpful to investigate a linear model. Knowledge about linear models is important in order to understand results found in non-linear models, especially for those cases which are weakly non-linear. For non-linear models describing the dynamic behavior of belts, one refers readers to Refs. [4–6]. In Ref. [6] the role played by the external frequency of the non-constant belt velocity and the bending stiffness was studied. It was found that, as the bending stiffness tends to zero, the system behaved more like a string and its dynamics became more complicated than the beam-like system.

Most belt studies involve mainly belts moving with a constant velocity. Recently in a series of papers [7–10] several authors considered the vibrations of belts moving with time-dependent velocities and the vibrations of tensioned pipes conveying fluid with time-dependent velocities. In fact in Refs. [7–10] Eqs. (1) or (2) have been studied, in which $V(t)$, as given by Eq. (3), is included. To find approximations of the displacement of the belt in the vertical direction, the authors of Refs. [7–10] used the eigenfunction expansion method, the Galerkin truncation method, and the multiple time-scales perturbation method as for instance described in Refs. [11,12]. To apply the method of eigenfunction expansions and the perturbation method, special attention has to be paid to the $\mathcal{O}(\varepsilon)$ terms involving v_x and v_{xt} in Eqs. (1) or (2). To apply the truncation method the internal resonances between the vibration modes have to be studied. In Refs. [7–10] the $\mathcal{O}(\varepsilon)$ terms in Eqs. (1) or (2) involving v_x and v_{xt} are not treated correctly by assuming that truncation to one mode (or a few modes) of the constant belt velocity system is allowable. In this paper one shows that this truncation is not allowable. In Refs. [7,9], no instabilities of the belt system (as described by Eq. (1)) were found using the truncation method when the velocity variation frequency Ω is equal to or close to the difference of two natural frequencies of the constant-velocity system. In this paper it will be shown that instabilities can also occur when Ω is equal to or close to the difference of two natural frequencies of the constant-velocity system. In Refs. [4,13–17] several remarks can be found on how and when truncation is allowable. In those papers weakly non-linear problems for wave and for beam equations have been studied.

In this paper one considers the vibrations of a belt modelled by a string moving with a non-constant velocity $V(t) = \varepsilon(V_0 + \alpha \sin \Omega t)$, where V_0, α, and Ω are constants with $V_0 > |\alpha|$. The velocity $V(t)$ can be considered as a periodically changing velocity such that the belt still moves in one direction. This variation in $V(t)$ can be considered as some kind of an excitation. In relation to excitations, some results in this area have been obtained in Refs. [18,19]. In Ref. [18], problems for a string moving with a constant velocity were considered when one of its ends (i.e., $x = L$) is subjected to an harmonic excitation. In Ref. [20], the vibrations of the string at $x = L$ is forced such that $v(x,t) = v_0 \cos \Omega t$. In Ref. [20], the author also studied the case where one end of the moving string was subjected to an harmonic excitation, representing the case of a belt travelling from an eccentric pulley to a smooth pulley, whereas the case where both ends of the string are excited has been studied in Ref. [21]. In that paper a moving string model was used to study the transverse vibrations of power transmission chains. In all of the papers [18–21], the belt velocity is assumed to be constant.

This paper is organized as follows. In Section 2, an equation is derived to describe the transversal vibrations of a belt (modelled as a string). Here one assumes that the belt moves with an arbitrary low velocity which varies harmonically, i.e., $V(t) = \varepsilon(V_0 + \alpha \sin \Omega t)$. In Section 3, the energy and the boundedness of the solution of the problem as derived in Section 2 is studied. In Section 4, the application of the two time-scales perturbation method is discussed to solve the equation. It turns out that there are infinitely many values of Ω that can cause internal resonances. In this paper, only the resonance case $\Omega = c\pi/L$ is investigated. All other resonance cases can be studied similarly. In this section it is also shown that the truncation method cannot be applied to this problem due to the distribution of energy among all vibration modes. In the last part of Section 4, a detuning case is also studied for $\Omega = c\pi/L$. Finally, in Section 5 some remarks will be made and some conclusions drawn.

2. A string model

In this section the dynamic behavior of a conveyor belt, modelled by a moving string, is studied (see Fig. 1). Since the belt is assumed to move with a speed $V(t)$ (which explicitly depends on t), one obtains for the time derivative of the transversal displacement $v(x,t)$ of the belt

$$\mathrm{D}v/\mathrm{D}t = \partial v/\partial t + (\partial v/\partial x)(\mathrm{d}x/\mathrm{d}t) = v_t + Vv_x, \tag{4}$$

and for the second order derivative with respect to time

$$\mathrm{D}^2v/\mathrm{D}t^2 = v_{tt} + 2Vv_{xt} + V^2v_{xx} + V_tv_x. \tag{5}$$

Accordingly, the equation of motion is

$$T_0v_{xx} = \rho\,\mathrm{D}^2v/\mathrm{D}t^2, \qquad c^2v_{xx} = v_{tt} + 2Vv_{xt} + V^2v_{xx} + V_tv_x, \tag{6}$$

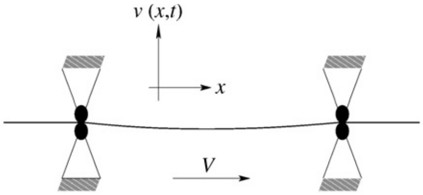

Fig. 1. Conveyor belt system.

where $c = \sqrt{T_0/\rho}$, in which T_0 and ρ are assumed to be the constant tension and the constant mass density of the string, respectively. At $x = 0$ and $x = L$ one assumes that the string is fixed in vertical direction, where L is the distance between the pulleys.

For $V(t)$ one uses $V(t) = \varepsilon(V_0 + \alpha \sin \Omega t)$ with $V_0 > 0$ and $V_0 > |\alpha|$. This velocity should be interpreted as low by comparison with the wave speed c of the belt. The condition $V_0 > |\alpha|$ guarantees that the belt will always move forward in one direction. Consequently, Eq. (6) becomes

$$c^2 v_{xx} - v_{tt} = \varepsilon[\alpha\Omega \cos(\Omega t) v_x + 2(V_0 + \alpha \sin(\Omega t)) v_{xt}] + \varepsilon^2[V_0 + \alpha \sin(\Omega t)]^2 v_{xx}, \tag{7}$$

where the boundary and initial conditions are given by

$$v(0, t; \varepsilon) = v(L, t; \varepsilon) = 0, \qquad v(x, 0; \varepsilon) = f(x), \qquad v_t(x, 0; \varepsilon) = g(x), \tag{8}$$

where $f(x)$ and $g(x)$ represent the initial displacement and the initial velocity of the belt, respectively. Throughout this paper it is assumed that f and g are sufficiently smooth such that a two-times continuously differentiable solution for the initial-boundary-value problem (7), (8) exists. Moreover, it is assumed that all series representations for the solution v (and its derivatives), and for the functions f and g are convergent. In this section the initial-boundary-value problem (7), (8) for $v(x, t)$ will be reduced to a system of infinitely many ordinary differential equations. This system will be studied further in Section 4 using a two time-scales perturbation method.

To satisfy the boundary conditions all functions should be expanded in Fourier sine series. So the solution is of the form $v(x, t; \varepsilon) = \sum_{n=1}^{\infty} v_n(t; \varepsilon) \sin(n\pi x/L)$. This is an odd function in x, both with regard to $x = 0$ and $x = L$. All functions on the right side of Eq. (7) should be extended properly to make them odd with respect to $x = 0$ and $x = L$, and periodic with period $2L$ thereof. Note that this extension or expansion process is not applied in Refs. [7–9] causing the occurrence of incorrect results in the critical values of Ω.

To make the right side of Eq. (7) odd, terms which are not already in Fourier sine series form in x are multiplied with (see also Refs. [13,16])

$$\mathscr{H}(x) = \left\{ \begin{array}{ll} 1 & \text{if } 0 < x < L \\ -1 & \text{if } -L < x < 0 \end{array} \right\} = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right). \tag{9}$$

Substituting Eq. (9) into Eq. (7) results in

$$\begin{aligned} c^2 v_{xx} - v_{tt} = {} & \varepsilon \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right) [\alpha\Omega \cos(\Omega t) v_x + 2(V_0 + \alpha \sin(\Omega t)) v_{xt}] \\ & + \varepsilon^2 (V_0 + \alpha \sin(\Omega t))^2 v_{xx}. \end{aligned} \tag{10}$$

Substitution of $v(x,t) = \sum_{n=1}^{\infty} v_n(t;\varepsilon)\sin(n\pi x/L)$ into Eq. (10) results in

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(-\left(\frac{cn\pi}{L}\right)^2 v_n - \ddot{v}_n\right)\sin\left(\frac{n\pi x}{L}\right)\\
&\quad = \varepsilon \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi}\sin\left(\frac{(2j+1)\pi x}{L}\right)\\
&\qquad \times\left(\alpha\Omega\cos(\Omega t)\frac{n\pi}{L}v_n\cos\left(\frac{n\pi x}{L}\right) + 2(V_0+\alpha\sin(\Omega t))\frac{n\pi}{L}\dot{v}_n\cos\left(\frac{n\pi x}{L}\right)\right)\\
&\qquad - \varepsilon^2\sum_{n=1}^{\infty}(V_0+\alpha\sin\Omega t)^2\left(\frac{n\pi}{L}\right)^2 v_n\sin\left(\frac{n\pi x}{L}\right).
\end{aligned} \tag{11}$$

By multiplying Eq. (11) with $\sin(k\pi x/L)$, and by integrating the so-obtained equation with respect to x from $x=-L$ to $x=L$, one obtains

$$\begin{aligned}
\ddot{v}_k + \left(\frac{ck\pi}{L}\right)^2 v_k &= \varepsilon\left[\sum_1 - \sum_2 - \sum_3\right]\frac{2n}{(2j+1)L}[\alpha\Omega\cos(\Omega t)v_n + 2(V_0+\alpha\sin(\Omega t))\dot{v}_n]\\
&\quad + \varepsilon^2(V_0+\alpha\sin(\Omega t))^2\left(\frac{k\pi}{L}\right)^2 v_k,
\end{aligned} \tag{12}$$

where $\sum_1 = \sum_{k=n-(2j+1)}$, $\sum_2 = \sum_{k=2j+1+n}$, and $\sum_3 = \sum_{k=2j+1-n}$. Eq. (12) will be studied further in Section 4.

3. Energy and boundedness of the solution

The concept of energy is used in many parts of the next sections. In this section the energy of the moving string as modelled by the wave equation is derived

$$c^2 v_{xx} = v_{tt} + 2Vv_{xt} + V^2 v_{xx} + V_t v_x. \tag{13}$$

On multiplying Eq. (13) with $(v_t + Vv_x)$ one obtains after some elementary calculations

$$\begin{aligned}
&(\tfrac{1}{2}v_t^2 + v_t V v_x + \tfrac{1}{2}c^2 v_x^2 + \tfrac{1}{2}V^2 v_x^2)_t\\
&\quad + (-c^2 v_x v_t - \tfrac{1}{2}c^2 V v_x^2 + V v_t^2 + V^2 v_x v_t + \tfrac{1}{2}V^3 v_x^2 - \tfrac{1}{2}V v_t)_x = 0.
\end{aligned} \tag{14}$$

Integrating Eq. (14) with respect to x from $x=0$ to $x=L$, and then integrating the so-obtained equation with respect to t from $t=0$ to t, one obtains

$$\int_0^L (\tfrac{1}{2}v_t^2 + Vv_t v_x + \tfrac{1}{2}(c^2+V^2)v_x^2)|_{t=0}^{t}\,\mathrm{d}x = \tfrac{1}{2}\int_0^t (c^2 - V^2)Vv_x^2|_{x=0}^{L}\,\mathrm{d}t. \tag{15}$$

The energy $E(t)$ of the moving string is now defined to be

$$E(t) = \tfrac{1}{2}\int_0^L ((v_t + Vv_x)^2 + c^2 v_x^2)\,\mathrm{d}x. \tag{16}$$

Thus, Eq. (15) can be written as

$$E(t) - E(0) = \tfrac{1}{2}\int_0^t (c^2 - V^2)Vv_x^2|_{x=0}^{L}\,\mathrm{d}t$$
$$\Leftrightarrow \frac{\mathrm{d}E}{\mathrm{d}t} = \tfrac{1}{2}(c^2 - V^2)V(v_x^2(L,t) - v_x^2(0,t)) \leqslant MV, \tag{17}$$

where M is the maximum of $\frac{1}{2}(c^2 - V^2)(v_x^2(L,t) - v_x^2(0,t))$, where it has been assumed that $v(x,t)$ is two-times continuously differentiable on $0 \leqslant x \leqslant L$ and $0 \leqslant t \leqslant T\varepsilon^{-1}$ for some positive constant $T < \infty$. It follows from Eq. (17) that $\mathrm{d}E/\mathrm{d}t \leqslant \mathcal{O}(\varepsilon)$ on $0 \leqslant t \leqslant T\varepsilon^{-1}$ since V is $\mathcal{O}(\varepsilon)$. And so, $E(t) - E(0) \leqslant \mathcal{O}(\varepsilon t)$ on $0 \leqslant t \leqslant T\varepsilon^{-1}$. The following estimate on $v(x,t)$ then also holds

$$|v(x,t)| = \left|\int_0^x v_x(x,t)\,\mathrm{d}x\right| \leqslant \int_0^x |v_x(x,t)|\,\mathrm{d}x \leqslant \int_0^L |v_x(x,t)|\,\mathrm{d}x$$
$$\leqslant \sqrt{\int_0^L 1^2\,\mathrm{d}x}\sqrt{\int_0^L 2\left(\tfrac{1}{2}(c^2v_x^2 + (v_t + Vv_x)^2)\right)\mathrm{d}x} = \sqrt{L}\sqrt{2E(t)}, \tag{18}$$

on $0 \leqslant t \leqslant T\varepsilon^{-1}$. One refers to Ref. [22] for more detailed descriptions of energetics of translating continua.

4. Application of the two time-scales perturbation method

Consider again Eq. (12). The application of a straightforward expansion method to solve Eq. (12) will result in the occurrence of so-called secular terms which causes the approximations to become unbounded on long time scales. To remove those secular terms, one introduces two time scales $t_0 = t$ and $t_1 = \varepsilon t$. The introduction of these two time scales defines the transformations

$$v_k(t;\varepsilon) = w_k(t_0, t_1;\varepsilon), \qquad \mathrm{d}v_k(t;\varepsilon)/\mathrm{d}t = \partial w_k/\partial t_0 + \varepsilon(\partial w_k/\partial t_1),$$

$$\mathrm{d}^2 v_k(t;\varepsilon)/\mathrm{d}t^2 = \partial^2 w_k/\partial t_0^2 + 2\varepsilon(\partial^2 w_k/\partial t_0 \partial t_1) + \varepsilon^2(\partial^2 w_k/\partial t_1^2). \tag{19}$$

By substituting Eq. (19) into Eq. (12) one obtains

$$\partial^2 w_k/\partial t_0^2 + 2\varepsilon(\partial^2 w_k/\partial t_0\partial t_1) + (ck\pi/L)^2 w_k$$
$$= \varepsilon\left[\sum_1 - \sum_2 - \sum_3\right]\frac{2n}{(2j+1)L}(\alpha\Omega\cos(\Omega t)w_n + 2[V_0 + \alpha\sin(\Omega t)(\partial w_n/\partial t_0)]) + \mathcal{O}(\varepsilon^2). \tag{20}$$

Assuming that $w_k(t_0, t_1;\varepsilon) = w_{k0}(t_0, t_1) + \varepsilon w_{k1}(t_0, t_1) + \cdots$, then in order to remove the secular terms up to $\mathcal{O}(\varepsilon)$, it is necessary to solve the problems

$$\mathcal{O}(1)\text{: } \partial^2 w_{k0}/\partial t_0^2 + (ck\pi/L)^2 w_{k0} = 0,$$

$$\mathcal{O}(\varepsilon)\text{: } \partial^2 w_{k1}/\partial t_0^2 + (ck\pi/L)^2 w_{k1} = -2\frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} + \left[\sum_1 - \sum_2 - \sum_3\right] \times \frac{2n}{(2j+1)L}\left(\alpha\Omega\cos(\Omega t_0) w_{n0} + 2(V_0 + \alpha\sin(\Omega t_0))\frac{\partial w_{n0}}{\partial t_0}\right). \quad (21)$$

The $\mathcal{O}(1)$ problem has as its solution

$$w_{k0}(t_0, t_1) = A_{k0}(t_1)\cos(ck\pi t_0/L) + B_{k0}(t_1)\sin(ck\pi t_0/L), \quad (22)$$

where A_{k0} and B_{k0} are still arbitrary functions that can be used to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$ problem.

From the $\mathcal{O}(\varepsilon)$ problem it can readily be seen that there are infinitely many values of Ω that can cause internal resonance. In fact these values are $(n+k)c\pi/L, (n-k)c\pi/L, (k-n)c\pi/L$ and $-(n+k)c\pi/L$, where $k = n-2j-1$, or $k = 2j+1-n$, or $k = n+2j+1$ (see also the summations in Eq. (12)). It is also easy to see that these values for Ω are always odd multiples of $c\pi/L$ (or are in an $\mathcal{O}(\varepsilon)$-neighbourhood of these odd multiples). In Refs. [7,9], the critical values of Ω are found to be even multiples of the natural frequency. These incorrect results in Refs. [7,9] for $\mathcal{O}(\varepsilon)$ belt velocities are due to the fact that certain terms in the PDE (that is, terms involving v_x and v_{xt} in Eq. (7)) are not extended or expanded correctly.

To show how the secular terms can be eliminated three cases will be considered: $\Omega = c\pi/L$, $\Omega = c\pi/L + \varepsilon\delta$ and the case in which Ω is not in a neighborhood of an odd multiple of $\Omega = c\pi/L$.

4.1. Case 1: $\Omega = c\pi/L$

In Appendix A the equations for $A_{k0}(t_1)$ and $B_{k0}(t_1)$ are derived in order that the approximations of the solution of the problem do not contain secular terms. It turns out that A_{k0} and B_{k0} have to satisfy

$$\begin{aligned} \mathrm{d}A_{k0}/\mathrm{d}\bar{t}_1 &= (k+1)B_{(k+1)0} + (k-1)B_{(k-1)0}, \\ \mathrm{d}B_{k0}/\mathrm{d}\bar{t}_1 &= -(k+1)A_{(k+1)0} - (k-1)A_{(k-1)0}, \end{aligned} \quad (23)$$

where $\bar{t}_1 = (\alpha/L)t_1$, and $k = 1, 2, 3, \ldots$. For $\Omega = m(c\pi/L)$, where m is odd, the same analysis as presented in Appendix A can be followed. It then follows that A_{k0} and B_{k0} have to satisfy $(k = 1, 2, 3, \ldots)$

$$\begin{aligned} \frac{\mathrm{d}A_{k0}}{\mathrm{d}\bar{t}_1} &= \frac{(k+m)(2k+2m-1)}{m(2k+m)}B_{(k+m)0} + \frac{(k-m)(2k-2m+1)}{m(2k-m)}B_{(k-m)0}, \\ \frac{\mathrm{d}B_{k0}}{\mathrm{d}\bar{t}_1} &= -\frac{(k+m)(2k+2m-1)}{m(2k+m)}A_{(k+m)0} - \frac{(k-m)(2k-2m+1)}{m(2k-m)}A_{(k-m)0}. \end{aligned}$$

It should be noticed that for $m = 1$ this system of ordinary differential equations is reduced to system (23). In this section system (23), which is a coupled system of infinitely many ordinary differential equations is studied.

4.1.1. Application of the truncation method

First, in trying to find an approximation of the solution of system (23) by using Galerkin's truncation method, the first few modes will be used and the higher order modes neglected. For example, in this case, by considering the first three modes, one obtains from Eq. (23)

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{24}$$

where

$$\mathbf{X} = \begin{pmatrix} B_{10} \\ A_{10} \\ B_{20} \\ A_{20} \\ B_{30} \\ A_{30} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

and where $\dot{\mathbf{X}}$ represents the derivative of $\mathbf{X}$ with respect to $\bar{t}_1$. This system has eigenvalues $2\sqrt{2}\mathrm{i}, -2\sqrt{2}\mathrm{i}$, and 0, all with multiplicity 2. Their associated eigenvectors are $(0, 1, \sqrt{2}\mathrm{i}, 0, 0, 1)$, $(1, 0, 0, -\sqrt{2}\mathrm{i}, 1, 0)$, $(1, 0, 0, \sqrt{2}\mathrm{i}, 1, 0)$, $(0, 1, -\sqrt{2}\mathrm{i}, 0, 0, 1)$, $(-3, 0, 0, 0, 1, 0)$ and $(0, -3, 0, 0, 0, 1)$, respectively. The solution of Eq. (24) is then given by

$$\begin{aligned}
B_{10}(t_1) &= C_3 \cos(2\sqrt{2}t_1) + C_4 \sin(2\sqrt{2}t_1) - 3C_5, \\
A_{10}(t_1) &= C_1 \cos(2\sqrt{2}t_1) + C_2 \sin(2\sqrt{2}t_1) - 3C_6, \\
B_{20}(t_1) &= -\sqrt{2}C_1 \sin(2\sqrt{2}t_1) + \sqrt{2}C_2 \cos(2\sqrt{2}t_1) - \sqrt{2}C_4 \cos(2\sqrt{2}t_1), \\
A_{20}(t_1) &= \sqrt{2}C_3 \sin(2\sqrt{2}t_1) - \sqrt{2}C_4 \cos(2\sqrt{2}t_1), \\
B_{30}(t_1) &= C_3 \cos(2\sqrt{2}t_1) + C_4 \sin(2\sqrt{2}t_1) + C_5, \\
A_{30}(t_1) &= C_1 \cos(2\sqrt{2}t_1) + C_2 \sin(2\sqrt{2}t_1) + C_6,
\end{aligned} \tag{25}$$

where $C_1, C_2, \ldots, C_6$ are all constants of integration. Note that all the bars in Eq. (25) has been dropped.

From the initial conditions (8), that is, $v(x, 0) = f(x)$ and $v_t(x, 0) = g(x)$ it follows that

$$\begin{aligned}
f(x) &= \sum_{k=1}^{\infty} v_k(0; \varepsilon) \sin\left(\frac{k\pi x}{L}\right) \;\Leftrightarrow\; v_k(0; \varepsilon) = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) \mathrm{d}x, \\
g(x) &= \sum_{k=1}^{\infty} \dot{v}_k(0; \varepsilon) \sin\left(\frac{k\pi x}{L}\right) \;\Leftrightarrow\; \dot{v}_k(0; \varepsilon) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) \mathrm{d}x.
\end{aligned} \tag{26}$$

Moreover, since $v_k(0; \varepsilon) = w_k(0, 0; \varepsilon) = w_{k0}(0, 0) + \varepsilon w_{k1}(0, 0) + \cdots$ and $\dot{v}_k(0; \varepsilon) = \dot{w}_k(0, 0; \varepsilon) = \dot{w}_{k0}(0, 0) + \varepsilon \dot{w}_{k1}(0, 0) + \cdots$ it follows that

$$w_{k0}(0, 0) = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) \mathrm{d}x, \qquad \dot{w}_{k0}(0, 0) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) \mathrm{d}x. \tag{27}$$

From Eqs. (22) and (27) one then obtains

$$A_{k0}(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x, \qquad B_{k0}(0) = \frac{2}{ck\pi}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x. \tag{28}$$

Eq. (28) can be used to calculate the constants in Eq. (25).

In summary, after all constants in Eq. (25) have been calculated, $w_{k0}(t_0, t_1)$ can be determined using Eq. (22). Then $v(x, t; \varepsilon)$ can be approximated by $\sum_{k=1}^{3} v_k(t; \varepsilon)\sin(k\pi x/L)$.

For example, using 1, 2, or 3 modes, respectively, with $f(x) = (-8/\pi^3)\sin(\pi x)$, $g(x) = 0$, $c = L = 1$ one finds as approximations for $v(x, t; \varepsilon)$:

$$v(x, t; \varepsilon) \approx (-8/\pi^3)\cos(\pi t_0)\sin(\pi x),$$

$$v(x, t; \varepsilon) \approx (-8/\pi^3)\cos(\sqrt{2}t_1)\cos(\pi t_0)\sin(\pi x) + (4\sqrt{2}/\pi^3)\sin(\sqrt{2}t_1)\sin(2\pi t_0)\sin(2\pi x),$$

$$\begin{aligned} v(x, t; \varepsilon) \approx\ & (-(2/\pi^3)\cos(2\sqrt{2}t_1) - 6/\pi^3)\cos(\pi t_0)\sin(\pi x) \\ & + (2\sqrt{2}/\pi^3)\sin(2\sqrt{2}t_1)\sin(2\pi t_0)\sin(2\pi x) \\ & + ((-2/\pi^3)\cos(2\sqrt{2}t_1) + 2/\pi^3)\cos(3\pi t_0)\sin(3\pi x). \end{aligned} \tag{29}$$

The graphs of these approximations for $v(x, t)$ for $x = 0.5$ and $\varepsilon = 0.01$ are depicted in Fig. 2.

For more than three modes, eigenvalues and eigenvectors become more and more difficult to compute by just using pencil and paper. Using the computer software package Maple, the eigenvalues of system (23) have been computed up to 20 modes and are listed in Table 1. From the table, it can be seen that the eigenvalues of the truncated system are always purely imaginary, each has multiplicity two, and for an odd number of modes one obtains an additional pair of zero eigenvalues. From approximations (29) and from Table 1 it can readily be seen that the truncation

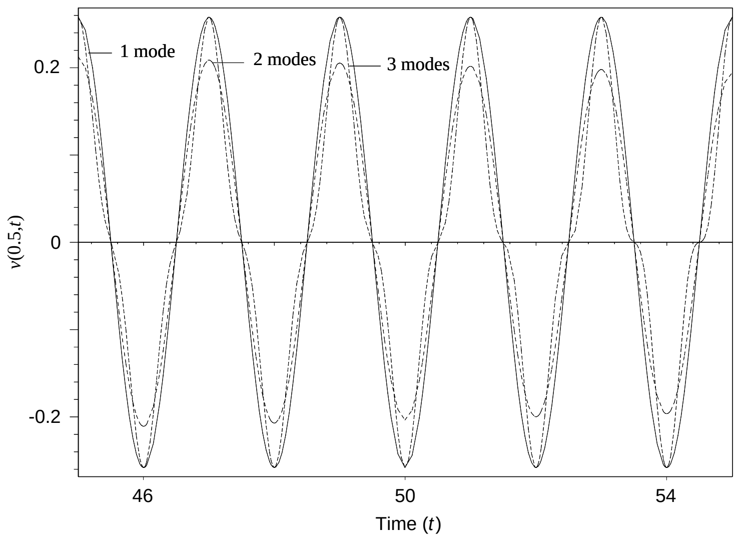

Fig. 2. Approximations for $v(x, t)$ with initial displacement $f(x) = (-8/\pi^3)\sin(\pi x)$ and initial velocity $g(x) = 0$. The graphs are given for $x = 0.5$, $t \in [45, 55]$, and $\varepsilon = 0.01$.

Table 1
Approximations of the eigenvalues of the truncated system (23)

No. of modes	Eigenvalues of matrix **A** (all multiplicity 2)	Dimension eigenspace of **A**
1	0	2
2	$\pm\sqrt{2}i$	4
3	$0, \pm 2\sqrt{2}i$	6
4	$\pm 1.13i, \pm 4.33i$	8
5	$0, \pm 2.30i, \pm 5.89i$	10
6	$\pm 7.50i, \pm 1.00i, \pm 3.56i$	12
7	$0, \pm 9.15i, \pm 2.05i, \pm 4.90i$	14
8	$\pm 10.83i, \pm 0.93i, \pm 3.18i, \pm 6.30i,$	16
9	$0, \pm 12.54i, \pm 1.89i, \pm 4.38i, \pm 7.74i$	18
10	$\pm 14.26i, \pm 0.87i, \pm 5.65i, \pm 9.23i, \pm 2.93i$	20
11	$0, \pm 16.01i, \pm 1.78i, \pm 4.05i, \pm 6.97i, \pm 10.76i$	22
12	$\pm 17.76i, \pm 0.83i, \pm 2.76i, \pm 5.22i, \pm 8.33i, \pm 12.31i$	24
13	$0, \pm 19.53i, \pm 1.70i, \pm 3.81i, \pm 6.45i, \pm 9.73i, \pm 13.88i, \pm 19.53i$	26
14	$\pm 21.31i, \pm 15.48i, \pm 0.80i, \pm 2.63i, \pm 4.92i, \pm 7.72i, \pm 11.16i$	28
15	$0, \pm 23.11i, \pm 17.10i, \pm 1.64i, \pm 3.63i, \pm 6.07i, \pm 9.03i, \pm 12.63i$	30
16	$\pm 24.91i, \pm 18.73i, \pm 0.78i, \pm 2.53i, \pm 4.68i, \pm 7.28i, \pm 10.38i, \pm 14.11i,$	32
17	$0, \pm 26.71i, \pm 20.38i, \pm 1.58i, \pm 3.49i, \pm 5.79i, \pm 8.52i, \pm 11.75i \pm 15.62i,$	34
18	$\pm 28.53i, \pm 22.05i, \pm 0.75i, \pm 2.45i, \pm 4.50i, \pm 6.93i, \pm 9.79i, \pm 13.16i, \pm 17.15i$	36
19	$0, \pm 30.35i, \pm 23.72i, \pm 1.54i, \pm 3.37i, \pm 5.55i, \pm 8.12i, \pm 11.10i, \pm 14.58i, \pm 18.70i$	38
20	$\pm 32.18i, \pm 25.41i, \pm 0.73i, \pm 2.38i, \pm 4.34i, \pm 6.65i, \pm 9.33i, \pm 12.43i, \pm 16.03i, \pm 20.27i$	40

method will not give accurate results on long time scales, that is, on time scales of order ε^{-1}. On the other hand it is well known in mathematics that if the truncated system has only purely imaginary eigenvalues and/or eigenvalues equal to zero then no conclusions can be drawn for the infinite dimensional system.

4.1.2. Analysis of the infinite dimensional system (23)

The previous subsection shows that if system (23) is truncated then the eigenvalues of the truncated system are always purely imaginary or zero. This section will show that the results obtained by applying the truncation method are not valid on time scales of order ε^{-1}.

By putting $kB_{k0}(t_1) = Y_{k0}(t_1)$ and $kA_{k0}(t_1) = X_{k0}(t_1)$, system (23) becomes

$$\mathrm{d}Y_{k0}/\mathrm{d}t_1 = k[-X_{(k+1)0} - X_{(k-1)0}], \qquad \mathrm{d}X_{k0}/\mathrm{d}t_1 = k[Y_{(k+1)0} + Y_{(k-1)0}], \tag{30}$$

for $k = 1, 2, 3, \ldots$, and $X_{00} = Y_{00} = 0$.

Accordingly

$$\begin{aligned} Y_{k0}\dot{Y}_{k0} &= -k[Y_{k0}X_{(k+1)0} + Y_{k0}X_{(k-1)0}], \\ X_{k0}\dot{X}_{k0} &= k[X_{k0}Y_{(k+1)0} + X_{k0}Y_{(k-1)0}]. \end{aligned} \tag{31}$$

By adding both equations in Eq. (31), and then summing from $k = 1$ to ∞

$$\tfrac{1}{2}\sum_{k=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t_1}(Y_{k0}^2 + X_{k0}^2) = \sum_{k=1}^{\infty}[X_{(k+1)0}Y_{k0} - Y_{(k+1)0}X_{k0}]. \tag{32}$$

By differentiating Eq. (32) with respect to t_1 one finds (see also Appendix B)

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}(Y_{k0}^2+X_{k0}^2)=2\sum_{k=1}^{\infty}(X_{k0}^2+Y_{k0}^2),\tag{33}$$

and so, by putting $\sum_{k=1}^{\infty}(X_{k0}^2+Y_{k0}^2)=W(t_1)$, finally

$$\frac{\mathrm{d}^2W(t_1)}{\mathrm{d}t_1^2}-4W(t_1)=0.\tag{34}$$

The solution of Eq. (34) is $W(t_1)=K_1\mathrm{e}^{2t_1}+K_2\mathrm{e}^{-2t_1}$, where K_1 and K_2 are constants. Note that $W(t_1)$ is a first integral of system (23). K_1 and K_2 are both positive numbers as is shown in the following calculation. From $W(t_1)=\sum_{k=1}^{\infty}[X_{k0}^2+Y_{k0}^2]$ it follows that

$$W(0)=\sum_{k=1}^{\infty}[X_{k0}^2(0)+Y_{k0}^2(0)]\geqslant 0\quad\Rightarrow\quad K_1+K_2\geqslant 0.\tag{35}$$

Differentiating $W(t_1)$ with respect to t_1 and then putting $t_1=0$

$$K_1-K_2=\sum_{k=1}^{\infty}[Y_{k0}(0)X_{(k+1)0}(0)-X_{k0}(0)Y_{(k+1)0}(0)].\tag{36}$$

From Eqs. (35) and (36) it then follows that

$$\begin{aligned}2K_1&=\sum_{k=1}^{\infty}[X_{k0}^2(0)+Y_{k0}^2(0)+Y_{k0}(0)X_{(k+1)0}(0)-X_{k0}(0)Y_{(k+1)0}(0)]\\&=\tfrac{1}{2}X_{10}^2(0)+\tfrac{1}{2}Y_{10}^2(0)+\tfrac{1}{2}(X_{10}(0)-Y_{20}(0))^2+\tfrac{1}{2}(Y_{10}(0)+X_{20}(0))^2\\&\quad+\tfrac{1}{2}(X_{20}(0)-Y_{30}(0))^2+\tfrac{1}{2}(Y_{20}(0)+X_{30}(0))^2+\cdots\\&\quad+\tfrac{1}{2}(X_{n0}(0)-Y_{(n+1)0}(0))^2+\tfrac{1}{2}(Y_{n0}(0)+X_{(n+1)0}(0))^2+\cdots\geqslant 0.\end{aligned}\tag{37}$$

So, $K_1\geqslant 0$ and 0 if and only if $X_{k0}(0)=Y_{k0}(0)=0$ for each $k=1,2,3,\ldots$. Using a similar method, K_2 also can be shown to be a non-negative number. Consequently, $W(t_1)$ is, in general, non-negative and increases as t_1 increases. This behavior is different from the behavior of $A_{k0}(t_1)$ and $B_{k0}(t_1)$ as obtained by applying the truncation method. If one applies the truncation method, one merely obtains sin and cos functions for A_{k0} and B_{k0} while the energy (see next subsection) is described by exponential functions. This means that the approximations obtained by applying the truncation method to system (23) are not accurate on long time scales, that is, on time scales of order ε^{-1}.

4.1.3. The energy

The energy $E(t)$ of the conveyor belt system can also be approximated using the function $W(t_1)$. Since

$$\begin{aligned}v(x,t)&=\sum_{k=1}^{\infty}v_k(t)\sin\left(\frac{k\pi x}{L}\right)\\&=\sum_{k=1}^{\infty}\left[A_{k0}(t_1)\cos\left(\frac{ck\pi t}{L}\right)+B_{k0}(t_1)\sin\left(\frac{ck\pi t}{L}\right)\right]\sin\left(\frac{k\pi x}{L}\right)+\mathcal{O}(\varepsilon),\end{aligned}\tag{38}$$

it follows that the energy $E(t)$ satisfies

$$\begin{aligned}
E(t) &= \tfrac{1}{2}\int_0^L [(v_t + Vv_x)^2 + c^2 v_x^2]\,\mathrm{d}x \\
&= \frac{c^2\pi^2}{4L}\sum_{k=1}^{\infty} k^2\Bigg[\left(-A_{k0}\sin\left(\frac{k\pi t}{L}\right) + B_{k0}\cos\left(\frac{k\pi t}{L}\right)\right)^2 \\
&\qquad + \left(A_{k0}\cos\left(\frac{ck\pi t}{L}\right) + B_{k0}\sin\left(\frac{ck\pi t}{L}\right)\right)^2\Bigg] + \mathcal{O}(\varepsilon) \\
&= \frac{c^2\pi^2}{4L}\sum_{k=1}^{\infty} [(kA_{k0})^2 + (kB_{k0})^2] + \mathcal{O}(\varepsilon) \\
&= \frac{c^2\pi^2}{4L}\sum_{k=1}^{\infty} [X_{k0}^2 + Y_{k0}^2] + \mathcal{O}(\varepsilon) = \frac{c^2\pi^2}{4L} W(t_1) + \mathcal{O}(\varepsilon) \qquad (39) \\
&= \frac{c^2\pi^2}{4L}(K_1 \mathrm{e}^{2t_1} + K_2 \mathrm{e}^{-2t_1}) + \mathcal{O}(\varepsilon). \qquad (40)
\end{aligned}$$

So, the energy increases, although it is bounded on a time scale of order $1/\varepsilon$.

4.2. Case 2: $\Omega = c\pi/L + \varepsilon\delta$

In this section detuning from $\Omega = c\pi/L$, will be studied in which the case $\Omega = c\pi/L + \varepsilon\delta$ where $\delta = \mathcal{O}(1)$ is considered. In order to avoid secular terms in the approximation, it can be shown (the calculation are similar to those in Section 4.1) that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\begin{aligned}
\mathrm{d}A_{k0}/\mathrm{d}\bar{t}_1 &= (k+1)[B_{(k+1)0}\cos(\delta\bar{t}_1) + A_{(k+1)0}\sin(\delta\bar{t}_1)] \\
&\quad + (k-1)[B_{(k-1)0}\cos(\delta\bar{t}_1) - A_{(k-1)0}\sin(\delta\bar{t}_1)], \\
\mathrm{d}B_{k0}/\mathrm{d}\bar{t}_1 &= -(k+1)[A_{(k+1)0}\cos(\delta\bar{t}_1) - B_{(k+1)0}\sin(\delta\bar{t}_1)] \\
&\quad - (k-1)[A_{(k-1)0}\cos(\delta\bar{t}_1) + B_{(k-1)0}\sin(\delta\bar{t}_1)], \qquad (41)
\end{aligned}$$

for $k = 1, 2, 3, \ldots$. It should be noticed that for $\delta = 0$ one obtains again system (23). For convenience, the bar from $\bar{t}_1$ is omitted.

The calculations as given in Section 4.1.2 can be followed again to obtain

$$\mathrm{d}^2 W(t_1)/\mathrm{d}t_1^2 + (\delta^2 - 4)W(t_1) = D_1\delta^2, \qquad (42)$$

where $W(t_1)$ is defined as in Section 4.1.2, and $D_1 = W(0)$. Elementary calculations then yield

$$\textit{for } |\delta| < 2\text{:}\ \ W(t_1) = \frac{D_1}{4-\delta^2}[4\cosh(t_1\sqrt{4-\delta^2}) - \delta^2] + \frac{\mathrm{D}_2}{\sqrt{4-\delta^2}}\sinh(t_1\sqrt{4-\delta^2});$$

$$\textit{for } |\delta| = 2\text{:}\ \ W(t_1) = D_1 + \mathrm{D}_2 t_1 + \tfrac{1}{2}D_1\delta^2 t_1^2;$$

$$\textit{for } |\delta| > 2\text{:}\ \ W(t_1) = \frac{D_1}{\delta^2-4}[\delta^2 - 4\cos(t_1\sqrt{\delta^2-4})] + \frac{\mathrm{D}_2}{\sqrt{\delta^2-4}}\sin(t_1\sqrt{\delta^2-4});$$

where $D_2 = \mathrm{d}W(0)/\mathrm{d}t_1$. The interesting features of these solutions are, that for $|\delta|<2$, $W(t_1)$ (and so the energy) increases exponentially. For $|\delta| = 2$, $W(t_1)$ increases polynomially, and finally for $|\delta| > 2$, $W(t_1)$ is bounded due to the trigonometric functions.

4.3. Case 3: the non-resonant case

If Ω is not within an order ε-neighborhood of the frequencies that cause internal resonance, that is, not within an order ε-neighborhood of $mc\pi/L$ (with m odd) then $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\mathrm{d}A_{k0}/\mathrm{d}t_1 = 0, \qquad \mathrm{d}B_{k0}/\mathrm{d}t_1 = 0, \tag{43}$$

in order to avoid secular terms. Consequently, $A_{k0}(t_1)$ and $B_{k0}(t_1)$ are constants, say $K1_{k0}$ and $K2_{k0}$ such that $v_{k0}(t_0, t_1) = K1_{k0}\cos(ck\pi t_0/L) + K2_{k0}\sin(ck\pi t_0/L)$. Since $v(x,t) = \sum_{k=1}^{\infty} v_k(t)\sin(k\pi x/L)$, where $v_k(t)$ is approximated by $w_{k0}(t_0, t_1)$, it follows from the initial conditions $v(x,0) = f(x)$ and $v_t(x,0) = g(x)$ that

$$K1_{k0} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x, \qquad K2_{k0} = \frac{2}{ck\pi}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x. \tag{44}$$

The energy $E(t)$ of the conveyor belt system for this case can be approximated from

$$v(x,t) \approx \sum_{k=1}^{\infty}\left(K1_{k0}\cos\left(\frac{ck\pi t_0}{L}\right) + K2_{k0}\sin\left(\frac{ck\pi t_0}{L}\right)\right)\sin\left(\frac{k\pi x}{L}\right) + \mathcal{O}(\varepsilon), \tag{45}$$

where $K1_{k0}$ and $K2_{k0}$ are given by Eq. (44). Then,

$$\begin{aligned} E(t) &= \int_0^L (v_t^2 + c^2 v_x^2)\,\mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= \sum_{k=1}^{\infty}\frac{(ck\pi)^2}{2L}(K1_{k0}^2 + K2_{k0}^2) + \mathcal{O}(\varepsilon) \\ &= \frac{c^2\pi^2}{2L}\sum_{k=1}^{\infty} k^2(K1_{k0}^2 + K2_{k0}^2) + \mathcal{O}(\varepsilon), \end{aligned} \tag{46}$$

Using Eq. (44), finally,

$$\begin{aligned} E(t) &= \frac{2c^2L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\int_0^L f''\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x\right]^2 + \frac{2L^3}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}\left[\int_0^L g''\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x\right]^2 + \mathcal{O}(\varepsilon) \\ &= constant + \mathcal{O}(\varepsilon). \end{aligned} \tag{47}$$

5. Conclusions

In this paper, initial-boundary-value problems which can be used as models to describe transversal vibrations of belt systems, have been studied. The belt was assumed to move with a non-constant, low velocity $V(t)$, that is, $V(t) = \varepsilon(V_0 + \alpha \sin(\Omega t))$, where $0 < \varepsilon \ll 1$ and V_0, α, Ω are constants. Formal approximations of the solution of the initial-boundary-value problem have been constructed. Also explicit approximations of the energy of the belt system have been given. It turns out that there are infinitely many values of Ω giving rise to internal resonances in the belt system. These values for Ω are $mc\pi/L + \varepsilon\delta$ where m is an arbitrary odd integer, $c\pi/L$ is the lowest natural frequency of the constant velocity system, and δ is a detuning parameter of $\mathcal{O}(1)$. For $\Omega = c\pi/L + \varepsilon\delta$ (that is, $m = 1$) the problem has been studied completely. The following interesting results have been found: for $|\delta| < 2$ the energy of the belt system increases exponentially, for $|\delta| = 2$ the energy increases polynomially and for $|\delta| > 2$ the energy is bounded and varies trigonometrically. When Ω is not in an order ε-neighborhood of $mc\pi/L$ (with m odd) the energy of the belt system is constant up to order ε. All the results found are valid on long time scales, that is, on time scales of order ε^{-1}.

One major conclusion of this paper is that the truncation method cannot be applied to obtain asymptotic results on long time scales (that is, on time scales of order ε^{-1}) when Ω is in an order ε-neighborhood of an odd multiple of the lowest natural frequency of the constant velocity system. Moreover, in this paper improvements have been suggested to the (incorrect) results and applied methods as for instance given and used in Refs. [7–10] for low-speed belt systems.

Appendix A

In order to remove secular terms in the approximation for $v(x, t; \varepsilon)$ this appendix will show that the function $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\begin{aligned} \mathrm{d}A_{k0}(t_1)/\mathrm{d}t_1 &= (k+1)B_{(k+1)0}(t_1) + (k-1)B_{(k-1)0}(t_1), \\ \mathrm{d}B_{k0}(t_1)/\mathrm{d}t_1 &= -(k+1)A_{(k+1)0}(t_1) - (k-1)A_{(k-1)0}(t_1), \end{aligned} \tag{A.1}$$

for $k = 1, 2, 3, \ldots$. This can be derived as follows. After introducing a slow and a fast time in Section 4, Eq. (21) with $\Omega = c\pi/L$ was obtained. The solution of the $\mathcal{O}(1)$ problem is $v_{k0}(t_0, t_1) = A_{k0}(t_1)\cos(ck\pi t_0/L) + B_{k0}(t_1)\sin(ck\pi t_0/L)$, where A_{k0} and B_{k0} can be determined from the $\mathcal{O}(\varepsilon)$ equation by removing terms on the right side of this equation causing secular terms in $v_{k1}(t_0, t_1)$.

The first term on the right side of the $\mathcal{O}(\varepsilon)$ equation causing secular terms is $-2\partial^2 v_{k0}/\partial t_0 \partial t_1 = 2(ck\pi/L)[(\mathrm{d}A_{k0}/\mathrm{d}t_1)\sin(ck\pi t_0/L) - (\mathrm{d}B_{k0}/\mathrm{d}t_1)\cos(ck\pi t_0/L)]$.

Separating those terms in the second term of the right side the $\mathcal{O}(\varepsilon)$ equation causing secular terms,

$$\begin{aligned} &\left[\sum_1 - \sum_2 - \sum_3\right] \frac{2n\alpha\Omega}{(2j+1)L} \cos(\Omega t_0) v_{n0} \\ &\quad = \left[\sum_1 - \sum_2 - \sum_3\right] \frac{2n\alpha\Omega}{(2j+1)L} \cos(\Omega t_0)\left[A_{n0}(t_1)\cos\left(\frac{cn\pi t_0}{L}\right) + B_{n0}(t_1)\sin\left(\frac{cn\pi t_0}{L}\right)\right] \end{aligned}$$

$$= \frac{\alpha c\pi}{L^2}\cos\left(\frac{ck\pi t_0}{L}\right)\left[(k+1)A_{(k+1)0} - (k-1)A_{(k-1)0} - \frac{k+1}{2k+1}A_{(k+1)0} - \frac{k-1}{2k-1}A_{(k-1)0}\right]$$
$$+ \frac{\alpha c\pi}{L^2}\sin\left(\frac{ck\pi t_0}{L}\right)\left[(k+1)B_{(k+1)0} - (k-1)B_{(k-1)0} - \frac{k+1}{2k+1}B_{(k+1)0} - \frac{k-1}{2k-1}B_{(k-1)0}\right]$$
$$+ \text{“terms not giving rise to secular terms in } v_{k1}\text{”}.$$

Similarly, for the third term,

$$\left[\sum_1 - \sum_2 - \sum_3\right]\frac{4n}{(2j+1)L}(V_0 + \alpha\sin(\Omega t_0))\frac{\partial v_{n0}}{\partial t_0}$$
$$= \left[\sum_1 - \sum_2 - \sum_3\right]\frac{4n}{(2j+1)L}(V_0 + \alpha\sin(\Omega t_0))\frac{cn\pi}{L}\left[B_{n0}\cos\left(\frac{cn\pi t_0}{L}\right) - A_{n0}\sin\left(\frac{cn\pi t_0}{L}\right)\right]$$
$$= \frac{\alpha c\pi}{L^2}\cos\left(\frac{ck\pi t_0}{L}\right)\left[-2(k+1)^2A_{(k+1)0} - 2(k-1)^2A_{(k-1)0}\right.$$
$$\left. + \frac{2(k+1)^2}{2k+1}A_{(k+1)0} - \frac{2(k-1)^2}{2k-1}A_{(k-1)0}\right] + \frac{\alpha c\pi}{L^2}\sin\left(\frac{ck\pi t_0}{L}\right)$$
$$\times\left[-2(k+1)^2B_{(k+1)0} - 2(k-1)^2B_{(k-1)0} + \frac{2(k+1)^2}{2k+1}B_{(k+1)0} - \frac{2(k-1)^2}{2k-1}B_{(k-1)0}\right]$$
$$+ \text{“terms not giving rise to secular terms in } v_{k1}\text{”}.$$

By collecting all terms on the right side of the $\mathcal{O}(\varepsilon)$ equation containing $\cos(ck\pi t_0/L)$ and those containing $\sin(ck\pi t_0/L)$, setting their coefficients equal to 0 in order to remove the secular terms, one obtains (A.1).

Appendix B

In this appendix one shows that

$$\tfrac{1}{2}\sum_{k=1}^{\infty}\frac{d^2}{dt_1^2}(Y_{k0}^2 + X_{k0}^2) = 2\sum_{k=1}^{\infty}(X_{k0}^2 + Y_{k0}^2). \tag{B.1}$$

From Eq. (30) it follows that

$$\tfrac{1}{2}\sum_{k=1}^{\infty}\frac{d}{dt_1}(Y_{k0}^2 + X_{k0}^2) = \sum_{k=1}^{\infty}[Y_{k0}\dot{Y}_{k0} + X_{k0}\dot{X}_{k0}]$$
$$= \sum_{k=1}^{\infty}[X_{(k+1)0}Y_{k0} - Y_{(k+1)0}X_{k0}].$$

Differentiating this expression with respect to t_1, and using Section 4.11

$$
\begin{aligned}
\frac{1}{2}\sum_{k=1}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}(Y_{k0}^2+X_{k0}^2) &= \sum_{k=1}^{\infty}[\dot{X}_{(k+1)0}Y_{k0}+X_{(k+1)0}\dot{Y}_{k0}-\dot{Y}_{(k+1)0}X_{k0}-Y_{(k+1)0}\dot{X}_{k0}]\\
&= \sum_{k=1}^{\infty}(k+1)[X_{k0}^2+Y_{k0}^2]-\sum_{m=2}^{\infty}(m-1)[X_{m0}^2+Y_{m0}^2]\\
&= 2(X_{10}^2+Y_{10}^2)+\sum_{k=2}^{\infty}(k+1)[X_{k0}^2+Y_{k0}^2]-\sum_{m=2}^{\infty}(m-1)[X_{m0}^2+Y_{m0}^2]\\
&= 2(X_{10}^2+Y_{10}^2)+\sum_{k=2}^{\infty}[(k+1)-(k-1)][X_{k0}^2+Y_{k0}^2]\\
&= 2(X_{10}^2+Y_{10}^2)+\sum_{k=2}^{\infty}2[X_{k0}^2+Y_{k0}^2]=2\sum_{k=1}^{\infty}[X_{k0}^2+Y_{k0}^2].
\end{aligned}
$$

And so, Eq. (B.1) has been proved.

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