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Damped vibration analysis of an elastically connected complex double-string system

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Abstract

The main purpose of the present paper is to consider theoretically damped transverse vibrations of an elastically connected double-string system. This system is treated as two viscoelastic strings with a Kelvin–Voigt viscoelastic layer between them. A theoretical analysis has been made for a simplified model of the system, in which assumed physical parameters make it possible to decouple the governing equations of motion by introducing the principal co-ordinates. Applying the method of separation of variables and the modal expansion method, exact analytical solutions for damped free and forced responses of the system subjected to arbitrarily distributed transverse continuous loads are determined in the case of arbitrary magnitude of linear viscous damping. It is important to note that the solutions obtained are explicitly expressed in terms of parameters characterizing the physical properties of the system under discussion. For the sake of completeness of the analysis, solutions for undamped free and forced vibrations are also formulated.

1. Introduction

The question of damping is one of the most important, as well difficult and complicated problems in vibration theory of mechanical systems. Mathematical troubles with damping begin in a discrete vibratory system having two degrees of freedom [1–13], and in a multi-degree-of-freedom system difficulties intensify [1–4]. For a simple distributed (continuous) system damped vibrations are usually determined under the assumption of small damping [1–7,12,13], and the general case of arbitrary damping is not considered. Most serious problems occur in damped complex continuous systems [14]. Investigations of their dynamical behaviours are of great theoretical and practical importance. A system of two parallel strings continuously coupled by a Winkler-type elastic layer constitutes the simplest model of one-dimensional complex continuous

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system. Undamped transverse vibrations of such a system have been discussed in the author's previous publications [14–17]. The vibration problem of an elastically connected double-string system with damping is difficult to solve in the general case. In Ref. [14], the author has determined the solutions for free and forced vibrations of the general string system in the case of low damping. The author's latest work [18] is devoted to damped free vibrations of the title system with arbitrary linear viscous damping. The present paper being an extension of this work contains the complete damped free and forced vibration analysis of a complex string system modelled as two viscoelastic strings connected by a Kelvin–Voigt viscoelastic layer. Exact analytical solutions are formulated for a certain simplified model of this system due to arbitrary magnitude of viscous damping.

It is proper to note that damped vibrations of technically important analogous double-beam systems have been studied by a number of authors: Oniszczuk [14,22], Dublin and Friedrich [19], Kessel and Raske [20], Lu and Douglas [21], Jacquot and Foster [23], Irie et al. [24], Nakai and Kitano [25], Vu [26], Aida et al. [27], Chen and Sheu [28,29], Chen and Lin [30], Kawazoe et al. [31], and Vu et al. [32], among others. In the investigation of a damped double-string continuous system, the vibration analysis of a two-degree-of-freedom discrete system with damping [33–35] can also be helpful because of evident analogy existing between both these systems [14,16,17].

2. Formulation of the problem

The transverse vibration problem of an elastically connected double-string complex system with damping was exactly formulated in Refs. [14,18]. The dynamic model of the system under consideration depicted in Fig. 1 consists of two parallel, homogeneous, uniform, viscoelastic strings attached together by a continuously distributed Kelvin–Voigt viscoelastic massless layer. As is well known this elastic foundation model, being a generalized Winkler one, is characterized by two parameters: stiffness modulus k and viscous damping coefficient c [3,14,36,37]. Both strings have the same length and are simply supported at their ends. The strings are stretched under suitable constant tensions and subjected to arbitrarily distributed transverse continuous loads. Small vibrations of the system are considered.

According to the Kelvin–Voigt foundation model, the damped transverse motion of an elastically connected double-string system due to general loading is described by a set of two

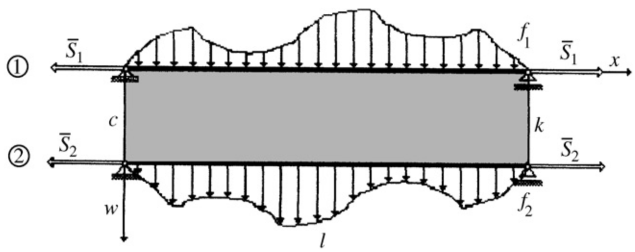

Fig. 1. The physical model of an elastically connected complex double-string complex system.

non-homogeneous partial differential equations [14,18] as

$$\begin{aligned} m_1\ddot{w}_1 + c_1\dot{w}_1 + c(\dot{w}_1 - \dot{w}_2) - S_1 w_1'' + k(w_1 - w_2) = f_1(x,t), \\ m_2\ddot{w}_2 + c_2\dot{w}_2 + c(\dot{w}_2 - \dot{w}_1) - S_2 w_2'' + k(w_2 - w_1) = f_2(x,t), \end{aligned} \tag{1}$$

where $w_i = w_i(x,t)$ is the transverse string deflection; x, t are the spatial co-ordinate and the time; $f_i = f_i(x,t)$ is the exciting distributed load; c_i is the viscous damping coefficient for the string; F_i is the cross-sectional area of the string; c, k are the viscous damping coefficient and the stiffness modulus of a Kelvin–Voigt viscoelastic layer, respectively; l is the string length; m_i is the string mass per unit length; S_i is the string tension; ρ_i is the mass density; $m_i = \rho_i F_i$, $\dot{w}_i = \partial w_i/\partial t$, $w_i' = \partial w_i/\partial x$, $i = 1,2$.

The boundary and initial conditions for this problem have the form

$$w_i(0,t) = w_i(l,t) = 0, \tag{2}$$

$$w_i(x,0) = w_{i0}(x), \quad \dot{w}_i(x,0) = v_{i0}(x), \quad i = 1,2. \tag{3}$$

Eqs. (1) constitute a coupled system of differential equations in two unknown functions $w_1(x, t)$ and $w_2(x,t)$, which is difficult to solve in a general form. Making certain simplifying assumptions, this system can be easily decoupled, which considerably facilitates finding the solutions. Therefore, the analysis of this problem is performed for a simplified system variant when the physical parameters, namely, the viscous damping coefficients c_i, the unit masses m_i, and the tension forces S_i are the same and assumed to be

$$c_i = C, \quad m_i = \rho_i F_i = m, \quad S_i = S, \quad i = 1,2. \tag{4}$$

It is seen that the string parameters F_i and ρ_i satisfying the corresponding relation (4) can be arbitrary to a certain degree. In the light of the above assumptions (4), Eqs. (1) can be rewritten in the form

$$\begin{aligned} m\ddot{w}_1 + C\dot{w}_1 + c(\dot{w}_1 - \dot{w}_2) - S w_1'' + k(w_1 - w_2) = f_1(x,t), \\ m\ddot{w}_2 + C\dot{w}_2 + c(\dot{w}_2 - \dot{w}_1) - S w_2'' + k(w_2 - w_1) = f_2(x,t). \end{aligned} \tag{5}$$

Introducing the new variables being the principal co-ordinates defined as

$$u_1(x,t) = \sum_{i=1}^{2} w_i(x,t), \quad u_2(x,t) = \sum_{i=1}^{2} a_i w_i(x,t), \quad a_1 = -a_2 = 1 \tag{6}$$

allows the decoupling of the differential equations (5). Adding and subtracting Eqs. (5) gives

$$m\ddot{u}_1 + C\dot{u}_1 - S u_1'' = F_1(x,t), \quad m\ddot{u}_2 + (C + 2c)\dot{u}_2 - S u_2'' + 2k u_2 = F_2(x,t), \tag{7}$$

where

$$F_1(x,t) = \sum_{i=1}^{2} f_i(x,t), \quad F_2(x,t) = \sum_{i=1}^{2} a_i f_i(x,t), \quad a_1 = -a_2 = 1. \tag{8}$$

The equations of motion (7) are now uncoupled, and each of them represents the damped transverse vibrations of a single string. Moreover, the second equation describes the oscillations of a string resting on a viscoelastic foundation. Eqs. (7) are accompanied by appropriate transformed boundary conditions (2)

$$u_i(0,t) = u_i(l,t) = 0, \quad i = 1,2 \tag{9}$$

and can be solved independently of each other to find the principal co-ordinates $u_i = u_i(x,t)$.

Finally, the unknown solutions of Eqs. (5) can be determined from the relationships

$$w_1(x,t) = 0.5\sum_{i=1}^{2} u_i(x,t), \quad w_2(x,t) = 0.5\sum_{i=1}^{2} a_i u_i(x,t), \quad a_1 = -a_2 = 1. \tag{10}$$

In the next section, solutions for damped free and forced responses are formulated. For the sake of completeness of the analysis, solutions for undamped vibrations are also presented.

3. Vibration analysis of the system

3.1. Undamped free vibrations

Application of the modal expansion method [1–14,16,38] for the damped vibration analysis of a continuous system usually requires knowing appropriate natural mode shapes of vibration. Therefore, the undamped free vibrations are first considered for the system shown in Fig. 2.

The undamped, free vibrations are described by the equations of motion (5) reduced (through omittion of the terms expressing damping and exciting loadings) to the form

$$m\ddot{w}_1 - Sw_1'' + k(w_1 - w_2) = 0, \quad m\ddot{w}_2 - Sw_2'' + k(w_2 - w_1) = 0. \tag{11}$$

In this connection, normal modes for the strings are found, by solving the alternative Eqs. (7) assumed in the homogeneous form in which damping is neglected, from

$$m\ddot{u}_1 - Su_1'' = 0, \quad m\ddot{u}_2 - Su_2'' + 2ku_2 = 0. \tag{12}$$

General solutions of these homogeneous partial differential equations are obtained by applying the method of separation of variables. The solutions are assumed to be in the form

$$u_1(x,t) = X_1(x)T_1(t), \quad u_2(x,t) = X_2(x)T_2(t). \tag{13}$$

Substituting expressions (13) into Eqs. (12), as a result of separation of variables, gives four homogeneous ordinary differential equations of the second order

$$\ddot{T}_1 + \omega_1^2 T_1 = 0, \quad T_2'' + \omega_2^2 T_2 = 0, \tag{14}$$

$$X_1'' + k_1^2 X_1 = 0, \quad X_2'' + k_2^2 X_2 = 0, \tag{15}$$

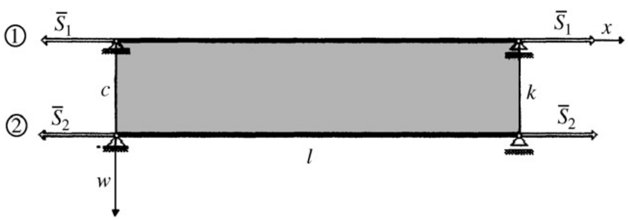

Fig. 2. The physical model of an elastically connected double-string system analyzed for free motions.

where

$$k_1^2 = m\omega_1^2 S^{-1}, \quad k_2^2 = (m\omega_2^2 - 2k)S^{-1}, \quad \omega_1^2 = Sk_1^2 m^{-1}, \quad \omega_2^2 = (Sk_2^2 + 2k)m^{-1}. \tag{16}$$

$\omega_{in}\,(i = 1, 2)$ are the separating constants denoting the natural frequencies of undamped free vibration of the system.

Eqs. (14) describe simple harmonic motions, their solutions being of the known form

$$T_1(t) = M_1 \sin(\omega_1 t) + N_1 \cos(\omega_1 t), \quad T_2(t) = M_2 \sin(\omega_2 t) + N_2 \cos(\omega_2 t). \tag{17}$$

Next, the eigenfunctions $X_1(x)$, $X_2(x)$ being the solutions of Eqs. (15) are

$$X_1(x) = A_1 \sin(k_1 x) + B_1 \cos(k_1 x), \quad X_2(x) = A_2 \sin(k_2 x) + B_2 \cos(k_2 x). \tag{18}$$

The unknown constants A_i and B_i $(i = 1, 2)$ are evaluated from the transformed boundary conditions (9)

$$X_i(0) = X_i(l) = 0, \quad i = 1, 2. \tag{19}$$

Solving the boundary value problems leads to the characteristic equations

$$\sin(k_i l) = 0, \quad i = 1, 2 \tag{20}$$

from which the eigenvalues of the boundary value problems can be found as

$$k_i = k_{in} = k_n = n\pi l^{-1}, \quad n = 1, 2, 3, \ldots. \tag{21}$$

Using relations (16)–(18), the time functions and natural mode shapes corresponding to the suitable mode shape coefficient k_n and corresponding to the natural frequencies $\omega_{in}\,(i = 1, 2)$ are determined, respectively,

$$T_{1n}(t) = M_{1n} \sin(\omega_{1n} t) + N_{1n} \cos(\omega_{1n} t), \quad T_{2n}(t) = M_{2n} \sin(\omega_{2n} t) + N_{2n} \cos(\omega_{2n} t), \tag{22}$$

$$X_{1n}(x) = X_n(x) = \sin(k_n x), \quad X_{2n}(x) = X_n(x) = \sin(k_n x), \tag{23}$$

$$\omega_{1n}^2 = Sk_n^2 m^{-1}, \quad \omega_{2n}^2 = (Sk_n^2 + 2k)m^{-1}. \tag{24}$$

Solving Eqs. (12) the principal co-ordinates $u_i = u_i(x, t)$ (Eq. (13)) are found in the form

$$\begin{aligned} u_1(x,t) &= \sum_{n=1}^{\infty} u_{1n}(x,t) = \sum_{n=1}^{\infty} X_{1n}(x) T_{1n}(t) = \sum_{n=1}^{\infty} X_n(x) T_{1n}(t) \\ &= \sum_{n=1}^{\infty} \sin(k_n x)[M_{1n} \sin(\omega_{1n} t) + N_{1n} \cos(\omega_{1n} t)], \\ u_2(x,t) &= \sum_{n=1}^{\infty} u_{2n}(x,t) = \sum_{n=1}^{\infty} X_{2n}(x) T_{2n}(t) = \sum_{n=1}^{\infty} X_n(x) T_{2n}(t) \\ &= \sum_{n=1}^{\infty} \sin(k_n x)[M_{2n} \sin(\omega_{2n} t) + N_{2n} \cos(\omega_{2n} t)]. \end{aligned} \tag{25}$$

The unknown solutions of Eqs. (11) $w_1(x,t)$ and $w_2(x,t)$ are then determined from relationships (10), and finally, the undamped free vibrations of the system can be represented by the formulae

$$w_1(x,t) = 0.5\sum_{i=1}^{2} u_i(x,t) = \sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2}[A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)],$$

$$w_2(x,t) = 0.5\sum_{i=1}^{2} a_i u_i(x,t) = \sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2}[A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)]a_{in}, \tag{26}$$

where

$$a_{1n} = a_1 = -a_{2n} = -a_2 = 1.$$

In order to determine the final form of free vibrations (26) the initial-value problem must be solved. The unknown constants A_{in} and B_{in} are calculated from the assumed initial conditions (3). In Ref. [15], the above solutions are obtained for more general system of two different strings.

3.2. Damped free vibrations

The damped free vibrations of the system are expressed by general solutions of the governing equations (7) taken in the homogeneous form

$$m\ddot{u}_1 + C\dot{u}_1 - Su_1'' = 0, \quad m\ddot{u}_2 + (C+2c)\dot{u}_2 - Su_2'' + 2ku_2 = 0. \tag{27}$$

Applying the modal expansion method, these solutions are assumed to be in the form of superposition of natural mode shapes (23), namely,

$$u_1(x,t) = \sum_{n=1}^{\infty} X_{1n}(x)S_{1n}(t) = \sum_{n=1}^{\infty} X_n(x)S_{1n}(t) = \sum_{n=1}^{\infty}\sin(k_n x)S_{1n}(t),$$

$$u_2(x,t) = \sum_{n=1}^{\infty} X_{2n}(x)S_{2n}(t) = \sum_{n=1}^{\infty} X_n(x)S_{2n}(t) = \sum_{n=1}^{\infty}\sin(k_n x)S_{2n}(t), \tag{28}$$

where $S_{1n}(t)$ and $S_{2n}(t)$ are the unknown time functions to be determined.

Substituting solutions (28) into Eqs. (27) results in the relationships

$$\sum_{n=1}^{\infty}[\ddot{S}_{1n} + 2h_1\dot{S}_{1n} + \omega_{1n}^2 S_{1n}]X_n = 0, \quad \sum_{n=1}^{\infty}[\ddot{S}_{2n} + 2h_2\dot{S}_{2n} + \omega_{2n}^2 S_{2n}]X_n = 0,$$

which give two independent infinite sequences of ordinary differential equations for the unknown time functions

$$\ddot{S}_{in} + 2h_i\dot{S}_{in} + \omega_{in}^2 S_{in} = 0, \quad i = 1,2, \quad n = 1,2,3,\ldots, \tag{29}$$

where

$$h_1 = 0.5Cm^{-1}, \quad h_2 = 0.5(C+2c)m^{-1}, \quad \omega_{1n}^2 = Sk_n^2 m^{-1}, \quad \omega_{2n}^2 = (Sk_n^2 + 2k)m^{-1}. \tag{30}$$

The quantities h_i and $\omega_{in}\,(i=1,2)$ denote the damping coefficients and undamped natural frequencies of vibration, respectively.

As is well known, Eqs. (29) have three types of solutions depending on the value of damping coefficients [1–13,18,39,40]. The following cases are usually considered:

(1) *Undercritical damping*: $h_i < \omega_{in}$,

$$S_{in}(t) = \mathrm{e}^{-h_i t}[C_{in}\sin(\Omega_{in}t) + D_{in}\cos(\Omega_{in}t)], \quad i = 1, 2, \tag{31}$$

where $\Omega_{in} = (\omega_{in}^2 - h_i^2)^{1/2}$ denotes the damped natural frequency,

$$\Omega_{1n} = (Sk_n^2 m^{-1} - 0.25C^2 m^{-2})^{1/2}, \quad \Omega_{2n} = [(Sk_n^2 + 2k)m^{-1} - 0.25(C + 2c)^2 m^{-2}]^{1/2}.$$

An underdamped case (small damping) is important in vibration analysis, because it is the unique case leading to an oscillatory motion. Solution (31) represents the damped free harmonic vibration of the system, which is performed at the frequency Ω_{in} and amplitude decreasing exponentially with time.

(2) *Critical damping*: $h_i = \omega_{in}$,

$$S_{in}(t) = \mathrm{e}^{-h_i t}[C_{in}t + D_{in}], \quad i = 1, 2, \tag{32}$$

(3) *Overcritical damping*: $h_i > \omega_{in}$,

$$S_{in}(t) = \mathrm{e}^{-h_i t}[C_{in}\sinh(\Psi_{in}t) + D_{in}\cosh(\Psi_{in}t)], \quad i = 1, 2, \tag{33}$$

where

$$\Psi_{in} = (h_i^2 - \omega_{in}^2)^{1/2},$$
$$\Psi_{1n} = (0.25C^2 m^{-2} - Sk_n^2 m^{-1})^{1/2}, \qquad \Psi_{2n} = [0.25(C + 2c)^2 m^{-2} - (Sk_n^2 + 2k)m^{-1}]^{1/2}.$$

Formulating solutions (10) for the damped free vibrations of the system discussed

$$\begin{aligned} w_1(x,t) &= 0.5\sum_{i=1}^{2} u_i(x,t) = 0.5\sum_{n=1}^{\infty} X_n(x)\sum_{i=1}^{2} S_{in}(t) = 0.5\sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2} S_{in}(t), \\ w_2(x,t) &= 0.5\sum_{i=1}^{2} a_i u_i(x,t) = 0.5\sum_{n=1}^{\infty} X_n(x)\sum_{i=1}^{2} a_{in}S_{in}(t) = 0.5\sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2} a_{in}S_{in}(t), \end{aligned} \tag{34}$$

where $a_{1n} = a_1 = -a_{2n} = -a_2 = 1$, nine possible cases depending on the mutual relations between the damping coefficients C and c, the stiffness modulus k, the string tension S, and the eigenvalue

k_n should be identified. They are presented below:

(1) $h_1 < \omega_{1r}, h_2 < \omega_{2s},$ *when* $C < 2(Sk_r^2 m)^{1/2}, c < [(Sk_s^2 + 2k)m]^{1/2} - 0.5C,$

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{ & \sum_{n=1}^{r-1} \sin(k_n x)[A_{1n} \sinh(\Psi_{1n} t) + B_{1n} \cosh(\Psi_{1n} t)] \\ & + \sum_{n=r}^{\infty} \sin(k_n x)[A_{1n} \sin(\Omega_{1n} t) + B_{1n} \cos(\Omega_{1n} t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{ & \sum_{n=1}^{s-1} \sin(k_n x)[A_{2n} \sinh(\Psi_{2n} t) + B_{2n} \cosh(\Psi_{2n} t)] \\ & + \sum_{n=s}^{\infty} \sin(k_n x)[A_{2n}\, sin(\Omega_{2n} t) + B_{2n} \cos(\Omega_{2n} t)]\}, \end{aligned} \tag{35}$$

(2) $h_1 < \omega_{1r}, h_2 = \omega_{2s},$ *when* $C < 2(Sk_r^2 m)^{1/2}, c = [(Sk_s^2 + 2k)m]^{1/2} - 0.5C,$

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{ & \sum_{n=1}^{r-1} \sin(k_n x)[A_{1n} \sinh(\Psi_{1n} t) + B_{1n} \cosh(\Psi_{1n} t)] \\ & + \sum_{n=r}^{\infty} \sin(k_n x)[A_{1n} \sin(\Omega_{1n} t) + B_{1n} \cos(\Omega_{1n} t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{ & \sum_{n=1}^{s-1} \sin(k_n x)[A_{2n} \sinh(\Psi_{2n} t) + B_{2n} \cosh(\Psi_{2n} t)] \\ & + \sin(k_s x)[A_{2s} t + B_{2s}] + \sum_{n=s+1}^{\infty} \sin(k_n x)[A_{2n} \sin(\Omega_{2n} t) + B_{2n} \cos(\Omega_{2n} t)]\Bigg\}, \end{aligned} \tag{36}$$

(3) $h_1 < \omega_{1r}, h_2 > \omega_{2s},$ *when* $C < 2(Sk_r^2 m)^{1/2}, c > [(Sk_s^2 + 2k)m]^{1/2} - 0.5C,$

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{ & \sum_{n=1}^{r-1} \sin(k_n x)[A_{1n} \sinh(\Psi_{1n} t) + B_{1n} \cosh(\Psi_{1n} t)] \\ & + \sum_{n=r}^{\infty} \sin(k_n x)[A_{1n} \sin(\Omega_{1n} t) + B_{1n} \cos(\Omega_{1n} t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{ & \sum_{n=1}^{s} \sin(k_n x)[A_{2n} \sinh(\Psi_{2n} t) + B_{2n} \cosh(\Psi_{2n} t)] \\ & + \sum_{n=s+1}^{\infty} \sin(k_n x)[A_{2n} \sin(\Omega_{2n} t) + B_{2n} \cos(\Omega_{2n} t)]\Bigg\}, \end{aligned} \tag{37}$$

(4) $h_1 = \omega_{1r}, h_2 < \omega_{2s}$, *when* $C = 2(Sk_r^2 m)^{1/2}, c < [(Sk_s^2 + 2k)m]^{1/2} - (Sk_r^2 m)^{1/2}$,

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{&\sum_{n=1}^{r-1} \sin(k_n x)[A_{1n}\sinh(\Psi_{1n}t) + B_{1n}\cosh(\Psi_{1n}t)] + \sin(k_r x)[A_{1r}t + B_{1r}] \\ &+ \sum_{n=r+1}^{\infty} \sin(k_n x)[A_{1n}\sin(\Omega_{1n}t) + B_{1n}\cos(\Omega_{1n}t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{&\sum_{n=1}^{s-1} \sin(k_n x)[A_{2n}\sinh(\Psi_{2n}t) + B_{2n}\cosh(\Psi_{2n}t)] \\ &+ \sum_{n=s}^{\infty} \sin(k_n x)[A_{2n}\sin(\Omega_{2n}t) + B_{2n}\cos(\Omega_{2n}t)]\Bigg\}, \end{aligned} \tag{38}$$

(5) $h_1 = \omega_{1r}, h_2 = \omega_{2s}$, *when* $C = 2(Sk_r^2 m)^{1/2}, c = [(Sk_s^2 + 2k)m]^{1/2} - (Sk_r^2 m)^{1/2}$,

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{&\sum_{n=1}^{r-1} \sin(k_n x)[A_{1n}\sinh(\Psi_{1n}t) + B_{1n}\cosh(\Psi_{1n}t)] + \sin(k_r x)[A_{1r}t + B_{1r}] \\ &+ \sum_{n=r+1}^{\infty} \sin(k_n x)[A_{1n}\sin(\Omega_{1n}t) + B_{1n}\cos(\Omega_{1n}t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{&\sum_{n=1}^{s-1} \sin(k_n x)[A_{2n}\sinh(\Psi_{2n}t) + B_{2n}\cosh(\Psi_{2n}t)] \\ &+ \sin(k_s x)[A_{2s}t + B_{2s}] + \sum_{n=s+1}^{\infty} \sin(k_n x)[A_{2n}\sin(\Omega_{2n}t) + B_{2n}\cos(\Omega_{2n}t)]\Bigg\}, \end{aligned} \tag{39}$$

(6) $h_1 = \omega_{1r}, h_2 > \omega_{2s}$, *when* $C = 2(Sk_r^2 m)^{1/2}, c > [(Sk_s^2 + 2k)m]^{1/2} - (Sk_r^2 m)^{1/2}$,

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{&\sum_{n=1}^{r-1} \sin(k_n x)[A_{1n}\sinh(\Psi_{1n}t) + B_{1n}\cosh(\Psi_{1n}t)] + \sin(k_r x)[A_{1r}t + B_{1r}] \\ &+ \sum_{n=r+1}^{\infty} \sin(k_n x)[A_{1n}\sin(\Omega_{1n}t) + B_{1n}\cos(\Omega_{1n}t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{&\sum_{n=1}^{s} \sin(k_n x)[A_{2n}\sinh(\Psi_{2n}t) + B_{2n}\cosh(\Psi_{2n}t)] \\ &+ \sum_{n=s+1}^{\infty} \sin(k_n x)[A_{2n}\sin(\Omega_{2n}t) + B_{2n}\cos(\Omega_{2n}t)]\Bigg\}, \end{aligned} \tag{40}$$

(7) $h_1 > \omega_{1r}, h_2 < \omega_{2s}$, *when* $C > 2(Sk_r^2 m)^{1/2}$, $c < [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{ & \sum_{n=1}^{r} \sin(k_n x)[A_{1n} \sinh(\Psi_{1n} t) + B_{1n} \cosh(\Psi_{1n} t)] \\ & + \sum_{n=r+1}^{\infty} \sin(k_n x)[A_{1n} \sin(\Omega_{1n} t) + B_{1n} \cos(\Omega_{1n} t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{ & \sum_{n=1}^{s-1} \sin(k_n x)[A_{2n} \sinh(\Psi_{2n} t) + B_{2n} \cosh(\Psi_{2n} t)] \\ & + \sum_{n=s}^{\infty} \sin(k_n x)[A_{2n} \sin(\Omega_{2n} t) + B_{2n} \cos(\Omega_{2n} t)]\Bigg\}, \end{aligned} \tag{41}$$

(8) $h_1 > \omega_{1r}, h_2 = \omega_{2s}$, *when* $C > 2(Sk_r^2 m)^{1/2}, c = [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{ & \sum_{n=1}^{r} \sin(k_n x)[A_{1n} \sinh(\Psi_{1n} t) + B_{1n} \cosh(\Psi_{1n} t)] \\ & + \sum_{n=r+1}^{\infty} \sin(k_n x)[A_{1n} \sin(\Omega_{1n} t) + B_{1n} \cos(\Omega_{1n} t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{ & \sum_{n=1}^{s-1} \sin(k_n x)[A_{2n} \sinh(\Psi_{2n} t) + B_{2n} \cosh(\Psi_{2n} t)] \\ & + \sin(k_s x)[A_{2s} t + B_{2s}] + \sum_{n=s+1}^{\infty} \sin(k_n x)[A_{2n} \sin(\Omega_{2n} t) + B_{2n} \cos(\Omega_{2n} t)]\Bigg\}, \end{aligned} \tag{42}$$

(9) $h_1 > \omega_{1r}, h_2 > \omega_{2s}$, *when* $C > 2(Sk_r^2 m)^{1/2}$, $c > [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$\begin{aligned} w_{1,2}(x,t) = \mathrm{e}^{-h_1 t}\Bigg\{ & \sum_{n=1}^{r} \sin(k_n x)[A_{1n} \sinh(\Psi_{1n} t) + B_{1n} \cosh(\Psi_{1n} t)] \\ & + \sum_{n=r+1}^{\infty} \sin(k_n x)[A_{1n} \sin(\Omega_{1n} t) + B_{1n} \cos(\Omega_{1n} t)]\Bigg\} \\ \pm \mathrm{e}^{-h_2 t}\Bigg\{ & \sum_{n=1}^{s} \sin(k_n x)[A_{2n} \sinh(\Psi_{2n} t) + B_{2n} \cosh(\Psi_{2n} t)] \\ & + \sum_{n=s+1}^{\infty} \sin(k_n x)[A_{2n} \sin(\Omega_{2n} t) + B_{2n} \cos(\Omega_{2n} t)]\Bigg\}, \end{aligned} \tag{43}$$

where $n, r, s = 1, 2, 3, \ldots$. In those cases where $r, s = 1$, the expressions $\sum_{n=1}^{r-1}(\cdots)$, $\sum_{n=1}^{s-1}(\cdots)$, must be assumed equal to zero.

In order to formulate the final form of free responses (35)–(43) the initial-value problem should be solved. The unknown integration constants A_{in}, B_{in} $(i = 1, 2)$ can be determined from the assumed initial conditions (3). The solutions obtained are expressed by the combinations of time functions describing the damped harmonic vibration (for the underdamped case) and damped aperiodic motion (for the critically damped and overdamped cases). It is seen that a double-string system executes two types of free vibrations (motions), synchronous and asynchronous. The synchronous vibrations correspond to the synchronous mode shapes, and are characterized by the parameters h_1, ω_{1n}, Ω_{1n}, Ψ_{1n}. These quantities are not functions of a damping coefficient c and stiffness modulus of viscoelastic layer k. The simplifying assumptions introduced cause the system to vibrate as a whole without any relative motion between two strings. This implies that the connecting layer is not deformed in the transverse direction. The asynchronous vibrations correspond to asynchronous mode shapes, and parameters characterizing this motion are h_2, ω_{2n}, Ω_{2n}, Ψ_{2n}. The deflections of both strings in the corresponding component motions are identical.

3.3. Undamped forced vibrations

For the sake of completeness, solutions of forced vibrations in the case of undamped system are also determined. The motion of such a system is governed by Eqs. (5) taken in the form

$$m\ddot{w}_1 - Sw_1'' + k(w_1 - w_2) = f_1(x, t), \quad m\ddot{w}_2 - Sw_2'' + k(w_2 - w_1) = f_2(x, t). \tag{44}$$

To this end, particular solutions of alternative non-homogeneous partial differential equations (7), in which terms expressing damping are omitted, should be found

$$m\ddot{u}_1 - Su_1'' = F_1(x, t), \quad m\ddot{u}_2 - Su_2'' + 2ku_2 = F_2(x, t). \tag{45}$$

These solutions can be sought by using the modal expansion method [1–14,16,38], and are assumed to be of the general form

$$\begin{aligned} u_1(x, t) &= \sum_{n=1}^{\infty} X_{1n}(x)P_{1n}(t) = \sum_{n=1}^{\infty} X_n(x)P_{1n}(t) = \sum_{n=1}^{\infty} \sin(k_n x)P_{1n}(t), \\ u_2(x, t) &= \sum_{n=1}^{\infty} X_{2n}(x)P_{2n}(t) = \sum_{n=1}^{\infty} X_n(x)P_{2n}(t) = \sum_{n=1}^{\infty} \sin(k_n x)P_{2n}(t), \end{aligned} \tag{46}$$

where $P_{1n}(t)$ and $P_{2n}(t)$ the unknown time functions corresponding to the natural frequencies ω_{in}, which are to be determined.

Substituting solutions (46) into Eqs. (45) results in the relationships

$$\sum_{n=1}^{\infty}[\ddot{P}_{1n} + \omega_{1n}^2 P_{1n}]X_n = m^{-1}F_1(x, t), \quad \sum_{n=1}^{\infty}[\ddot{P}_{2n} + \omega_{2n}^2 P_{2n}]X_n = m^{-1}F_2(x, t).$$

The above relations are multiplied by the eigenfunction X_m, then they are integrated with respect to x from 0 to l. Next, the classical orthogonality condition is applied

$$\int_0^l X_m X_n \, \mathrm{d}x = \int_0^l \sin(k_m x)\sin(k_n x) \, \mathrm{d}x = d\delta_{mn},$$
$$d = d_n^2 = \int_0^l X_n^2 \, \mathrm{d}x = \int_0^l \sin^2(k_n x) \, \mathrm{d}x = 0.5l, \tag{47}$$

where δ_{mn} is the Kronecker delta function: $\delta_{mn} = 0$ for $m \neq n$, and $\delta_{mn} = 1$ for $m = n$.

Finally, the two independent infinite sequences of ordinary differential equations for the unknown time functions are obtained from

$$\ddot{P}_{in} + \omega_{in}^2 P_{in} = H_{in}(t), \quad i = 1, 2, \quad n = 1, 2, 3, \ldots, \tag{48}$$

where

$$H_{in}(t) = (dm)^{-1} \int_0^l F_i(x,t)\sin(k_n x) \, \mathrm{d}x = 2M^{-1} \int_0^l F_i(x,t)\sin(k_n x) \, \mathrm{d}x, \quad M = lm. \tag{49}$$

The solutions of Eqs. (48) satisfying homogeneous initial conditions are [14,16]

$$P_{in}(t) = \omega_{in}^{-1} \int_0^t H_{in}(\tau)\sin[\omega_{in}(t-\tau)] \, \mathrm{d}\tau$$
$$= 2(M\omega_{in})^{-1} \int_0^t \left[\int_0^l F_i(x,\tau)\sin(k_n x) \, \mathrm{d}x \right] \sin[\omega_{in}(t-\tau)] \, \mathrm{d}\tau, \quad i = 1, 2. \tag{50}$$

Then, solutions (46) take the form

$$u_1(x,t) = \sum_{n=1}^{\infty} X_n(x) P_{1n}(t) = \sum_{n=1}^{\infty} \omega_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau)\sin[\omega_{1n}(t-\tau)] \, \mathrm{d}\tau,$$
$$u_2(x,t) = \sum_{n=1}^{\infty} X_n(x) P_{2n}(t) = \sum_{n=1}^{\infty} \omega_{2n}^{-1} \sin(k_n x) \int_0^t H_{2n}(\tau)\sin[\omega_{2n}(t-\tau)] \, \mathrm{d}\tau. \tag{51}$$

The undamped forced vibrations of a double-string system are described by the versatile formulae

$$w_1(x,t) = 0.5 \sum_{i=1}^{2} u_i(x,t) = 0.5 \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} \omega_{in}^{-1} \int_0^t H_{in}(\tau)\sin[\omega_{in}(t-\tau)]\mathrm{d}\tau,$$
$$w_2(x,t) = 0.5 \sum_{i=1}^{2} a_i u_i(x,t) = 0.5 \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} a_{in}\omega_{in}^{-1} \int_0^t H_{in}(\tau)\sin[\omega_{in}(t-\tau)]\mathrm{d}\tau, \tag{52}$$

where $a_{1n} = a_1 = -a_{2n} = -a_2 = 1$.

In Ref. [16], analogous solutions are obtained for more general system of two different strings.

3.4. Damped forced vibrations

The damped forced responses of the system due to arbitrarily distributed transverse continuous loads are represented by the particular solutions of the governing non-homogeneous partial differential equations (5). These solutions will be found after solving a derived auxiliary uncoupled set of Eqs. (7)

$$m\ddot{u}_1 + C\dot{u}_1 - Su_1'' = F_1(x,t), \quad m\ddot{u}_2 + (C+2c)\dot{u}_2 - Su_2'' + 2ku_2 = F_2(x,t). \tag{53}$$

Applying the modal expansion method, the solutions are assumed to be in the following form:

$$u_1(x,t) = \sum_{n=1}^{\infty} X_{1n}(x)P_{1n}(t) = \sum_{n=1}^{\infty} X_n(x)P_{1n}(t) = \sum_{n=1}^{\infty} \sin(k_n x)P_{1n}(t),$$
$$u_2(x,t) = \sum_{n=1}^{\infty} X_{2n}(x)P_{2n}(t) = \sum_{n=1}^{\infty} X_n(x)P_{2n}(t) = \sum_{n=1}^{\infty} \sin(k_n x)P_{2n}(t), \tag{54}$$

where $P_{1n}(t)$ and $P_{2n}(t)$ are the unknown time functions to be determined.

Substituting solutions (54) into Eqs. (53) results in the relationships

$$\sum_{n=1}^{\infty}[\ddot{P}_{1n} + 2h_1\dot{P}_{1n} + \omega_{1n}^2 P_{1n}]X_n = m^{-1}F_1(x,t),$$
$$\sum_{n=1}^{\infty}[\ddot{P}_{2n} + 2h_2\dot{P}_{2n} + \omega_{2n}^2 P_{2n}]X_n = m^{-1}F_2(x,t).$$

Multiplying the above relations by the eigenfunction X_m, then integrating them with respect to x from 0 to l and applying the orthogonality condition (47), one gets the differential equations for the unknown time functions

$$\ddot{P}_{in} + 2h_i\dot{P}_{in} + \omega_{in}^2 P_{in} = H_{in}(t), \quad i = 1, 2, \quad n = 1, 2, 3, \ldots, \tag{55}$$

where

$$H_{in}(t) = (dm)^{-1}\int_0^l F_i(x,t)\sin(k_n x)\,\mathrm{d}x = 2M^{-1}\int_0^l F_i(x,t)\sin(k_n x)\,\mathrm{d}x. \tag{56}$$

Searching for their particular solutions, three possible cases must be considered [1–13,39,40].

(1) *Undercritical damping:* $h_i < \omega_{in}$,

$$P_{in}(t) = \Omega_{in}^{-1}\int_0^t H_{in}(\tau)\mathrm{e}^{-h_i(t-\tau)}\sin[\Omega_{in}(t-\tau)]\mathrm{d}\tau, \quad i = 1, 2, \tag{57}$$

where

$$\Omega_{in} = (\omega_{in}^2 - h_i^2)^{1/2},$$
$$\Omega_{1n} = (Sk_n^2 m^{-1} - 0.25C^2 m^{-2})^{1/2}, \qquad \Omega_{2n} = (Sk_n^2 + 2k)m^{-1} - 0.25(C+2c)^2 m^{-2}]^{1/2}.$$

(2) *Critical damping:* $h_i = \omega_{in}$,

$$P_{in}(t) = \int_0^t H_{in}(\tau)\mathrm{e}^{-h_i(t-\tau)}(t-\tau)\mathrm{d}\tau, \quad i = 1, 2. \tag{58}$$

(3) *Overcritical damping:* $h_i > \omega_{in}$,

$$P_{in}(t) = \Psi_{in}^{-1} \int_0^t H_{in}(\tau)\mathrm{e}^{-h_i(t-\tau)}\sinh[\Psi_{in}(t-\tau)]\mathrm{d}\tau, \quad i = 1, 2, \tag{59}$$

where

$$\Psi_{in} = (h_i^2 - \omega_{in}^2)^{1/2},$$
$$\Psi_{1n} = (0.25C^2m^{-2} - Sk_n^2m^{-1})^{1/2}, \qquad \Psi_{2n} = [0.25(C+2c)^2m^{-2} - (Sk_n^2 + 2k)m^{-1}]^{1/2}.$$

Setting solutions (10) for the damped forced vibrations of the system

$$\begin{aligned} w_1(x,t) &= 0.5\sum_{i=1}^{2} u_i(x,t) = 0.5\sum_{n=1}^{\infty} X_n(x)\sum_{i=1}^{2} P_{in}(t) = 0.5\sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=1}^{2} P_{in}(t), \\ w_2(x,t) &= 0.5\sum_{i=1}^{2} a_iu_i(x,t) = 0.5\sum_{n=1}^{\infty} X_n(x)\sum_{i=1}^{2} a_{in}P_{in}(t) = 0.5\sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=1}^{2} a_{in}P_{in}(t), \end{aligned} \tag{60}$$

where $a_{1n} = a_1 = -a_{2n} = -a_2 = 1$, the following nine possible cases have to be shown:
(1) $h_1 < \omega_{1r}, h_2 < \omega_{2s}$, *when* $C < 2(Sk_r^2m)^{1/2}$, $c < [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$\begin{aligned} w_{1,2}(x,t) = 0.5\Bigg\{ &\sum_{n=1}^{r-1}\Psi_{1n}^{-1}\sin(k_nx)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sinh[\Psi_{1n}(t-\tau)]\mathrm{d}\tau \\ &+ \sum_{n=r}^{\infty}\Omega_{1n}^{-1}\sin(k_nx)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sin[\Omega_{1n}(t-\tau)]\mathrm{d}\tau\Bigg\} \\ \pm 0.5\Bigg\{ &\sum_{n=1}^{s-1}\Psi_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau \\ &+ \sum_{n=s}^{\infty}\Omega_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\Bigg\}, \end{aligned} \tag{61}$$

(2) $h_1 < \omega_{1r}, h_2 = \omega_{2s}$, *when* $C < 2(Sk_r^2 m)^{1/2}$, $c = [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$w_{1,2}(x,t) = 0.5\left\{\sum_{n=1}^{r-1} \Psi_{1n}^{-1}\sin(k_n x)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sinh[\Psi_{1n}(t-\tau)]\mathrm{d}\tau + \sum_{n=r}^{\infty} \Omega_{1n}^{-1}\sin(k_n x)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sin[\Omega_{1n}(t-\tau)]\mathrm{d}\tau\right\}$$
$$\pm 0.5\left\{\sum_{n=1}^{s-1} \Psi_{2n}^{-1}\sin(k_n x)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau + \sin(k_s x)\int_0^t H_{2s}(\tau)\mathrm{e}^{-h_2(t-\tau)}(t-\tau)\mathrm{d}\tau + \sum_{n=s+1}^{\infty} \Omega_{2n}^{-1}\sin(k_n x)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\right\}, \quad (62)$$

(3) $h_1 < \omega_{1r}, h_2 > \omega_{2s}$, *when* $C < 2(Sk_r^2 m)^{1/2}$, $c > [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$w_{1,2}(x,t) = 0.5\left\{\sum_{n=1}^{r-1} \Psi_{1n}^{-1}\sin(k_n x)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sinh[\Psi_{1n}(t-\tau)]\mathrm{d}\tau + \sum_{n=r}^{\infty} \Omega_{1n}^{-1}\sin(k_n x)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sin[\Omega_{1n}(t-\tau)]\mathrm{d}\tau\right\}$$
$$\pm 0.5\left\{\sum_{n=1}^{s} \Psi_{2n}^{-1}\sin(k_n x)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau + \sum_{n=s+1}^{\infty} \Omega_{2n}^{-1}\sin(k_n x)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\right\}, \quad (63)$$

(4) $h_1 = \omega_{1r}, h_2 < \omega_{2s}$, *when* $C = 2(Sk_r^2 m)^{1/2}$, $c < [(Sk_s^2 + 2k)m]^{1/2} - (Sk_r^2 m)^{1/2}$,

$$w_{1,2}(x,t) = 0.5\left\{\sum_{n=1}^{r-1} \Psi_{1n}^{-1}\sin(k_n x)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sinh[\Psi_{1n}(t-\tau)]\mathrm{d}\tau + \sin(k_r x)\int_0^t H_{1r}(\tau)\mathrm{e}^{-h_1(t-\tau)}(t-\tau)\mathrm{d}\tau + \sum_{n=r+1}^{\infty} \Omega_{1n}^{-1}\sin(k_n x)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sin[\Omega_{1n}(t-\tau)]\mathrm{d}\tau\right\}$$
$$\pm 0.5\left\{\sum_{n=1}^{s-1} \Psi_{2n}^{-1}\sin(k_n x)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau + \sum_{n=s}^{\infty} \Omega_{2n}^{-1}\sin(k_n x)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\right\}, \quad (64)$$

(5) $h_1 = \omega_{1r}, h_2 = \omega_{2s}$, *when* $C = 2(Sk_r^2 m)^{1/2}$, $c = [(Sk_s^2 + 2k)m]^{1/2} - (Sk_r^2 m)^{1/2}$,

$$
\begin{aligned}
w_{1,2}(x,t) = 0.5\Bigg\{ & \sum_{n=1}^{r-1} \Psi_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau) \mathrm{e}^{-h_1(t-\tau)} \sinh[\Psi_{1n}(t-\tau)] \mathrm{d}\tau \\
& + \sin(k_r x) \int_0^t H_{1r}(\tau) \mathrm{e}^{-h_1(t-\tau)} (t-\tau) \mathrm{d}\tau \\
& + \sum_{n=r+1}^{\infty} \Omega_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau) \mathrm{e}^{-h_1(t-\tau)} \sin[\Omega_{1n}(t-\tau)] \mathrm{d}\tau \Bigg\} \\
\pm 0.5\Bigg\{ & \sum_{n=1}^{s-1} \Psi_{2n}^{-1} \sin(k_n x) \int_0^t H_{2n}(\tau) \mathrm{e}^{-h_2(t-\tau)} \sinh[\Psi_{2n}(t-\tau)] \mathrm{d}\tau \\
& + \sin(k_s x) \int_0^t H_{2s}(\tau) \mathrm{e}^{-h_2(t-\tau)} (t-\tau) \mathrm{d}\tau \\
& + \sum_{n=s+1}^{\infty} \Omega_{2n}^{-1} \sin(k_n x) \int_0^t H_{2n}(\tau) \mathrm{e}^{-h_2(t-\tau)} \sin[\Omega_{2n}(t-\tau)] \mathrm{d}\tau \Bigg\},
\end{aligned} \tag{65}
$$

(6) $h_1 = \omega_{1r}, h_2 > \omega_{2s}$, *when* $C = 2(Sk_r^2 m)^{1/2}$, $c > [(Sk_s^2 + 2k)m]^{1/2} - (Sk_r^2 m)^{1/2}$,

$$
\begin{aligned}
w_{1,2}(x,t) = 0.5\Bigg\{ & \sum_{n=1}^{r-1} \Psi_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau) \mathrm{e}^{-h_1(t-\tau)} \sinh[\Psi_{1n}(t-\tau)] \mathrm{d}\tau \\
& + \sin(k_r x) \int_0^t H_{1r}(\tau) \mathrm{e}^{-h_1(t-\tau)} (t-\tau) \mathrm{d}\tau \\
& + \sum_{n=r+1}^{\infty} \Omega_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau) \mathrm{e}^{-h_1(t-\tau)} \sin[\Omega_{1n}(t-\tau)] \mathrm{d}\tau \Bigg\} \\
\pm 0.5\Bigg\{ & \sum_{n=1}^{s} \Psi_{2n}^{-1} \sin(k_n x) \int_0^t H_{2n}(\tau) \mathrm{e}^{-h_2(t-\tau)} \sinh[\Psi_{2n}(t-\tau)] \mathrm{d}\tau \\
& + \sum_{n=s+1}^{\infty} \Omega_{2n}^{-1} \sin(k_n x) \int_0^t H_{2n}(\tau) \mathrm{e}^{-h_2(t-\tau)} \sin[\Omega_{2n}(t-\tau)] \mathrm{d}\tau \Bigg\},
\end{aligned} \tag{66}
$$

(7) $h_1 > \omega_{1r}, h_2 < \omega_{2s}$, *when* $C > 2(Sk_r^2 m)^{1/2}$, $c < [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$
\begin{aligned}
w_{1,2}(x,t) = 0.5\Bigg\{ & \sum_{n=1}^{r} \Psi_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau) \mathrm{e}^{-h_1(t-\tau)} \sinh[\Psi_{1n}(t-\tau)] \mathrm{d}\tau \\
& + \sum_{n=r+1}^{\infty} \Omega_{1n}^{-1} \sin(k_n x) \int_0^t H_{1n}(\tau) \mathrm{e}^{-h_1(t-\tau)} \sin[\Omega_{1n}(t-\tau)] \mathrm{d}\tau \Bigg\}
\end{aligned}
$$

$$\pm 0.5\left\{\sum_{n=1}^{s-1}\Psi_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau\right.$$
$$\left.+\sum_{n=s}^{\infty}\Omega_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\right\}, \tag{67}$$

(8) $h_1 > \omega_{1r}, h_2 = \omega_{2s}$, *when* $C > 2(Sk_r^2m)^{1/2}$, $c = [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$w_{1,2}(x,t) = 0.5\left\{\sum_{n=1}^{r}\Psi_{1n}^{-1}\sin(k_nx)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sinh[\Psi_{1n}(t-\tau)]\mathrm{d}\tau\right.$$
$$\left.+\sum_{n=r+1}^{\infty}\Omega_{1n}^{-1}\sin(k_nx)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sin[\Omega_{1n}(t-\tau)]\mathrm{d}\tau\right\}$$
$$\pm 0.5\left\{\sum_{n=1}^{s-1}\Psi_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau\right.$$
$$+\sin(k_sx)\int_0^t H_{2s}(\tau)\mathrm{e}^{-h_2(t-\tau)}(t-\tau)\mathrm{d}\tau$$
$$\left.+\sum_{n=s+1}^{\infty}\Omega_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\right\}, \tag{68}$$

(9) $h_1 > \omega_{1r}, h_2 > \omega_{2s}$, *when* $C > 2(Sk_r^2m)^{1/2}$, $c > [(Sk_s^2 + 2k)m]^{1/2} - 0.5C$,

$$w_{1,2}(x,t) = 0.5\left\{\sum_{n=1}^{r}\Psi_{1n}^{-1}\sin(k_nx)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sinh[\Psi_{1n}(t-\tau)]\mathrm{d}\tau\right.$$
$$\left.+\sum_{n=r+1}^{\infty}\Omega_{1n}^{-1}\sin(k_nx)\int_0^t H_{1n}(\tau)\mathrm{e}^{-h_1(t-\tau)}\sin[\Omega_{1n}(t-\tau)]\mathrm{d}\tau\right\}$$
$$\pm 0.5\left\{\sum_{n=1}^{s}\Psi_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sinh[\Psi_{2n}(t-\tau)]\mathrm{d}\tau\right.$$
$$\left.+\sum_{n=s+1}^{\infty}\Omega_{2n}^{-1}\sin(k_nx)\int_0^t H_{2n}(\tau)\mathrm{e}^{-h_2(t-\tau)}\sin[\Omega_{2n}(t-\tau)]\mathrm{d}\tau\right\}. \tag{69}$$

These formulae for the case of arbitrarily distributed transverse continuous loads are sufficiently versatile, to allow the solutions for any type of non-inertial stationary or moving loading to be found.

4. Conclusions

The paper deals with the theoretical investigation of damped vibrations for a system of two viscoelastic strings connected by a viscoelastic layer of a Kelvin–Voigt type. The motion of this system is described by a coupled set of two non-homogeneous partial differential equations. The introduction of corresponding principal co-ordinates leads to the decoupling of the differential equations of motion which are easily solved by application of two classical fundamental mathematical methods: the method of separation of variables and the modal expansion method. Exact analytical solutions for damped free and forced responses of strings subjected to generally distributed transverse continuous loads and due to arbitrary damping are formulated. It is relevant to note that coefficients shaping the solutions are explicitly expressed in terms of the physical parameters characterizing the system. Nine possible different solutions for free motion are described by the combinations of time functions expressing the damped harmonic vibrations (for undercritical damping cases), as well as the damped aperiodic motions (for critical and overcritical damping cases), according to the mutual relations between physical parameters of the system. In the case of forced vibrations, nine possible solutions are also determined. Although solutions relating to underdamped cases are only useful in vibration analysis, but consideration of all complete possible solutions of the problem makes possible a better understanding of the vibration phenomena occurring in damped complex continuous systems. The solutions obtained in this paper can be a basis for the formulation of damped responses for more general double-string systems characterized by arbitrary geometrical and physical parameters.

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