Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 264 (2003) 225–234
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Frequency analysis of a rectangular plate with attached discrete systems

S. Kukla*

Technical University of Częstochowa, Dąbrowskiego 73, 42-200 Częstochowa, Poland

Received 4 July 2001; accepted 29 July 2002

1. Introduction

Vibration problems of beams or plates with discrete systems are very important because of the practical applications of the vibration analysis of this type of compound systems. The problems of vibrations of plates with attachments have been the subject of many papers (e.g., the Refs. [1–15]). Most of the papers are devoted to the vibrations of plates with rigidly mounted concentrated masses or supports located at the edge or at interior points. The vibration of plates with elastically mounted masses are considered in Refs. [2,6,14]. The investigations lead to conclusions concerning the effect of the attachments on vibrations as well as the utility of the applied methods to solutions of the stated vibration problems.

In Ref. [1] the exact solution of the problem of vibration of a rectangular plate with rigid supports at inner points has been given. The considerations concern the plate with two opposite edges simply supported. Ref. [2] is devoted to the dynamics of plates and shells with concentrated masses. In a part of the work the authors deal with the problem of the free vibration of plates and shells with oscillators. The frequency analysis of the system with the use of the Green's function method is presented. The method of superposition was exploited in Refs. [3,4] to solve free vibration problems of plates with concentrated attachments. The technique has been also successfully employed in many vibration problems concerning various cases of rectangular plates. In Refs. [5–9] the Rayleign–Ritz method was applied. The method was used in the cases of uniform rectangular plates with attachments for different boundary conditions as well as plates of polygonal shape. The flexibility function approach has been applied in Refs. [10,11] in the analysis of point supported rectangular plates. In this approach a fictitious foundation simulating the points supports in plates was introduced. In Ref. [12] the free vibration analysis of plates with elastic point supports, line supports and uniformly distributed supports is presented. The analysis was performed by using the finite strip element method. The application of the Green's function method to the free vibration problem of plates with attachments was presented in Refs. [13–15].

*Corresponding author.
E-mail address: kukla@matinf.pcz.czest.pl (S. Kukla).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01180-X

The analytical form of the frequency equations and mode shapes was obtained for rectangular plates with two opposite edges simply supported.

The problem of the transverse free vibrations of a rectangular plate with discrete systems attached at arbitrary points of the plate if considered here. The exact solution of the problem is obtained by using a Green's function method. In contrast to numerical method (e.g., the finite element method), the presented method leads to an analytical form of the solution and permits qualitative estimation of the influence of the parameters characterizing the system on its free vibration. Because the problems of vibrations of plates with elastically mounted masses and with elastic supports can be treated as particular cases of the ones considered here, the presented solution comprises a wide range of issues. The effect of the location of a discrete system attached to a plate on the eigenfrequencies of the combined system has been numerically investigated.

2. Theory

Consider a rectangular plate with N discrete systems attached to it at points (x_i, y_i), $i = 1, 2, \ldots, N$. The ith discrete system is composed of n_i masses and springs combined in series (see Fig. 1). The plate deflection $w(x,y,t)$ is governed by the following differential equation:

$$D\nabla^4 w + \rho w_{tt} = \sum_{i=1}^{N} k_{i1}[z_{i1}(t) - w(x_i, y_i, t)]\delta(x - x_i)\delta(y - y_i), \tag{1}$$

where D is the flexural rigidity, ρ is the mass density per unit area of the plate, k_{i1} are the stiffness coefficients of the translational springs, $\nabla^4 = \partial^4/\partial x^4 + 2\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4$ is the biharmonic operator and $\delta(\ )$ is the Dirac delta function. Eq. (1) is accompained by appropriate boundary conditions.

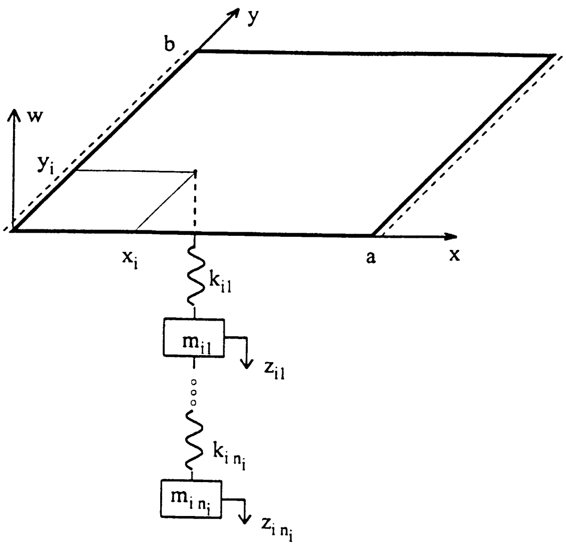

Fig. 1. A sketch of the system considered: the jth discrete system of n_j degree-of-freedom attached at the plate point (x_j, y_j).

The displacements $z_{ij}(t)$ of the masses m_{ij} ($i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n_i$) are governed by equations, which are written for three cases:

1. If $n_i = 1$, then

$$m_{i1}\frac{d^2 z_{i1}(t)}{dt^2} + k_{i1}(z_{i1}(t) - w(x_i, y_i, t)) = 0. \tag{2}$$

2. If $n_i = 2$, then

$$m_{i1}\frac{d^2 z_{i1}(t)}{dt^2} + k_{i1}(z_{i1}(t) - w(x_i, y_i, t)) + k_{i2}(z_{i1}(t) - z_{i2}(t)) = 0, \tag{3}$$

$$m_{i2}\frac{d^2 z_{i2}(t)}{dt^2} + k_{i2}(z_{i2}(t) - z_{i1}(t)) = 0. \tag{4}$$

3. If $n_i > 2$, then

$$m_{i1}\frac{d^2 z_{i1}(t)}{dt^2} + k_{i1}(z_{i1}(t) - w(x_i, y_i, t)) + k_{i2}(z_{i1}(t) - z_{i2}(t)) = 0, \tag{5}$$

$$m_{ij}\frac{d^2 z_{ij}(t)}{dt^2} + k_{ij}(z_{ij}(t) - z_{ij-1}(t)) + k_{ij+1}(z_{ij}(t) - z_{ij+1}(t)) = 0 \quad \text{for } j = 2, \ldots, n_i - 1, \tag{6}$$

$$m_{in_i}\frac{d^2 z_{in_i}(t)}{dt^2} + k_{in_i}(z_{in_i}(t) - z_{in_i-1}(t)) = 0. \tag{7}$$

In order to find the natural frequencies of the system, ω, one assumes that

$$w(x, t) = \bar{W}(x)\cos \omega t, \quad z_{ij}(t) = \bar{Z}_{ij} \cos \omega t. \tag{8}$$

Taking Eq. (8) into account in Eqs. (1)–(7) and introducing dimensionless quantities, one obtains (the case of $n_i > 2$ is presented below):

$$\frac{\partial^4 W}{\partial \eta^4} + 2\Phi^2\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \Phi^4\frac{\partial^4 W}{\partial \xi^4} - \Phi^4\lambda^4 W = -\Phi^3 \sum_{i=1}^{N} K_{i1}(W(\xi_i, \eta_i) - Z_{i1})\delta(\xi - \xi_i)\delta(\eta - \eta_i), \tag{9}$$

$$-\lambda^4 Z_{i1} + \beta_{i1}^4[Z_{i1} - W(\xi_i, \eta_i)] + \gamma_{i1}\beta_{i1}^4(Z_{i1} - Z_{i2}) = 0, \tag{10}$$

$$-\lambda^4 Z_{ij} + \beta_{ij}^4[Z_{ij} - Z_{ij-1}] + \gamma_{ij}\beta_{ij}^4(Z_{ij} - Z_{ij+1}) = 0, \quad j = 2, \ldots, n_i - 1, \tag{11}$$

$$-\lambda^4 Z_{in_i} + \beta_{in_i}^4[Z_{in_i} - Z_{in_i-1}] = 0, \tag{12}$$

where $\xi = x/a$, $\eta = y/b$, $W = \bar{W}/a$, $\xi_i = x_i/a$, $\eta_i = y_i/b$, $\Phi = b/a$, $\lambda^4 = \omega^2 a^4 \rho/D$, $Z_{ij} = \bar{Z}_{ij}/a$, $K_{ij} = k_{ij}a^2/D$, $M_{ij} = m_{ij}/a^2\rho$, $\gamma_{ij} = K_{ij+1}/K_{ij}$ and $\beta_{ij}^4 = (\rho a^4/D)(k_{ij}/m_{ij})$.

The expression $W(\xi_i, \eta_i) - Z_{i1}$, which occurs in Eq. (9), may be written by using Eqs. (10)–(12) in the following form:

$$W(\xi_i, \eta_i) - Z_{i1} = Q_{in_i} W(\xi_i, \eta_i). \tag{13}$$

Here the coefficients Q_{in_i} for $n_i = 1$, 2 and 3, are [16]:

$$Q_{i1} = 1 + \frac{1}{\kappa_{i1}}, \quad Q_{i2} = 1 + \frac{1}{\kappa_{i1} - \gamma_{i1}(1 + 1/\kappa_{i2})},$$
$$Q_{i3} = 1 + \frac{1}{\kappa_{i1} - \gamma_{i1}[1 + 1/(\kappa_{i2} - \gamma_{i2}(1 + 1/\kappa_{i3}))]}, \tag{14}$$

where $\kappa_{ij} = \lambda^4/\beta_{ij}^4 - 1$. The expressions Q_{i2} and Q_{i3} may be written symbolically, by applying the notation used for continued fractions [17], as

$$Q_{i2} = 1 + \frac{1}{\kappa_{i1} - \gamma_{i1}} \, \underset{+}{} \, \frac{-\gamma_{i1}}{\kappa_{i2}}, \quad Q_{i3} = 1 + \frac{1}{\kappa_{i1} - \gamma_{i1}} \, \underset{+}{} \, \frac{-\gamma_{i1}}{\kappa_{i2} - \gamma_{i2}} \, \underset{+}{} \, \frac{-\gamma_{i2}}{\kappa_{i3}}.$$

Generally, one has

$$Q_{in_i} = 1 + \frac{1}{\kappa_{i1} - \gamma_{i1}} \, \underset{+}{} \, \frac{-\gamma_{i1}}{\kappa_{i2} - \gamma_{i2}} \, \underset{+}{} \, \frac{-\gamma_{i2}}{\kappa_{i3} - \gamma_{i3}} \, \underset{+}{} \cdots \underset{+}{} \, \frac{-\gamma_{in_i-1}}{\kappa_{in_i}}. \tag{15}$$

For determination of the vibration frequencies of the system the Green's function G of the corresponding differential problem has been applied. The function is a solution of the differential equation

$$\frac{\partial^4 G}{\partial \eta^4} + 2\Phi^2 \frac{\partial^4 G}{\partial \xi^2 \partial \eta^2} + \Phi^4 \frac{\partial^4 G}{\partial \xi^4} - \Phi^4 \lambda^4 G = \delta(\xi - \zeta)\delta(\eta - \theta) \tag{16}$$

and in respect to variables ξ and η, it satisfies the same boundary conditions as the function W. The Green's functions for Levy isotropic plates are given in Ref. [13], and for orthotropic plates they are presented in Ref. [18].

The following equation has been obtained by using the properties of the Green's function and Eqs. (9) and (13):

$$W(\xi, \eta) = -\Phi^3 \sum_{i=1}^{N} K_{i1} Q_{in_i} W(\xi_i, \eta_i) G(\xi, \eta, \xi_i, \eta_i; \lambda). \tag{17}$$

Substituting now $(\xi, \eta) = (\xi_k, \eta_k)$ for $k = 1, 2, \ldots, N$, successively into Eq. (17), one obtains a system of N homogeneous, linear equations with respect to displacements $W(\xi_i, \eta_i)$, $i = 1, 2, \ldots, N$. This equation system written in the matrix notation has the form

$$\mathbf{AW} = \mathbf{0}, \tag{18}$$

where $\mathbf{A} = [a_{ik}]_{1 \leqslant i,k \leqslant N}$, $a_{ik} = \Phi^3 K_{k1} Q_{kn_k} G(\xi_i, \eta_i, \xi_k, \eta_k; \lambda) + \delta_{ik}$ and δ_{ik} is the Kronecker delta. The requirement of a non-trivial solution of Eq. (18) yields the eigenvalue equation

$$\det \mathbf{A} = 0. \tag{19}$$

This equation with respect to non-dimensional frequencies is then solved numerically.

3. Discussion and numerical examples

Consider the system consisting of a rectangular plate and a single oscillator attached to this plate at point (ξ_1, η_1). The frequency equation for the system, which is obtained from Eq. (19), has the form

$$\Phi^3 K_{11} Q_{1n_1} G(\xi_1, \eta_1, \xi_1, \eta_1; \lambda) + 1 = 0, \tag{20}$$

where Q_{in_1} for $n_1 = 1$, 2 and 3, are given by Eq. (14), and for arbitrary n_1, by Eq. (15). The mode shapes corresponding to the eigenfrequencies $\lambda_{mn} = a\sqrt[4]{\omega_{mn}^2 \rho / D}, \quad m, n = 1, 2, \ldots$, determined from Eq. (20), are obtained on the basis of Eq. (17) in the form

$$W_{mn}(\xi, \eta) = C_{mn} G(\xi, \eta, \xi_1, \eta_1; \lambda_{mn}), \tag{21}$$

where C_{mn} is a constant. If K_{11} tends to infinity in Eq. (20) for $n_1 = 1$, then the obtained frequency equation corresponds to a system of a plate with rigidly attached mass. This equation is as follows:

$$\Phi^4 \lambda^4 M_{11} G(\xi_1, \eta_1, \xi_1, \eta_1; \lambda) - 1 = 0. \tag{22}$$

Similarly, if M_{11} tends to infinity, then Eq. (20) for $n_1 = 1$, assumes the form of the frequency equation for a plate with intermediate elastic support

$$\Phi^3 K_{11} G(\xi_1, \eta_1, \xi_1, \eta_1; \lambda) + 1 = 0. \tag{23}$$

An elastic support or a concentrated mass attached to a plate, causes changes of its natural frequencies. It is well known that an elastic support leads to the increase of the plate frequencies, and a rigidly attached mass decreases the frequencies [19]. Comparing Eq. (20) with Eqs. (22) and (23) one can show, that a single oscillator increases or decreases the frequencies of the plate depending of the sign of the expression Q_{1n_1}. An oscillator added to the plate decreases the frequencies of the original system when $Q_{1n_1} < 0$, and increases the frequencies when $Q_{1n_1} > 0$. For instance, the condition $Q_{11} > 0$, for $\lambda > \beta_{11}$ is satisfied. Therefore, the frequencies greater than β_{11} are increased and the other are decreased in comparison with frequencies of the plate without oscillator.

The effect of the location of a discrete system attached to a plate on the eigenfrequencies of the combined system has been numerically investigated. In the first example, the calculation was performed for the system, which consists of an oscillator and an isotropic, quadratic plate simply supported at two opposite edges ($\xi = 0$ and 1) and free at the other (S–F–S–F). The oscillator, which is characterised by dimensionless quantities: $M_1 = 1.0$ and $K_1 = 1000$, is mounted at the point (ξ_1, η_1) of the plate. The Poisson coefficient is assumed $\nu = 0.3$. The dimensionless free vibration frequencies of the isolated plate, $\Omega_{mn} = \lambda_{mn}^2$, are: $\Omega_{11} = 9.6314$, $\Omega_{12} = 16.1348$, $\Omega_{13} = 36.7256$, $\Omega_{21} = 38.9449$ (the indices show the mode of vibration), and the frequency parameter of the isolated oscillator is $\Omega_1^* = \beta_{11}^2 = 31.6228$. The contour plots of the functions $\Omega_{mn} = \Omega_{mn}(\xi_1, \eta_1)$ are presented in Fig. 2. Figs. 2(a), (b) and (d) correspond to the vibration modes of the plate: (1,1), (1,2) and (1,3), respectively, and Fig. 2(c) corresponds to the vibration mode of the oscillator.

If $\xi_1 = 0$ or $\xi_1 = 1$ then the frequencies of the plate without an oscillator (Figs. 2(a), (b) and (d)) or the frequency of the grounded oscillator (Fig. 2(c)), are obtained. Two frequencies of the combined system are lower than the frequency of the isolated oscillator. These frequencies are decreased in comparison to the corresponding frequencies of the plate without the oscillator (Figs. 2(a) and (b)). All the other frequencies of the plate are greater than the frequency Ω_1^* and the frequencies of the combined system corresponding to them are increased. The thick-line in Fig. 2(c) appoints the characteristic locations of the oscillator on the plate: the free vibration

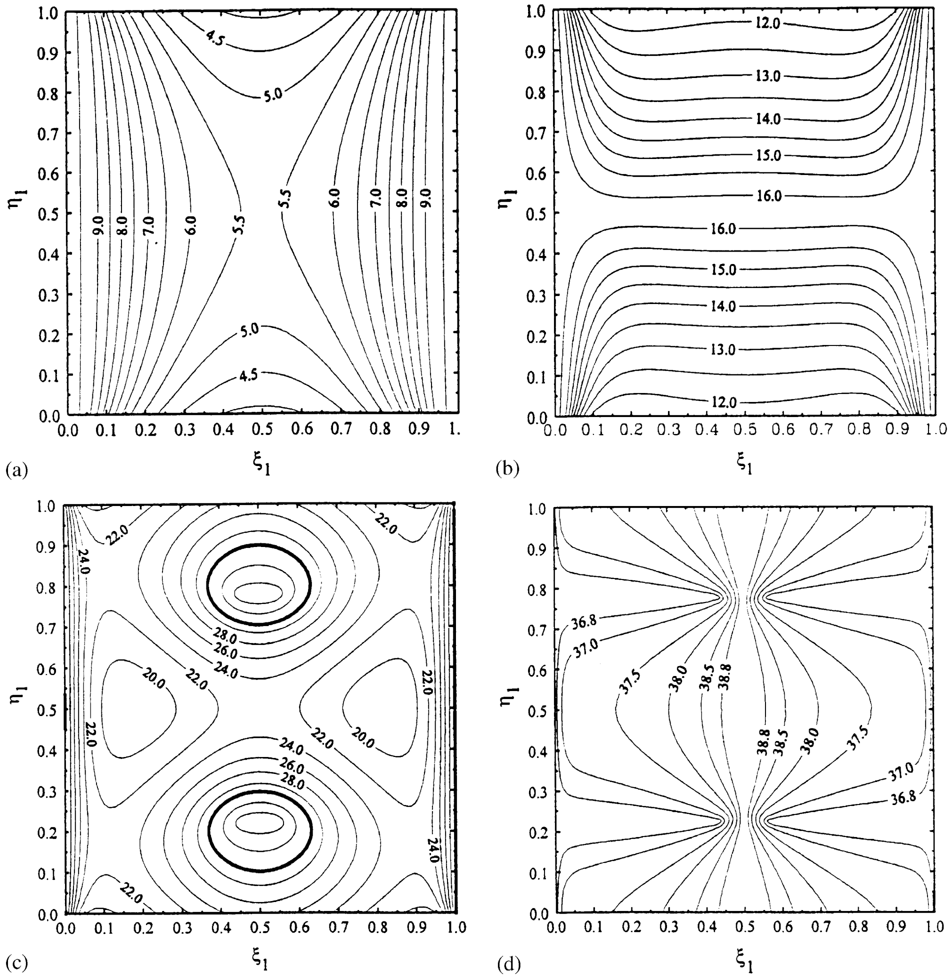

Fig. 2. Contour lines for the functions $\Omega_n = \Omega_n(\xi_1,\eta_1)$, $n = 1,\ldots,4$, corresponding to first four mode vibration of a square S–F–S–F plate with one-degree-of-freedom discrete system mounted at plate point (ξ_1,η_1); $M_1 = 1.0$, $K_1 = 1000$.

frequency of the combined system with the oscillator attached at the arbitrary point of this line is equal to the frequency of the isolated oscillator. In Ref. [20] it has been shown, that in this case, the oscillator is attached at a node of a normal mode of the structure. Here it results from Eqs. (20) and (21).

The four mode shapes of the plate corresponding to the system of the S–F–S–F plate with the oscillator mounted at an established point (ξ_1,η_1) are presented in Fig. 3. It is assumed that the free vibration frequency of the isolated oscillator is $\Omega_1^* = 31.6222$ and $\eta_1 = 0.21$. The abscissa of the location point $\xi_1 \approx 0.37$ is obtained from the equation: $G(\xi_1, \eta_1, \xi_1, \eta_1; \sqrt{\Omega_1^*}) = 0$. This plate point belongs to the distinguished curve in Fig. 2(c) and it belongs also to the node line of the mode shape corresponding to the frequency Ω_1^*. The oscillator causes shifting of the frequencies in

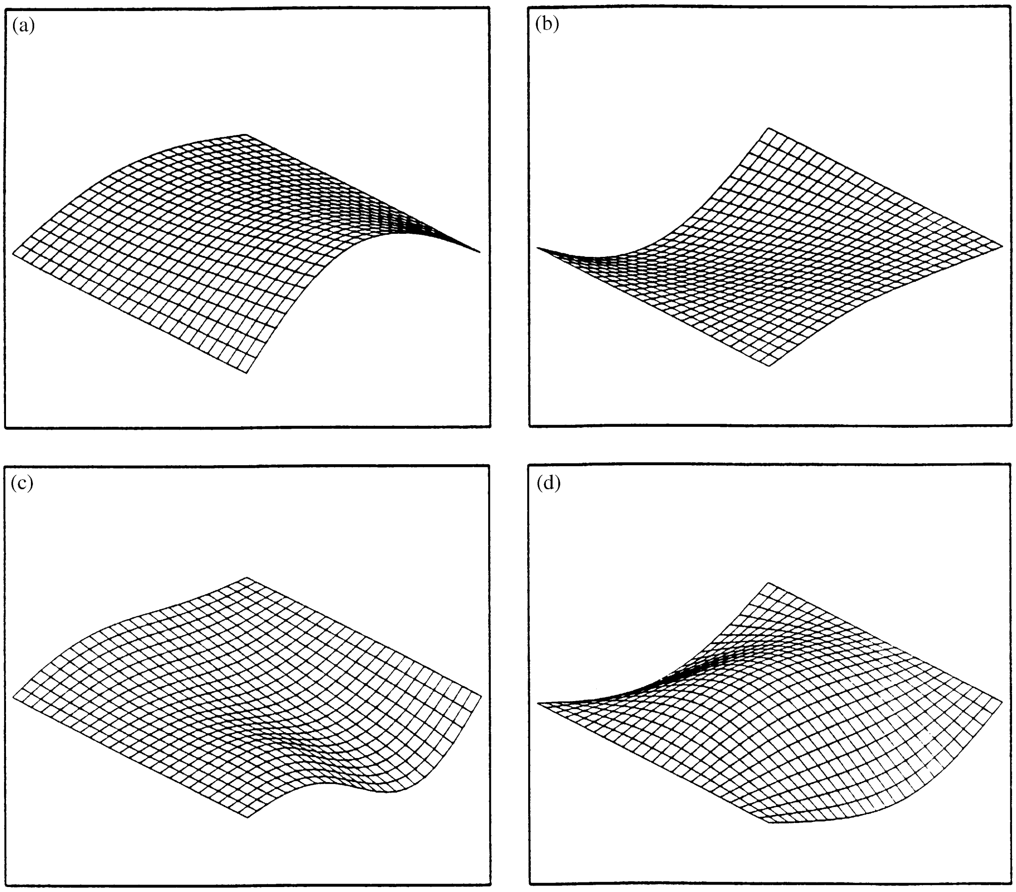

Fig. 3. Mode shapes of the square plate with an oscillator attached at the plate point $(\xi_1,\eta_1)=(0.37;0.21)$.

comparing with those obtained for the plate without attachments as it is shown above. For the assumed location of the oscillator, the five dimensionless free vibration frequencies of the compound system are: $\Omega_{11}=5.2416$, $\Omega_{12}=13.3870$, $\Omega_1^*=31.6228$, $\Omega_{13}=36.7613$ and $\Omega_{21}=43.0500$.

The second example is concerned with the system of the S–F–S–F quadratic plate with two-degree-of-freedom discrete system, mounted at the point (ξ_1,η_1) of the plate. In this case $(n_1=2)$, on the basis of Eq. (14b), we have

$$Q_{12}=\frac{\lambda^4(\lambda^4-\mu^4)}{(\lambda^4-\beta_1^{*4})(\lambda^4-\beta_2^{*4})},$$

where $\mu^4=\gamma_1\beta_{11}^4+\beta_{12}^4$ and $\beta_{1,2}^{*2}=\sqrt{\frac{1}{2}[\beta_{11}^4+\mu^4\mp\sqrt{(\beta_{11}^4+\mu^4)^2-4\beta_{11}^4\beta_{12}^4}]}$ are the dimensionless frequencies of the isolated discrete system. Because $\beta_1^*<\mu<\beta_2^*$, then the frequencies of the plate with one discrete system with two degrees of freedom attached, within two intervals, $\beta_1^*<\lambda<\mu$ and $\lambda>\beta_2^*$ are increased, and the remaining frequencies are decreased (as compared with the frequencies of the plate without the discrete system). The discrete system considered here is

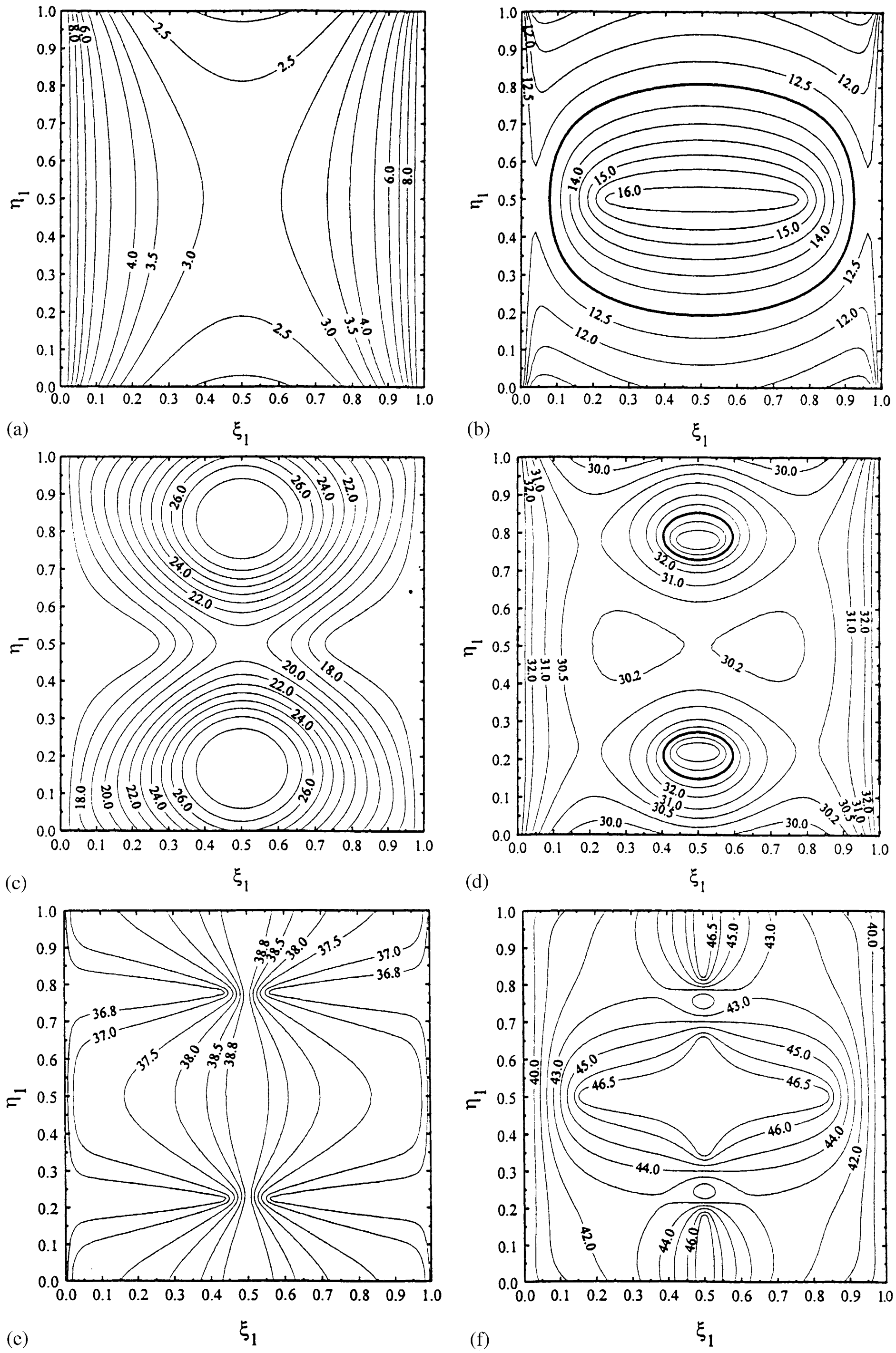

Fig. 4. Contour lines for the functions $\Omega_n = \Omega_n(\xi_1, \eta_1)$, $n = 1, \ldots, 6$, corresponding to first six mode vibration of a square S–F–S–F plate with two-degree-of-freedom discrete system mounted at plate point (ξ_1, η_1); $M_1 = M_2 = 1.0$, $K_1 = K_2 = 441$.

characterized by the dimensionless quantities: $M_1 = M_2 = 1.0$ and $K_1 = K_2 = 441$. It follows that the dimensionless free vibration frequencies of the isolated discrete system are : $\Omega_1^* = (\beta_{11})^2 = 12.9787$, $\Omega_2^* = (\beta_{12})^2 = 33.9787$ and $\mu^2 = 29.6985$. The contour plots of the functions $\Omega_{mn} = \Omega_{mn}(\xi_1, \eta_1)$ for six eigenfrequencies of the compound system are presented in Fig. 4. According to the earlier considerations, the frequencies of the compound system within either of two intervals $\Omega_1^* < \Omega < \mu^2$ and $\Omega > \Omega_2^*$ are greater than or equal to the relevant plate frequencies (Figs. 4(c), (e) and (f)). In this case, two "additional" frequencies appear (except the frequencies corresponding to the plate eigenfrequencies). They correspond to the discrete system eigenfrequencies (Figs. 4(b) and (d)). If the discrete system is mounted at the plate points marked by the thick line in Figs. 4(b) and (d) then the free vibration frequencies of the compound system are equal to the first or second frequencies of isolated discrete system, respectively.

4. Conclusions

The closed-form solution of the problem of free vibration of a rectangular plate with attached discrete systems has been presented. The rectangular plate with two opposite edges simply supported is assumed. Each discrete system of finite degrees of freedom consists of spring–mass systems. Although the number of discrete systems considered in the numerical examples was limited to two, the solution may be used for an arbitrary number of discrete systems attached to the plate. The solution of the problem is obtained by using the Green's function method.

The discrete system mounted to a plate may cause increases or decreases in the eigenfrequencies of the combined system as compared with those of the without discrete systems attached. In the case of a single spring–mass system attached at an interior point of the plate, the frequencies of the system lower than the spring–mass frequency are decreased and the higher ones—increased, as compared with corresponding plate frequencies (except at points when the frequencies are unchanged). The frequencies of a combined system of two-degree-of-freedom discrete system attached to a plate, are decreased in two finite intervals. Similarly, the frequencies of the *n*-degree-of-freedom mounted to a plate determine the intervals of decreasing of the plate frequencies. At the same time, the remaining frequencies are increased as compared with the plate frequencies.

References

[1] W. Nowacki, Vibration and buckling of rectangular plates simply supported at the periphery and at several points inside, Archiwum Mechaniki Stosowanej 5 (3) (1953) 437–454 (in polish).
[2] L.W. Andreev, A.L. Dyshko, I.D. Pawlenko, Dynamics of Plates and Shells with Concentrated Masses, Mashinostroenie, Moscow, 1988 (in Russian).
[3] D.J. Gorman, Free Vibration Analysis of Rectangular Plates, North-Holland, New York, 1982.
[4] D.J. Gorman, An analytical solution for the free vibration analysis of rectangular plates resting on symmetrically distributed point supports, Journal of Sound and Vibration 79 (1981) 561–574.
[5] P.A.A. Laura, C.P. Filipich, V.H. Cortinez, Vibrations of beams and plates carrying concentrated masses, Journal of Sound and Vibration 117 (1987) 459–465.
[6] H.A. Larrondo, D.R. Avalos, P.A.A. Laura, Transverse vibrations of simply supported anisotropic rectangular plates carrying an elastically mounted concentrated mass, Journal of Sound and Vibration 215 (1998) 1195–1202.

[7] G.-B. Chai, Frequency analysis of a S-C-S-C plate carrying a concentrated mass, Journal of sound and Vibration 179 (1995) 170–177.
[8] K.M. Liew, K.Y. Lam, Effects of arbitrarily distributed elastic point constraints on vibrational behaviour of rectangular plates, Journal of Sound and Vibration 174 (1994) 23–36.
[9] K.Y. Lam, G. Amrutharaj, Natural frequencies of rectangular stepped plates using polynomial beam functions with subsectioning, Applied Acoustics 44 (1995) 325–340.
[10] A.V. Bapat, S. Suryanarayan, Free vibrations of rectangular plates with interior point supports, Journal of Sound and Vibration 134 (1989) 291–313.
[11] A.V. Bapat, S. Suryanarayan, The fictitious foundation approach to vibration analysis of plates with interior point supports, Journal of Sound and Vibration 155 (1992) 325–341.
[12] M.H. Huang, D.P. Thambiratnam, Free vibration analysis of rectangular plates on elastic intermediate supports, Journal of Sound and Vibration 240 (2001) 567–580.
[13] L.A. Bergman, J.K. Hall, G.G.G. Lueschen, D.M. McFarland, Dynamic Green's functions for Levy plates, Journal of Sound and Vibration 162 (1993) 281–310.
[14] S. Kukla, Free vibration of a rectangular plate with oscillators, XVIII Symposium "Vibrations in Physical Systems", Poznań-Błażejewko, 1998, pp. 175–176.
[15] S. Kukla, Die Methode der Greenschen Funktion in der Frequenzanalyse von Platten, Zeitschrift für Angewandte Mathematik und Mechanik 76 (1992) T19–T22.
[16] S. Kukla, Application of Green's functions in frequency anlaysis of Timoshenko beams with oscillators, Journal of Sound and Vibration 205 (1997) 355–363.
[17] W.B. Jones, W.J. Thron, Continued Fractions. Analytic Theory and Applications, Addison-Wesley, Reading, MA, 1980.
[18] S. Kukla, Application of Green's function in free vibration analysis of a system of line connected rectangular plates, Journal of Sound and Vibration 217 (1998) 1–15.
[19] E.H. Dowell, On some general properties of combined dynamical systems, Journal of Applied Mechanics 46 (1979) 206–209.
[20] P.D. Cha, C. Peirre, Frequency analysis of a linear elastic structure carrying a chain of oscillators, Journal of Engineering Mechanics 125 (1999) 587–591.