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Some problems of analysis and optimization of plates and shells

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Abstract

The classical optimization problems of plates and shells to satisfy *a priori* given geometry and dynamical characteristics are considered. Orthotropic plates and shells with variable thickness and low transverse stiffness are analyzed. First, some useful theorems and their proofs are given. Then the finite approximation of the problem related to optimization of free vibrations of shells with transverse deformation and rotary inertia is discussed. The varational iteration (MVI) and Bubnov-Galerkin (MB) methods are applied, and their convergence and suitability for application to plates and shells analysis are discussed and numerically evaluated.

1. Introduction

Plates and shells of moderate thickness are often used in many engineering structures, such as aircraft fuselages, turbine discs, reinforced aircraft bosses and wings. Free vibrations of rectangular moderately thick plates and shells are of great importance, because the dynamic characteristics are needed to carry out proper optimization of the geometry of their cross-section in order to achieve the required structural performance.

Recently the control of spatial structures composed of plates and shells have attracted the attention of many engineers and researchers. Although the concept of control appears to be simple in theory [1,2] a new construction of a spatial structure (plate, shell) together with the sensors and actuators made from ceramics or polymers requires both careful modelling and numerical investigations. The use of modern smart materials such as piezoceramics and polymers [3] as sensors and actuators modifies the shape of the spatial structure to be controlled.

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The modified spatial structure can therefore be considered as one with variable thickness. This problem has recently been addressed in Refs. [4,5], where assumptions and hypotheses for a 3-D theory for orthotropic shallow shells with attached masses (additives) have been proposed, and the validity of the 2-D theories range has been estimated.

Aspects of optimization are as follows: (a) a practical realization of a construction with minimal masses, but capable of sustaining a given dynamic load; (b) a practical realization of a construction with *a priori* a specified frequency of oscillations or other dynamical properties.

Several types of optimization are described in the literature, and some devoted to optimization of plates and shells are briefly mentioned here. The optimal shell configurations has been illustrated and classified in Refs. [6,7], which include a bibliography up to 1991. It has been pointed out that it is difficult for to provide a general formulation of the optimization tasks for a class of shells, but that each situation should be solved separately. Mass optimization of ribbed thin-walled structures has been presented in Ref. [8]. Optimization of various constructions from the point of view of stability has been presented in Ref. [9]. Optimizations of three layers constructions have also been presented in Ref. [10], whereas a general formulation of optimizations of shells using FEM and non-linear programming has been given in Ref. [11].

Many fundamental problems of design optimizations have been addressed in refs. [12–15], and they are helpful while formulating optimization problems of constructions with shell members.

Although many optimization problems are described in books and references, many researchers point out that the application of optimization is complicated even for a relatively simple design and that it requires strict formulation of a rigorous mathematical background (see for example Refs. [16,17]).

2. Free vibrations of orthotropic plates and shells with variable thickness and low transverse stiffness

In this Section different frequency spectra of shells with transverse deformation and rotary inertia will be described and methods to analyze free oscillation solutions of rectangular shells will be given. The theoretical results are applied to the analysis of plates with either constant or variable thickness and with low transverse stiffness.

2.1. Frequency spectra properties of orthotropic shells with variable thickness

It is known that a shell is a continuous medium composed of an infinite number of degrees of freedom. It means that the number of frequencies is infinite. Stiffness and curvature coefficients as well as the thickness distribution along a plate surface have an essential influence on the frequency spectra distribution.

Oscillations of shallow shells with transverse deformation and rotary inertia as well as shells with variable thickness are considered below [18,19].

Suppose that a shell has a finite area Ω with a border S. A kinematic model for the governing equations for displacements using an isotropic shell with constant thickness with transverse deformation and rotary inertia has been introduced by Naghdi [20, 21]. In the present case, free oscillations of an orthotropic shell with variable thickness are governed by the following more generalized equations (see also Refs. [22, 23]):

$$
\begin{aligned}
&K_y\frac{\partial^2 F}{\partial x^2}+K_x\frac{\partial^2 F}{\partial y^2}+\tfrac{2}{3}\frac{\partial}{\partial x}\left[A_{1313}h\left(\gamma_x+\frac{\partial w}{\partial x}\right)\right]\\
&\quad+\tfrac{2}{3}\frac{\partial}{\partial y}\left[A_{2323}h\left(\gamma_y+\frac{\partial w}{\partial y}\right)\right]+\omega^2 hw=0,\\
&\tfrac{2}{3}\frac{\partial}{\partial x}\left[h^3\left(A_{1111}\frac{\partial\gamma_x}{\partial x}+A_{1122}\frac{\partial\gamma_y}{\partial y}\right)\right]+\tfrac{2}{3}\frac{\partial}{\partial y}\left[A_{1212}h^3\left(\frac{\partial\gamma_x}{\partial y}+\frac{\partial\gamma_y}{\partial x}\right)\right]\\
&\quad-\tfrac{2}{3}A_{1313}h\left(\gamma_x+\frac{\partial w}{\partial x}\right)+\tfrac{2}{3}\omega^2h^3\gamma_x=0,\\
&\tfrac{2}{3}\frac{\partial}{\partial y}\left[h^3\left(A_{2222}\frac{\partial\gamma_y}{\partial y}+A_{1122}\frac{\partial\gamma_x}{\partial x}\right)\right]+\tfrac{2}{3}\frac{\partial}{\partial x}\left[A_{1212}h^3\left(\frac{\partial\gamma_x}{\partial y}+\frac{\partial\gamma_y}{\partial x}\right)\right]\\
&\quad-\tfrac{2}{3}A_{2323}h\left(\gamma_y+\frac{\partial w}{\partial y}\right)+\tfrac{2}{3}\omega^2h^3\gamma_y=0, \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(a_{1111}h^{-1}\frac{\partial^2 F}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}\left(a_{1122}h^{-1}\frac{\partial^2 F}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(a_{1122}h^{-1}\frac{\partial^2 F}{\partial x^2}\right)\\
&\quad+\frac{\partial^2}{\partial y^2}\left(a_{2222}h^{-1}\frac{\partial^2 F}{\partial y^2}\right)-\frac{\partial^2}{\partial x\partial y}\left(a_{1212}h^{-1}\frac{\partial^2 F}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}(k_xw)+\frac{\partial^2}{\partial x^2}(k_yw)=0.
\end{aligned}
$$

Eqs. (1) have been given in a hybrid form. Shear forces, moments and membrane forces F, w, γ_x, γ_y satisfy the following conditions:

$$
\begin{aligned}
M_{11}&=\tfrac{2}{3}h^3\left(A_{1111}\frac{\partial\gamma_x}{\partial x}+A_{1122}\frac{\partial\gamma_y}{\partial y}\right)\quad (\overset{\leftrightarrows}{x,y})\\
M_{12}&=\tfrac{2}{3}h^3A_{1212}\left(\frac{\partial\gamma_x}{\partial y}+\frac{\partial\gamma_y}{\partial x}\right)\quad S=-\frac{\partial^2F}{\partial x\partial y}=\frac{1}{2h}a_{1212}\varepsilon_{12},\\
Q_1&=\tfrac{2}{3}A_{1313}h\left(\gamma_x+\frac{\partial w}{\partial x}\right)\quad T_1=\frac{1}{2h}(a_{2222}\varepsilon_1+a_{1122}\varepsilon_2)\quad (\overset{\leftrightarrows}{x,y}).
\end{aligned}
$$

The boundary conditions on S boundary will be formulated for a general case. Let $S=S_1+S_2+S_3$, where S_1 is a free support, S_2 is a roller support and S_3 is a movable support. Then

$$Q_n=M_n=M_\tau=0 \quad \text{on } S_1, \qquad (2)$$

$$w=M_n=\gamma_\tau=0 \quad \text{on } S_2, \qquad (3)$$

$$w=\gamma_n=\gamma_\tau=0 \quad \text{on } S_3, \qquad (4)$$

$$F=\frac{\partial^2 F}{\partial n^2}=0 \quad \text{on } S. \qquad (5)$$

The following conditions related to stiffness coefficients and $h(x, y)$ function are assumed:
(a)

$$0 < A_H \leqslant A_{ijmk}(x, y) \leqslant A_B, \quad 0 < a_H \leqslant a_{ijmk}(x, y) \leqslant a_B, \tag{6}$$

and $A_{ijmk}(x, y)$ and $a_{ijmk}(x, y)$ are limited functions in Ω space $(i, j, m, k = 1, 2, 3)$;
(b) for an arbitrary $(x, y) \in \Omega$ and $\xi, \eta \in R^1$ there exists a constant $c_0 > 0$ such that

$$A_{1111}(x, y)\xi^2 + 2A_{1122}(x, y)\xi\eta + A_{2222}(x, y)\eta^2 \geqslant c_0(\xi^2 + \eta^2), \tag{7}$$

(c) there exists a constant $c_1 > 0$, so that for all $(x, y) \in \Omega$ and $\xi, \eta \in R^1$ the following inequality holds:

$$a_{1111}(x, y)\xi^2 + (2a_{1122}(x, y) - a_{1212}(x, y))\xi\eta + a_{2222}(x, y)\eta^2 \geqslant c_1(\xi^2 + \eta^2), \tag{8}$$

(d) $h(x, y)$ is the function bounded on Ω, and for (an arbitrary) $(x, y) \in \Omega$

$$0 < h_H \leqslant h(x, y) \leqslant h_B. \tag{9}$$

Since the frequency ω does not occur in the fourth equation of Eqs. (1), then F may be reduced.

Consider separately the fourth equation of Eqs. (1) with the boundary conditions (5). It may be presented in the following form:

$$\frac{\partial^2}{\partial x^2}\left(a_{1111}h^{-1}\frac{\partial^2 F}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\left(a_{1122}h^{-1}\frac{\partial^2 F}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(a_{1122}h^{-1}\frac{\partial^2 F}{\partial x^2}\right) + \frac{\partial^2}{\partial y^2}\left(a_{2222}h^{-1}\frac{\partial^2 F}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(a_{1212}h^{-1}\frac{\partial^2 F}{\partial x \partial y}\right) = -\frac{\partial^2}{\partial x^2}(k_y w) - \frac{\partial^2}{\partial y^2}(k_x w). \tag{10}$$

Let a differential operator standing on the left-hand side of Eq. (10) be $G(\bullet)$. The following functional space $H_0^2(\Omega)$ is introduced, which is the closure of the function set:

$$V = \left\{F \in C^\infty(\Omega) \mid F = \frac{\partial^2 F}{\partial n^2} = 0 \text{ on } S\right\}.$$

If it is possible to prove that for an arbitrary function w from an energy space of the problem, Eq. (10) is solvable because of F in $H_0^2(\Omega)$, then F may be extracted from system (1) and therefore the problem dimension may be reduced (see Appendix A).

In order to analyze oscillations of shells with transverse deformation and rotary inertia and in order to take into account a rotation energy the following bilinear form is introduced:

$$b_h(\vec{u}, \vec{v}) = \int_\Omega \left[hw\tilde{w} + \tfrac{2}{3}h^3\gamma_x\tilde{\gamma}_x + \tfrac{2}{3}h^3\gamma_y\tilde{\gamma}_y\right] d\Omega, \tag{11}$$

which is proportional to a shell kinetic energy.

It is simple to check using Eq. (11) that the form $b_h(\vec{u}, \vec{v})$ is also symmetric, positive and continuous.

Consider now the problem related to free oscillations of shells with transverse deformation and rotary inertia from another point of view. The functions w, γ_x, γ_y (which are simultaneously not identically equal to zero) can be found such that the full energy on an arbitrarily taken and kinematically allowed virtual displacement $\vec{v} = (\tilde{w},\ \tilde{\gamma}_x,\ \tilde{\gamma}_y)$ is equal to zero. It means that

$$a_h(\vec{u},\ \vec{v}) = \omega^2 b_h(\vec{u},\ \vec{v}) \quad \text{for every} \ \vec{v} \in V_0. \tag{12}$$

Therefore, the problem of finding the free vibrations spectra of a shell is reduced to the classical problem of eigenvalues [24]. The results, which are similar to those given in Ref. [25] for a plate, are given in Appendix B.

2.2. Finite-dimensional approximation of shells with transverse deformation and rotary inertia

2.2.1. General remarks and comparison of the methods

The definition of dynamic characteristics, and particularly of frequency spectra of free oscillations of plates and shells using the Kirchhoff–Love model leads to a complicated problem [25]. After transition to more appropriate models such as shells with transverse deformation and rotary inertia the difficulties even increase. These are caused by complexity of the differential equations. It is mainly because the latter has two more unknown functions (γ_x and γ_y) in comparison to the Kirchhoff-Love equations.

Since shells with transverse deformation and rotary inertia have four unknowns, the number of algebraic equations increases four times, which leads to some serious numerical difficulties.

For all problems considered in this paper it is assumed that the coefficients A_{ijmk}, a_{ijmk} and the curvatures k_x, k_y do not depend on $(x, y) \in \Omega$, and the thickness function $h(x,\ y)$ is symmetric in relation to the co-ordinate axes which originate from the centre of a plate.

It seems that the most suitable methods of solving the mentioned problems are the Kantorovitch method, the variational iterations method (MVI) and the Bubnov–Galerkin method with high order approximations (MB) [25]. The first two methods reduce the original two-dimensional problem to one-dimensional. Therefore, it is possible to decrease the number of the algebraic equations. In order to solve one-dimensional problems, the finite difference, as well as the finite element, methods may be used. Two important MVI and MB methods for analysis of plates and shells with variable thickness will now be discussed.

2.2.2. MVI approximation

The method of variational iterations applied to problems related to dynamics consists of the following steps.

Suppose that free vibration frequencies of a shell is to be determined from the equations

$$Z[h]\vec{u} - \omega^2 M[h]\vec{u} = 0, \tag{13}$$

using the corresponding boundary conditions. $\vec{u} \in V_0$ are kinematics allowed displacements and let $\Omega = [0, l_1] \times [0, l_2]$. Assume that the solution being sought has the following form:

$$\vec{u}(x, y) \approx \sum_{i=1}^{N} \vec{u}_i^1(x) \otimes \vec{u}_i^2(y), \quad N = 1,\ 2, \ldots, \tag{14}$$

where $\vec{u}_i^1(x)\otimes\vec{u}_i^2(y)$ is the vector with the components $w_i^1(x)w_i^2(y)$, $\gamma_{xi}^1(x)\gamma_{xi}^2(y)$, $\gamma_{yi}^1(x)\gamma_{yi}^2(y)$. Substituting Eq. (14) into Eq. (13) gives

$$\sum_{i=1}^{N} Z[h]\vec{u}_i^1(x)\otimes\vec{u}_i^2(y) = \omega^2 \sum_{i=1}^{N} M[h]\vec{u}_i^1(x)\otimes\vec{u}_i^2(y). \tag{15}$$

Applying the Bubnov–Galerkin method, Eqs. (15) are projected on the functions $\{\vec{u}_j^1(x)\}_{j=1}^N$ and $\{\vec{u}_j^2(y)\}_{j=1}^N$. As a result, this gives $2N$ differential equations of the form

$$\sum_{i=1}^{N}\left(\int_0^{l_1} Z[h]\vec{u}_i^1(x)\otimes\vec{u}_i^2(y)\cdot\vec{u}_j^1(x)\,\mathrm{d}x - \omega^2\int_0^{l_1} M[h]\vec{u}_i^1(x)\otimes\vec{u}_i^2(y)\cdot\vec{u}_j^1(x)\,\mathrm{d}x\right) = 0, \tag{16}$$

$$\sum_{i=1}^{N}\left(\int_0^{l_2} Z[h]\vec{u}_i^1(x)\otimes\vec{u}_i^2(y)\cdot\vec{u}_j^2(y)\,\mathrm{d}y - \omega^2\int_0^{l_2} M[h]\vec{u}_i^1(x)\otimes\vec{u}_i^2(y)\cdot\vec{u}_j^2(y)\,\mathrm{d}y\right) = 0$$
$$(j = 1, 2, \ldots, N). \tag{17}$$

In order to solve Eqs. (16) and (17) the iteration method will be used. As the first step, an initial approximation of $\vec{u}_i^1(x)$, $i = 1, 2, \ldots, N$ is carried out and then they are substituted into Eq. (16). Thus, a set of N linear ordinary differential equations with respect to the unknown functions $\vec{u}_i^2(y)$, $i = 1, 2, \ldots, N$, is obtained. The system obtained has a non-zero solution only for certain discrete values of ω_k^2, which can be treated as the first approximation to the free oscillation frequencies of a shell being sought. The corresponding modes are denoted by $\{\vec{u}_{ik}^2(y)\}_{i=1}^N$. The obtained functions $\{\vec{u}_{ik}^2(y)\}_{i=1}^N$ for certain k (temporarily fixed) are substituted into Eq. (17). As a result, N linear ordinary homogeneous equations with respect to $\{\vec{u}_i^1(x)\}_{i=1}^N$ are obtained, which give ω_{km}^2. For a fixed m number the corresponding functions $\{\vec{u}_{im}^1(x)\}_{i=1}^N$ are substituted into Eq. (16) and then the process is repeated.

The iteration are carried out in two steps. The iteration convergence process may be controlled during each step, or after the step when ω_{km}^2 is obtained.

The iteration result for each k and m is ω_{km}^2 and the corresponding vibration modes are

$$\vec{u}_{km}(x,\ y) = \sum_{i=1}^{N}\vec{u}_{im}^1(x)\otimes\vec{u}_{ik}^2(y). \tag{18}$$

$Z[h]$ operator includes the $G^{-1}(\nabla_{,k}^2 w)$ one. It means that during each step of the variational iterations the relation defined by Eq. (A.2) should be taken into account. For the approximation of that equation the same variational iteration algorithm is used.

2.2.3. *Bubnov–Galerkin approximation*

In the variational iteration method the basis functions $\vec{u}_i^1(x)$ and $\vec{u}_i^2(y)$ are not given *a priori* and they are found by means of optimal solutions. On the other hand, the Bubnov–Galerkin method assumes

$$\vec{u}(x,y) \approx \sum_{i,j=1}^{N} \vec{f}_{ij} \vec{u}_i^1(x) \otimes \vec{u}_j^2(y), \tag{19}$$

where $\vec{u}_i^1(x)$ and $\vec{u}_j^2(y)$ are *a priori* given systems of independent linear functions satisfying specific boundary conditions and closure properties. Substituting Eq. (19) into Eq. (13) and projecting the system of equations obtained in $\vec{u}_l^1(x) \otimes \vec{u}_p^2(y)$ the following system of homogeneous algebraic equations are obtained:

$$\begin{aligned} &\sum_{i,j=1}^{N} \vec{f}_{ij} \int_{\Omega} Z[h] \vec{u}_i^1(x) \otimes \vec{u}_j^2(y) \cdot \vec{u}_p^1(x) \otimes \vec{u}_p^2(y) \, \mathrm{d}\Omega \\ &\quad - \omega^2 \sum_{i,j=1}^{N} \vec{f}_{ij} \int_{\Omega} M[h] \vec{u}_i^1(x) \otimes \vec{u}_j^2(y) \cdot \vec{u}_p^1(x) \otimes \vec{u}_p^2(y) \, \mathrm{d}\Omega = 0 \\ &\quad (l,\ p = 1,\ 2, \dots, N). \end{aligned} \tag{20}$$

The condition of a non-zero solution to system (20) gives N^2 approximate values of the ω_{km} plate frequencies $(k, m = 1, 2, \dots, N)$.

2.2.4. Convergence of MVI and MB approximations

Convergence of the MVI method applied to free vibration analysis problems has not yet been investigated. The MVI convergence for positively defined and symmetric operators has been already proved [27, 28]. The MVI convergence in the problems related to free vibration analysis is ilustrated. Let the operators $Z[h]$ and $M[h]$ be positively defined in earlier Eq. (13). According to Theorem A.2 (see Appendix A), this condition is satisfied. It has been shown earlier that the problem has been reduced to the definition of the eigenfunction of ordinary differential equations (16) and (17).

Following Ref. [27] the MVI calculation step is taken from $\{\vec{u}_i^1(x)\}^{(p)}$ (or $\{\vec{u}_i^2(y)\}^{(p)}$) to $\{\vec{u}_i^2(y)\}^{(p+1)}$ (or $\{\vec{u}_i^1(x)\}^{(p+1)}$), respectively.

For a given and fixed N after p steps (analogous to Eq. (18)) free oscillation frequencies $\omega_{km}^{(p,N)}$ and the corresponding eigenfunctions are of the form

$$\vec{u}_{km}^{(p,N)}(x,\ y) = \sum_{i=1}^{N} \vec{u}_{im}^{1(p-1)}(x) \otimes \vec{u}_{ik}^{2(p)}(y). \tag{21}$$

They satisfy the following equations:

$$\begin{aligned} &\sum_{i=1}^{N} \left\{ \int_0^{l_1} Z[h] \vec{u}_{im}^{1(p-1)}(x) \otimes \vec{u}_{ik}^{2(p)}(y) \cdot \vec{u}_{jm}^{1(p-1)}(x) \, \mathrm{d}x \right. \\ &\quad \left. - \left(\omega_{km}^{(p,N)}\right)^2 \int_0^{l_1} M[h] \vec{u}_{im}^{1(p-1)}(x) \otimes \vec{u}_{ik}^{2(p)}(y) \cdot \vec{u}_{jm}^{1(p-1)}(x) \, \mathrm{d}x \right\} = 0 \\ &\quad (j = 1,\ 2, \dots, N). \end{aligned} \tag{22}$$

The frequencies are estimated using Theorem C.1 (see Appendix C).

Using the results included in Appendix C it may be concluded that in all cases where MB is convergent, MVI is also convergent but slower than MB one (see Eq. (C.4)).

The formulated theoretical basis serves for preparation of suitable algorithms for MB and MVI.

2.3. Algorithms for MB and MVI

In order to find the free vibrations of a rectangular $\Omega = [0, l_1] \times [0, l_2]$ for shells with transverse deformation and rotary inertia, both MB and MVI methods will be applied and the corresponding algorithms will be formulated with the assumption that $h = h(x, y)$.

Consider Eqs. (13). Introduce the following dimensionless quantities [22, 23]:

$$\bar{x} = x/l_1, \quad \bar{y} = y/l_2, \quad \lambda = l_1/l_2, \quad \bar{h} = h/h_0, \quad \lambda_1 = l_1/2h_0, \quad \lambda_2 = l_2/2h_0,$$
$$\bar{w} = w/2h_0, \quad \bar{\gamma}_x = \lambda_1\gamma_x, \quad \bar{\gamma}_y = \lambda_2\gamma_y, \quad \bar{A}_{ijkl} = A_{ijkl}/A_{1111},$$
$$\bar{a}_{ijkl} = a_{ijkl} \cdot A_{1111}, \quad \bar{\lambda}_1 = \lambda_2^2\bar{A}_{1313}, \quad \bar{\lambda}_2 = \lambda_1^2\bar{A}_{2323}, \quad \bar{k}_x = k_x l_1^2/2h_0,$$
$$\bar{k}_y = k_y l_2^2/2h_0, \quad \bar{F} = F/\left(8h_0^3 A_{1111}\right), \quad \bar{\omega}^2 = l_1^2 l_2^2 \rho\omega^2/\left(8h_0^3 A_{1111}\right).$$

The non-dimensional set of equations obtained has the following form (bars have been omitted):

$$K_y\frac{\partial^2 F}{\partial x^2} + K_x\frac{\partial^2 F}{\partial y^2} + \tfrac{2}{3}\bar{\lambda}_1\frac{\partial}{\partial x}\left[h\left(\gamma_x + \frac{\partial w}{\partial x}\right)\right] + \tfrac{2}{3}\bar{\lambda}_2\frac{\partial}{\partial y}\left[h\left(\gamma_y + \frac{\partial w}{\partial y}\right)\right] + \omega^2 hw = 0, \tag{23}$$

$$\begin{aligned}&\tfrac{1}{12}\frac{\partial}{\partial x}\left[\lambda^{-2}h^3A_{1111}\frac{\partial\gamma_x}{\partial x} + h^3A_{1122}\frac{\partial\gamma_y}{\partial y}\right] + \tfrac{1}{12}A_{1212}\frac{\partial}{\partial y}\left[h^3\left(\frac{\partial\gamma_x}{\partial y}+\frac{\partial\gamma_y}{\partial x}\right)\right]\\&\quad - \tfrac{2}{3}\bar{\lambda}_1 h\left(\gamma_x + \frac{\partial w}{\partial x}\right) + \tfrac{1}{12}\lambda_1^{-2}\omega^2h^3\gamma_x = 0,\end{aligned} \tag{24}$$

$$\begin{aligned}&\tfrac{1}{12}\frac{\partial}{\partial y}\left[\lambda^2h^3A_{2222}\frac{\partial\gamma_y}{\partial y} + h^3A_{1122}\frac{\partial\gamma_x}{\partial x}\right] + \tfrac{1}{12}A_{1212}\frac{\partial}{\partial x}\left[h^3\left(\frac{\partial\gamma_y}{\partial x}+\frac{\partial\gamma_x}{\partial y}\right)\right]\\&\quad - \tfrac{2}{3}\bar{\lambda}_2 h\left(\gamma_y + \frac{\partial w}{\partial y}\right) + \tfrac{2}{3}\lambda_2^{-2}\omega^2h^3\gamma_y = 0,\end{aligned} \tag{25}$$

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}(k_y w) + \frac{\partial^2}{\partial y^2}(k_x w) + \frac{\partial^2}{\partial x^2}\left[\lambda^{-4}a_{1111}\frac{\partial^2 F}{\partial x^2} + h^{-1}a_{1122}\frac{\partial^2 F}{\partial y^2}\right]\\&\quad + \frac{\partial^2}{\partial y^2}\left[\lambda^4h^{-1}a_{2222}\frac{\partial^2 F}{\partial y^2} + h^{-1}a_{1122}\frac{\partial^2 F}{\partial x^2}\right] - a_{1212}\frac{\partial^2}{\partial x\partial y}\left(h^{-1}\frac{\partial^2 F}{\partial x\partial y}\right) = 0.\end{aligned} \tag{26}$$

Consider the following boundary conditions:

(a) a movable support: $w = \gamma_x = \gamma_y = F = \left.\frac{\partial^2 F}{\partial n^2}\right|_S = 0$;

(b) a rolling support: $w = F = \left.\dfrac{\partial^2 F}{\partial n^2}\right|_S = 0$, $\gamma_y = M_{11} = 0$ for $x = \text{const}$, $\gamma_x = M_{22} = 0$ for $y = \text{const}$, and assume that

$$\begin{gathered} w = w_1(x)\cdot w_2(y), \quad \gamma_x = \varphi_1(x)\cdot\varphi_2(y), \quad \gamma_y = \psi_1(x)\cdot\psi_2(y), \\ F = F_1(x)\cdot F_2(y). \end{gathered} \tag{27}$$

Substituting Eq. (27) into Eqs. (23)–(26) gives

$$\begin{aligned} &k_x F_1 F_{2,yy} + k_y F_2 F_{1,xx} + \tfrac{2}{3}\,\bar\lambda_1 \varphi_2 (h\varphi_1)_{,x} + \tfrac{2}{3}\,\bar\lambda_1 w_2 (h w_{1,x})_{,x} \\ &\quad + \tfrac{2}{3}\,\bar\lambda_2 \psi_1 (h\psi_2)_{,y} + \tfrac{2}{3}\,\bar\lambda_2 w_1 (h w_{2,y})_{,y} + \omega^2 h w_1 w_2 = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} &\tfrac{1}{12}\lambda^{-2} A_{1111}\varphi_2 (h^3\varphi_{1,x})_{,x} + \tfrac{1}{12} A_{1122}\psi_{2,y}(h^3\psi_1)_{,x} + \tfrac{1}{12} A_{1212}\psi_{1,x}(h^3\psi_2)_{,y} + \tfrac{1}{12}A_{1212}\varphi_1(h^3\varphi_{2,y})_{,y} \\ &\quad - \tfrac{2}{3}\bar\lambda_1 h w_{1,x} w_2 - \tfrac{2}{3}\bar\lambda_1 h\varphi_1\varphi_2 + \tfrac{1}{12}\lambda_1^{-2}\omega^2 h^3\varphi_1\varphi_2 = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &\tfrac{1}{12}\lambda^{2} A_{2222}\psi_1 (h^3\psi_{2,y})_{,y} + \tfrac{1}{12} A_{1122}\varphi_{1,x}(h^3\varphi_2)_{,y} + \tfrac{1}{12} A_{1212}\varphi_{2,y}(h^3\varphi_1)_{,x} + \tfrac{1}{12}A_{1212}\psi_2(h^3\psi_{1,x})_{,x} \\ &\quad - \tfrac{2}{3}\bar\lambda_2 h w_1 w_{2,y} - \tfrac{2}{3}\bar\lambda_2 h\psi_1\psi_2 + \tfrac{1}{12}\lambda_2^{-2}\omega^2 h^3\psi_1\psi_2 = 0, \end{aligned} \tag{30}$$

$$\begin{aligned} &w_1(k_x w_2)_{,yy} + w_2(k_y w_1)_{,xx} + \lambda^{-4} a_{1111} F_2 (h^{-1}F_{1,xx})_{,xx} + a_{1122}F_{2,yy}(h^{-1}F_1)_{,xx} + a_{1122}F_{1,xx}(h^{-1}F_2)_{,yy} \\ &\quad + \lambda^4 a_{2222} F_1 (h^{-1}F_{2,yy})_{,yy} - a_{1212}(h^{-1}F_{1,x}F_{2,y})_{,xy} = 0. \end{aligned} \tag{31}$$

Let w_1, φ_1, ψ_1 and F_1 be known. Then, multiplying Eqs. (28)–(31) by w_1, φ_1, ψ_1, F_1, respectively and integrating in the interval [0, 1] in regard to x, one gets the following differential equations with respect to w_2, φ_2, ψ_2 and F_2:

$$\begin{aligned} &F_{2,yy}\int_0^1 k_x F_1 w_1\,\mathrm{d}x + F_2\int_0^1 k_y F_{1,xx} w_1\,\mathrm{d}x + \tfrac{2}{3}\bar\lambda_1\varphi_2\int_0^1 (h\varphi_1)_{,x} w_1\,\mathrm{d}x \\ &\quad + \tfrac{2}{3}\bar\lambda_1 w_2 \int_0^1 (h w_{1,x})_{,x} w_1\,\mathrm{d}x + \tfrac{2}{3}\bar\lambda_2\left(\psi_2\int_0^1 h\psi_1 w_1\,\mathrm{d}x\right)_{,y} \\ &\quad + \tfrac{2}{3}\bar\lambda_2\left(w_{2,y}\int_0^1 h w_1 w_1\,\mathrm{d}x\right)_{,y} + \omega^2 w_2\int_0^1 h w_1 w_1\,\mathrm{d}x = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} &\tfrac{1}{12}\lambda^{-2}A_{1111}\varphi_2\int_0^1 (h^3\varphi_{1,x})_{,x}\varphi_1\,\mathrm{d}x + \tfrac{1}{12}A_{1122}\psi_{2,y}\int_0^1 (h^3\psi_1)_{,x}\varphi_1\,\mathrm{d}x \\ &\quad + \tfrac{1}{12}A_{1212}\left(\psi_2\int_0^1 h^3\psi_{1,x}\varphi_1\,\mathrm{d}x\right)_{,y} + \tfrac{1}{12}A_{1212}\left(\varphi_{2,y}\int_0^1 h^3\varphi_1\varphi_1\,\mathrm{d}x\right)_{,y} \\ &\quad - \tfrac{2}{3}\bar\lambda_1\varphi_2\int_0^1 h\varphi_1\varphi_1\,\mathrm{d}x - \tfrac{2}{3}\bar\lambda_1 w_2\int_0^1 h w_{1,x}\varphi_1\,\mathrm{d}x + \frac{\omega^2}{12}\lambda_1^{-2}\varphi_2\int_0^1 h^3\varphi_1\varphi_1\,\mathrm{d}x = 0, \end{aligned} \tag{33}$$

$$\tfrac{1}{12}\lambda^2 A_{2222}\left(\psi_{2,y}\int_0^1 h^3\psi_1\psi_1\,\mathrm{d}x\right)_{,y}+\tfrac{1}{12}A_{1122}\left(\varphi_2\int_0^1 h^3\varphi_{1,x}\psi_1\,\mathrm{d}x\right)_{,y}$$
$$+\tfrac{1}{12}A_{1212}\varphi_{2,y}\int_0^1 (h^3\varphi_1)_{,x}\psi_1\,\mathrm{d}x+\tfrac{1}{12}A_{1212}\psi_2\int_0^1 (h^3\psi_{1,x})_{,x}\psi_1\,\mathrm{d}x$$
$$-\tfrac{2}{3}\bar{\lambda}_2\psi_2\int_0^1 h\,\psi_1\psi_1\,\mathrm{d}x-\tfrac{2}{3}\bar{\lambda}_2 w_{2,y}\int_0^1 hw_1\psi_1\,\mathrm{d}x+\frac{\omega^2}{12}\lambda_2^{-2}\psi_2\int_0^1 h^3\psi_1\psi_1\,\mathrm{d}x=0, \qquad (34)$$

$$\left(w_2\int_0^1 k_x w_1 F_1\,\mathrm{d}x\right)_{,yy}+w_2\int_0^1 (k_y w_1)_{,xx}F_1\,\mathrm{d}x+\lambda^{-4}a_{1111}F_2\int_0^1 (h^{-1}F_{1,xx})_{,xx}F_1\,\mathrm{d}x$$
$$+a_{1122}F_{2,yy}\int_0^1 (h^{-1}F_1)_{,xx}F_1\,\mathrm{d}x+a_{1122}\left(F_2\int_0^1 h^{-1}F_{1,xx}F_1\,\mathrm{d}x\right)_{,yy}$$
$$+\lambda^4 a_{2222}\left(F_{2,yy}\int_0^1 h^{-1}F_1F_1\,\mathrm{d}x\right)_{,yy}-a_{1212}\left(F_{2,y}\int_0^1 (h^{-1}F_{1,x})_{,x}F_1\,\mathrm{d}x\right)_{,y}=0. \qquad (35)$$

Proceeding in a similar way, the following equation system, with w_1, φ_1, ψ_1 and F_1 being sought is obtained assuming that w_2, φ_2, ψ_2 and F_2 are known:

$$F_1\int_0^1 k_x F_{2,yy}w_2\,\mathrm{d}y+F_{1,xx}\int_0^1 k_y F_2 w_2\,\mathrm{d}y+\tfrac{2}{3}\bar{\lambda}_1\left(\varphi_1\int_0^1 h\varphi_2 w_2\,\mathrm{d}y\right)_{,x}$$
$$+\tfrac{2}{3}\bar{\lambda}_1\left(w_{1,x}\int_0^1 hw_2w_2\,\mathrm{d}y\right)_{,x}+\tfrac{2}{3}\bar{\lambda}_2\psi_1\int_0^1 (h\psi_2)_{,y}w_2\,\mathrm{d}y$$
$$+\tfrac{2}{3}\bar{\lambda}_2 w_1\int_0^1 (hw_{2,y})_{,y}w_2\,\mathrm{d}y+\omega^2 w_1\int_0^1 hw_2w_2\,\mathrm{d}y=0, \qquad (36)$$

$$\tfrac{1}{12}\lambda^{-2}A_{1111}\left(\varphi_{1,x}\int_0^1 h^3\varphi_2\varphi_2\,\mathrm{d}y\right)_{,x}+\tfrac{1}{12}A_{1122}\left(\psi_1\int_0^1 h^3\psi_{2,y}\varphi_2\,\mathrm{d}y\right)_{,x}$$
$$+\tfrac{1}{12}A_{1212}\psi_{1,x}\int_0^1 (h^3\psi_2)_{,y}\varphi_2\,\mathrm{d}y+\tfrac{1}{12}A_{1212}\varphi_1\int_0^1 (h^3\varphi_{2,y})_{,y}\varphi_2\,\mathrm{d}y$$
$$-\tfrac{2}{3}\bar{\lambda}_1\varphi_1\int_0^1 h\varphi_2\varphi_2\,\mathrm{d}y-\tfrac{2}{3}\bar{\lambda}_1 w_{1,x}\int_0^1 hw_2\varphi_2\,\mathrm{d}y+\frac{\omega^2}{12}\lambda_1^{-2}\varphi_1\int_0^1 h^3\varphi_2\varphi_2\,\mathrm{d}y=0, \qquad (37)$$

$$\tfrac{1}{12}\lambda^2 A_{2222}\psi_1\int_0^1 (h^3\psi_{2,y})_{,y}\psi_2\,\mathrm{d}y+\tfrac{1}{12}A_{1122}\varphi_{1,x}\int_0^1 (h^3\varphi_2)_{,y}\psi_2\,\mathrm{d}y$$
$$+\tfrac{1}{12}A_{1212}\left(\varphi_1\int_0^1 h^3\varphi_{2,y}\psi_2\,\mathrm{d}y\right)_{,x}+\tfrac{1}{12}A_{1212}\left(\psi_{1,x}\int_0^1 h^3\psi_2\psi_2\,\mathrm{d}y\right)_{,x}$$
$$-\tfrac{2}{3}\bar{\lambda}_2\,\psi_1\int_0^1 h\psi_2\psi_2\,\mathrm{d}y-\tfrac{2}{3}\bar{\lambda}_2 w_1\int_0^1 hw_{2,y}\psi_2\,\mathrm{d}y+\frac{\omega^2}{12}\lambda_2^{-2}\psi_1\int_0^1 h^3\psi_2\psi_2\,\mathrm{d}y=0, \qquad (38)$$

$$w_1 \int_0^1 (k_x w_2)_{,yy} F_2 \, dy + \left(w_1 \int_0^1 k_y w_2 F_2 \, dy \right)_{,xx} + \lambda^{-4} a_{1111} \left(F_{1,xx} \int_0^1 h^{-1} F_2 F_2 \, dy \right)_{,xx}$$
$$+ a_{1122} \left(F_1 \int_0^1 h^{-1} F_{2,yy} F_2 \, dy \right)_{,xx} + a_{1122} F_{1,xx} \int_0^1 (h^{-1} F_2)_{,yy} F_2 \, dy$$
$$+ \lambda^4 a_{2222} F_1 (h^{-1} F_{2,yy})_{,yy} F_2 \, dy - a_{1212} \left(F_{1,x} \int_0^1 (h^{-1} F_{2,y})_{,y} F_2 \, dy \right)_{,x} = 0. \quad (39)$$

Eight equations (32)–(39) form a non-linear system of the integral–differential equations with respect to the unknown functions w_1, φ_1, ψ_1, F_1, w_2, φ_2, ψ_2, F_2. In order to abbreviate the equations the following notation in used:

$$A(f,\ u,\ v,\ k,\ m) = \int_0^1 f \frac{d^k u d^m v}{dx^k dx^m} \, dx. \quad (40)$$

Integrating by parts (when it is possible), and taking into account the boundary conditions and Eq. (40), Eqs. (32)–(39) can be re-written in a more suitable form

$$F_1 A(k_x,\ F_2,\ w_2,\ 2,\ 0) + F_{1,xx} A\big(k_y,\ F_2,\ w_2,\ 0,\ 0\big) + \tfrac{2}{3}\bar{\lambda}_1 (\varphi_1 A(h,\ \varphi_2,\ w_2,\ 0,\ 0))_{,x}$$
$$+ \tfrac{2}{3}\bar{\lambda}_1 (w_{1,x} A(h,\ w_2,\ w_2,\ 0,\ 0))_{,x} - \tfrac{2}{3}\bar{\lambda}_2 \psi_1 A\big(h,\ \psi_2,\ w_2,\ 0,\ 1\big)$$
$$- \tfrac{2}{3}\bar{\lambda}_2 w_1 A(h,\ w_2,\ w_2,\ 1,\ 1) + \omega^2 w_1 A(h,\ w_2,\ w_2,\ 0,\ 0) = 0, \quad (41)$$

$$\tfrac{1}{12}\lambda^{-2} A_{1111} (\varphi_{1,x} A(h^3,\ \varphi_2,\ \varphi_2,\ 0,\ 0))_{,x} + \tfrac{1}{12} A_{1122} (\psi_1 A(h^3,\ \psi_2,\ \varphi_2,\ 1,\ 0))_{,x}$$
$$- \tfrac{1}{12} A_{1212} \psi_{1,x} A(h^3,\ \psi_2,\ \varphi_2,\ 0,\ 1) - \tfrac{1}{12} A_{1212} \varphi_1 A(h^3,\ \varphi_2,\ \varphi_2,\ 1,\ 1)$$
$$- \tfrac{2}{3}\bar{\lambda}_1 \varphi_1 A(h,\ \varphi_2,\ \varphi_2,\ 0,\ 0) - \tfrac{2}{3}\bar{\lambda}_1 w_{1,x} A(h,\ w_2,\ \varphi_2,\ 0,\ 0)$$
$$+ \tfrac{1}{12}\lambda_1^{-2} \omega^2 \varphi_2 A(h^3,\ \varphi_2,\ \varphi_2,\ 0,\ 0) = 0, \quad (42)$$

$$\tfrac{1}{12} A_{1212} (\psi_{1,x} A(h^3,\ \psi_2,\ \psi_2,\ 0,\ 0))_{,x} - \tfrac{1}{12} A_{1122} \varphi_{1,x} A(h^3,\ \varphi_2,\ \psi_2,\ 0,\ 1)$$
$$+ \tfrac{1}{12} A_{1212} (\varphi_1 A(h^3,\ \varphi_2,\ \psi_2,\ 1,\ 0))_{,x} - \tfrac{1}{12}\lambda^2 A_{2222} \psi_1 A(h^3,\ \psi_2,\ \psi_2,\ 1,\ 1)$$
$$- \tfrac{2}{3}\bar{\lambda}_2\ \psi_1 A(h,\ \psi_2,\ \psi_2,\ 0,\ 0) - \tfrac{2}{3}\bar{\lambda}_2 w_1 A(h,\ w_2,\ \psi_2,\ 1,\ 0)$$
$$+ \tfrac{1}{12}\lambda_2^{-2} \omega^2 \psi_1 A(h^3,\ \psi_2,\ \psi_2,\ 0,\ 0) = 0, \quad (43)$$

$$w_1 A(k_x, w_2, F_2, 0, 2) + (w_1 A(k_y, w_2, F_2, 0, 0))_{,xx}$$
$$+ \lambda^{-4} a_{1111} (F_{1,xx} A(h^{-1}, F_2, F_2, 0, 0))_{,xx}$$
$$+ a_{1122} (F_1 A(h^{-1}, F_2, F_2, 2, 0))_{,xx} + a_{1122} F_{1,xx} A(h^{-1}, F_2, F_2, 0, 2)$$
$$+ \lambda^4 a_{2222} F_1 A(h^{-1}, F_2, F_2, 2, 2) - a_{1212} (F_{1,x} A(h^{-1}, F_2, F_2, 1, 1))_{,x} = 0, \quad (44)$$

$$
\begin{aligned}
&F_{2,yy}A(k_x,F_1,w_1,0,0)+F_2A\big(k_y,F_1,w_1,2,0\big)-\tfrac{2}{3}\bar{\lambda}_1\varphi_2A(h,\varphi_1,w_1,0,1)\\
&\quad-\tfrac{2}{3}\bar{\lambda}_1w_2A(h,w_1,w_1,1,1)+\tfrac{2}{3}\bar{\lambda}_2(\psi_2A(h,\psi_1,w_1,0,0))_{,y}\\
&\quad+\tfrac{2}{3}\bar{\lambda}_2(w_{2,y}A(h,w_1,w_1,0,0))_{,y}+\omega^2w_2A(h,w_1,w_1,0,0)=0,
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&-\tfrac{1}{12}\lambda^{-2}A_{1111}\varphi_2A(h^3,\varphi_1,\varphi_1,1,1)-\tfrac{1}{12}A_{1122}\psi_{2,y}A(h^3,\psi_1,\varphi_1,0,1)\\
&\quad+\tfrac{1}{12}A_{1212}(\psi_2A(h^3,\psi_1,\varphi_1,1,0))_{,y}\\
&\quad+\tfrac{1}{12}A_{1212}(\varphi_{2,y}A(h^3,\varphi_1,\varphi_1,0,0))_{,y}-\tfrac{2}{3}\bar{\lambda}_1\varphi_2A(h,\varphi_1,\varphi_1,0,0)\\
&\quad-\tfrac{2}{3}\bar{\lambda}_1w_2A(h,w_1,\varphi_1,1,0)+\tfrac{1}{12}\omega^2\lambda_1^{-2}\varphi_2A(h^3,\varphi_1,\varphi_1,0,0)=0,
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&-\tfrac{1}{12}A_{1212}\psi_2A(h^3,\psi_1,\psi_1,1,1)+\tfrac{1}{12}A_{1122}(\varphi_2A(h^3,\varphi_1,\psi_1,1,0))_{,y}\\
&\quad-\tfrac{1}{12}A_{1212}\varphi_{2,y}A(h^3,\varphi_1,\psi_1,0,1)\\
&\quad+\tfrac{1}{12}\lambda^2A_{2222}(\psi_{2,y}A(h^3,\psi_1,\psi_1,0,0))_{,y}-\tfrac{2}{3}\bar{\lambda}_2\ \psi_2A(h,\psi_1,\psi_1,0,0)\\
&\quad-\tfrac{2}{3}\bar{\lambda}_2w_{2,y}A(h,w_1,\psi_1,0,0)+\tfrac{1}{12}\omega^2\lambda_2^{-2}\psi_2A(h^3,\psi_1,\psi_1,0,0)=0,
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&(w_2A(k_x,w_1,F_1,0,0))_{,yy}+w_2A(k_y,w_1,F_1,0,2)\\
&\quad+\lambda^{-4}a_{1111}F_2A(h^{-1},F_1,F_1,2,2)+a_{1122}F_{2,yy}A(h^{-1},F_1,F_1,0,2)\\
&\quad+a_{1122}(F_2A(h^{-1},F_1,F_1,2,0))_{,yy}+\lambda^4a_{2222}(F_{2,y}A(h^{-1},F_1,F_1,0,0))_{,yy}\\
&\quad-a_{1212}(F_{2,y}A(h^{-1},F_1,F_1,1,1))_{,y}=0.
\end{aligned}
\tag{48}
$$

The finite element method of the second order [26, 28, 29] is used to solve the ordinary differential equations obtained. A detailed description of application of FEM to this problem is given in Appendix D.

During formulation of the stiffness and mass matrices (see C_i^e and D_i^e in Appendix D) it is necessary to calculate the integrals in the rectangular spaces $\Omega^e=[0,1]\times\Delta e$:

$$
\int_{\Delta e}A(f,u,v,p,q)\xi_k^e\xi_m^{e'}\mathrm{d}y=\int_{\Delta e}\int_0^1 f(x,y)\frac{\partial^p u(x)\partial^q v(x)}{\partial x^p\quad\partial x^q}\xi_k^e(y)\xi_m^{e'}(y)\ \mathrm{d}y\ \mathrm{d}x,
$$

with the corresponding change of the variables in $f\,(x,y)$. For this purpose, a two-dimensional formula (similar to the Simpson formula) may be used.

The algorithm presented has been used to define the elements of the local matrices of stiffness C_i^e and mass D_i^e.

3. Numerical results

In Fig. 1 a distribution for four elements in interval [0, 1], where 'e' is the element number, is shown. For each element local numbers are given above, whereas below the global numbers are presented below. As it has been shown (in the one-dimensional case) the global numbers are

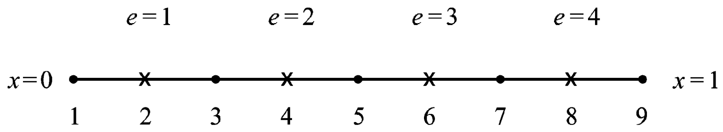

Fig. 1. Local and global numeration of the [0, 1] partition into 4 elements.

related to the local numbers and to the finite element number in the following manner:

$$i_g = i_l + 2\,(e-1). \tag{49}$$

The above formula is used for an arrangement of the obtained elements of the local matrices C_i^e and D_i^e obtained on the corresponding places in the global matrices C_i and D_i to realize their calculation sequence. The algorithm action is composed of one iteration variational step and FEM results in matrices C_i and D_i for $i = 1$ or $i = 2$. The boundary conditions are realized by cancelling the columns and the rows corresponding to the given variable w, γ_x, γ_y, F on the boundaries from the C_i and D_i matrices. The boundary conditions of the type $M_{11} = M_{22} = 0$, $\partial^2 F/\partial n^2 = 0$ are introduced automatically, because the algorithm uses a stationary energy deformation. It means that the algorithm is variational.

The eigenvalues of the generalized problems are obtained from the equation

$$C_i\vec{u}_i - \omega^2 D_i\vec{u}_i = 0, \quad i = 1 \text{ or } i = 2, \tag{50}$$

where $\vec{u}_i = \left(w_i^1, w_i^2, \dots, w_i^N, \varphi_i^1, \varphi_i^2, \dots, \varphi_i^N, \psi_i^1, \dots, \psi_i^N, F_i^1, \dots, F_i^N\right)$ are the functions w_i, φ_i, ψ_i, and F_i in the global nodes $j = 1, 2, \dots, N$, being sought.

In order to solve Eq. (50) the Schwartz iteration method has been used [30]. This method is particularly effective for low-frequency vibrations.

Suppose that fundamental (the lowest) free vibration frequencies are to be determined. Assuming the $\vec{u}_i^{(n)}$ value the following equation can be solved:

$$C_i\vec{u}_i^{(n+1)} = D_i\vec{u}_i^{(n)}. \tag{51}$$

Substituting the solution obtained again into the right-hand side of Eq. (51), gives $\vec{u}_i^{(n+2)}$, and so on. In each step the approximated value of the fundamental frequency ω_n^2 may be obtained from the Rayleigh formula:

$$\omega_n^2 = \frac{\vec{u}_i^{(n+1)} C_i \vec{u}_i^{(n+1)}}{\vec{u}_i^{(n+1)} D_i \vec{u}_i^{(n+1)}}. \tag{52}$$

The process is controlled because of the difference between ω_n^2 and ω_{n+1}^2. The algorithm has been described in detail in Ref. [24], where its convergence has been proved.

The convergence of the finite elements method results from the bilinear forms $a_h(\vec{u}, \vec{v})$ and $b_h(\vec{u}, \vec{v})$ properties, proved in Theorem A.2 (see Appendix A) and the respective theorems for the second order finite elements [28, 30].

A convergence of MB has been illustrated in Figs. 2 and 3, where a solid line corresponds to a rolling support, and a dashed line corresponds to a clamped support.

As it is seen from the figures, the convergence is practically achieved for $N = 12$ for MVI and for $M = 18$ for MB. However, an error of frequency determination for the MVI method with

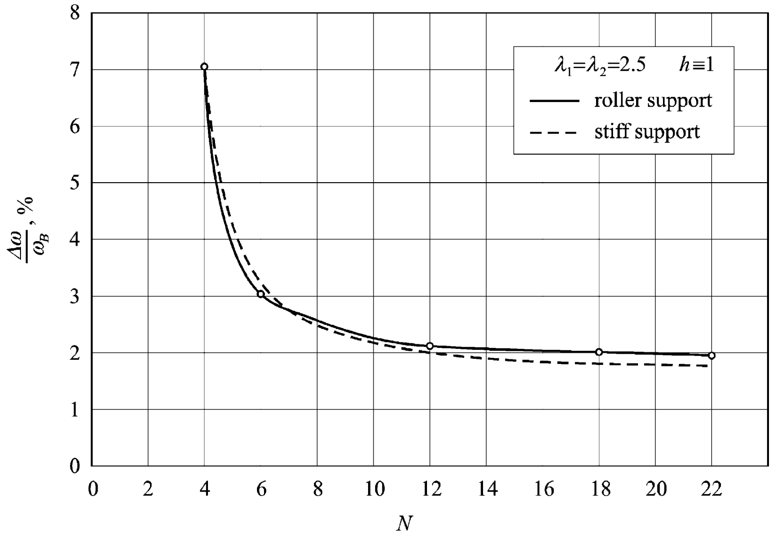

Fig. 2. Estimation error of fundamental frequency using MVI in relation to frequency ω_B defined by MB.

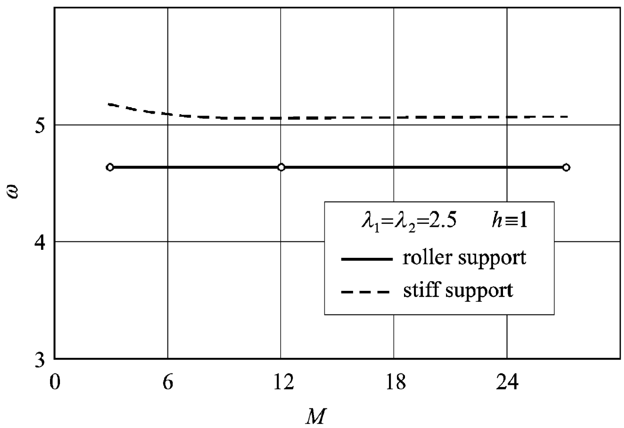

Fig. 3. Illustration of MB method convergence for a clamped and free supported plate.

regard to the MB method (Fig. 2) does not tend towards zero, but it achieves 1.9% for a clamped (stiff) support, and 2.3% for the rolling support. The reason is that in MVI only one factor in the function distribution (29) has been taken into account.

In order to verify the applications range and to compare the free vibration frequencies of a plate with a rolling support the calculations using the MVI and MB methods have been carried out for the physical-geometrical parameter $\bar{\lambda}_1 = \bar{\lambda}_2$ within the interval [0.25; 2500] for $v = 0.25$ and $\lambda_1 = \lambda_2$ within the interval [100,1]. The obtained dependence has been shown in Fig. 4.

Analysis of the results obtained leads to the following general conclusions. The frequencies, obtained using the first approximation of the MVI beginning from $\bar{\lambda}_1 = \bar{\lambda}_2 = 25$, differ by no more than 5%, since only one term has been taken into account by MVI. However, for $\bar{\lambda}_1 =$

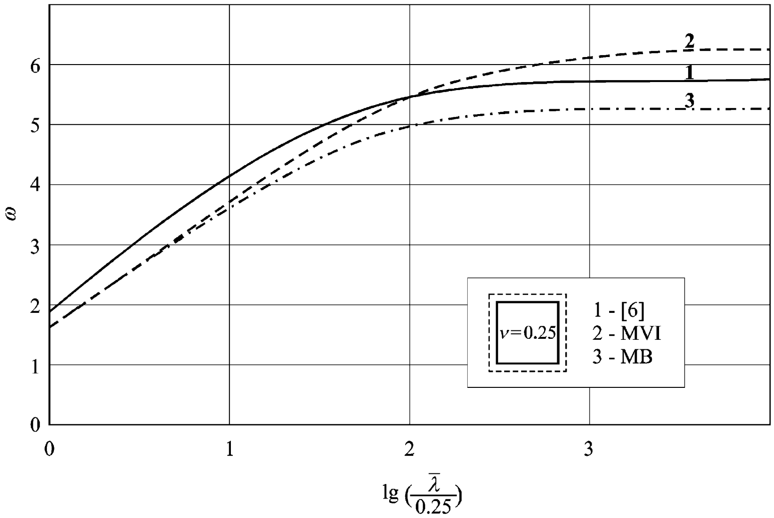

Fig. 4. Comparison of different methods using free support for $\bar{\lambda} \in (0.25;\ 2500)$.

$\bar{\lambda}_2 \leqslant 25$ they are very close to each other and for $\bar{\lambda}_1 = \bar{\lambda}_2 = 0.25$ they fully overlap. It means that for plates with low transverse stiffness one can apply MVI, which needs lower time computations in comparison to MB.

3.1. Test of algorithms

In order to carry out algorithm tests relating to the free vibration frequencies obtained using the MVI and the MB methods, first the plates with constant thickness (equal to 1) for $\lambda_1 = \lambda_2 = 2, 5$, and for different$G_{13}/E = G_{23}/E$ and for $G/E \in [0, 4; 0, 01]$ have been analyzed. The parameters considered correspond to the properties of composite materials currently used [31], such as, for the plastic-glass material $G/E = \frac{1}{7}$ [32], for the borplastic $G/E = \frac{1}{25}$, for the graphitoplastic $G/E = \frac{1}{40}$. A material for which $G/E = \frac{1}{100}$ [32, 33] also exists.

During the analysis of the algorithm convergence, fundamental frequencies of a vibrating plate with constant thickness for $\nu = 0.25, \lambda_1 = \lambda_2 = 2.5$ for the isotropic material ($G/E = 0.4$) have been found ($\bar{\lambda}_1 = \bar{\lambda}_2 = 2.34375$).

In Fig. 2 an error dependence on the fundamental mode calculation using the MVI in the first approximation versus the finite elements number N along the x- and y-axis, has been shown. The ω_B denotes a frequency of the fundamental mode obtained using the MB for $M = 27$ (9 terms for each function).

4. Plates and shells mass optimization with free vibration frequency constraints

Assume that a shell with variable thickness occupies a bounded space Ω with a piecewise smooth edge S. Free vibration of this shell are governed by Eqs (1) with boundary conditions (2)–(5).

Consider the following problem. Among all possible material distributions within a shell, described by the thickness function $h(x, y)$ it is required to determine the $h^*(x, y)$ from a given set of functions U_∂ (defined later) in order to minimize the shell's mass and to get the fundamental frequency is the same as for the shell with constant thickness equal to 1.

This problem may also be formulated in a slightly different manner. For a given shell mass m^* it is required to find the distribution $h^*_\partial(x, y)$ in order to achieve the maximum frequency of the fundamental mode. It can be proved that for shells with transverse deformation and rotary inertia the two approaches described are equivalent.

Therefore, the problem is reduced to that of minimalization with the constraints

$$h^*(x, y) : \to \min_{h \in U_\partial} \int_\Omega \rho \mathrm{d}\Omega, \quad \omega = \omega_0, \tag{53}$$

where ω denotes the fundamental mode frequency found from Eqs. (2)–(5), and ω_0 is free vibration frequency of the plate with constant thickness $h(x, y) \equiv 1$.

Using a penalty method it is possible to avoid the explicit constraint $\omega = \omega_0$. As a result, problem (53) is reduced to the following:

$$h^*_\varepsilon(x, y) : \to \min_{h \in U_\partial} \left[\int_\Omega \rho h \, \mathrm{d}\Omega + \frac{1}{\varepsilon}(\omega - \omega_0)^2 \right], \tag{54}$$

where ω again is defined by Eqs. (2)–(5). Using the approach presented in Ref. [34] it can be shown that for a defined choice of the set U_∂ the solution $h^*_\varepsilon(x, y)$ of Eq. (54) tends to $h^*(x, y)$ for $\varepsilon \to 0$.

However, in the majority of cases occurring in practice it is not necessary to have the exact relation $\omega = \omega_0$, and therefore without the condition of $\varepsilon \to 0$ there is still a practically valid relation. Similar problems are called the ε-optimization problems or the rational design problems and they will be considered further.

Define a set of admissible functions $h(x, y)$, called the set of admissible control

$$U_\partial = \left\{ h(x, y) \mid h \in H^1(\Omega), |h|_{1,\Omega} \leqslant \text{const}, 0 < h_H \leqslant h(x, y) \leqslant h_B \right\}. \tag{55}$$

In the case of the admissible functions, the functions with piecewise continuous first derivatives and bounded above and below are taken.

A choice of the admissible control (55) guarantees a solution to problem (54), which is defined by the following theorem (proof is omitted).

Theorem 1. *If assumptions* (6)–(8) *are satisfied and* k_x *and* k_y *are bounded on* Ω, *then the rational design problem* (54) *defined on* U_∂ *according to* Eq. (55) *has at least one solution for an arbitrarily taken* $\varepsilon > 0$.

The obtained result is valid not only from the point of view of different designs, but also from the point of view of the $h(x, y)$ approximation. If $h(x, y) \in U_\partial$ then during a construction of the finite approximation $h(x, y)$ the approximation function is required to belong to the U_∂ interior. Otherwise, the process may not be convergent.

5. Finite-dimensional approximations of the rational design of plates and shells

5.1. Finite-dimensional approximation

The problem dealing with the finite-dimensional approximation is particularly important for optimal design of plates and shells. It is evident that there are no analytical solutions in the majority of tasks devoted to design [35]. The only possible solutions are approximate ones obtained using the numerical methods.

In the following considerations problem (53) is omitted and the rational design defined by Eq. (54) for a given ε is considered. A solution to problem (53), as has been mentioned earlier, may be obtained for $\varepsilon \to 0$. For an engineering calculation purpose it is sufficient to take a small number as ε. Now, the questions considered are concerned with the occurrence of approximated solutions on the finite-dimensional subspaces U_∂ and their convergence.

It will be shown that the frequency of the fundamental mode of a shells with transverse deformation and rotary inertia is a weakly continuous functional for all $h \in U_\partial$ (U_∂ is defined by Eq. (55)). Taking into account that ω^2 is defined by the Rayleigh formula (52), this result will be formulated as the following theorem (proof is omitted).

Theorem 2. *The functional*

$$\varphi(h) = \omega^2 = \inf_{\vec{u} \in V_0} \frac{a_h(\vec{u}, \vec{u})}{b_h(\vec{u}, \vec{u})}$$

is a weakly continuous transformation from U_∂ *to* R^1.

Thus, the problem is reduced to a construction of the finite-dimensional subspaces of functions from U_∂ and having convergence properties. Otherwise, a situation may occur for which it will be extremely difficult to achieve a minimum of the functional

$$J(h) = \int_\Omega \rho h \, \mathrm{d}\Omega + \frac{1}{\varepsilon}\left(\omega^2 - \omega_0^2\right)^2. \tag{56}$$

Select a series of the finite-dimensional subspaces $\{H_n\}_{n=1}^{\infty}$ in such a way that for an arbitrary n $H_n \subset H^1(\Omega)$ and

$$\lim_{n \to \infty} \inf_{g \in H_n \cap U_\partial} |g - h_1|_{1,\Omega} = 0, \quad \forall h_1 \in U_\partial. \tag{57}$$

In other words, an arbitrary function $h_1 \in U_\partial$ should approach the norm of $H^1(\Omega)$ through the functions $g_n \in H_n \cap U_\partial$ with an arbitrarily given accuracy for $n \to \infty$. Now, reduce the problem of the rational design of a plate in regard to its mass with a given constraint for its fundamental frequency ω to the following finite-dimensional programming problem. Find $h_n^* \in H_n \cap U_\partial$ and the corresponding ω_n, $\vec{u}_n \in V_0$ so that

$$J\left(h_n^*\right) = \inf_{h \in H_n \cap U_\partial} J(h), \quad n = 1, 2, \ldots, \tag{58}$$

where $J(h)$ is defined by Eq. (56). A rigorous discussion of solution to problem (58) is omitted here.

It often happens that from a point of view of technology engineering imagination and the second order constraints, the U_∂ set of the thickness distribution functions is chosen as finite dimensional. It deals with those functions depending on finite numbers of parameters or the functions represented by finite series pieces because of the known function choice [25]. At the first glance, the cases occurring are already finite dimensional, and therefore, a question related to convergence and approximation does not appear. However, it is not entirely true. Let us take into account the function $h(x, y)$ in one of the earlier discussed forms and substitute it in system (1). Applying one of the numerical methods for the determination of ω and $\vec{u}$, the necessity of approximating $h(x, y)$ is approached, as is a given set of parameters which define this function. Which conditions should be satisfied for the assumed approximation scheme? The approximation should satisfy condition (57).

The methods which satisfy these conditions are: the finite element method, the mesh method with $O(h^\alpha)$ approximation, $\alpha \geqslant 1$, and many others. In the case of application of the Kirchhoff–Love model, it is necessary to apply the finite elements securing not only $h(x, y)$ continuity but also continuity of its first partial derivatives, which of course is difficult to be realized numerically.

6. Conclusions

The conclusions include the following main parts:

(i) *Investigation of inertial effect influence:* Comparing the two results obtained with the results given in Ref. [23] (the dashed line in Fig. 4) obtained using the MB method with higher approximations and without inertial effects related to rotation, the following conclusions are drawn. The inertial effect related to rotation decreases the frequency by 12% for $\bar{\lambda}_1 = \bar{\lambda}_2 = 0.25$ and by 9% for $\bar{\lambda}_1 = \bar{\lambda}_2 = 2500$. It means that this effect must be taken into account during transversal-isotropic plates analysis.

Analogous results have been obtained for plates with changing thickness according to the exponential law

$$h(x, y) = z_1 + z_2 e^{z_3(x^2+y^2)}. \tag{59}$$

In Fig. 5 the curves of such dependences with the rolling plate support for $\bar{\lambda}_1 = \bar{\lambda}_2$, $z_1 = 0.4$, $z_2 = 0.02$, $z_3 = 3$ obtained using the MB (dashed line) and the MVI (solid line) for $\lambda_1 = \lambda_2 = 2.5$ and $\bar{\lambda}_1 = \bar{\lambda}_2 \in [2.5; 0.0625]$ have been shown.

(ii) *Investigation of boundary conditions influence for plates having low transverse stiffness:* Decreasing transversal stiffness should cause a decrease of the area of boundary effects. It is expected that the influence of the boundary conditions on free vibration frequencies should also be decreased. This behaviour of the fundamental mode for plates with $\bar{\lambda}_1 = \bar{\lambda}_2 \in [0.0625;\ 0.25]$ is analyzed for different thicknesses and for $\lambda_1 = \lambda_2 = 2.5$. In Fig. 6 the dependences ω versus $\bar{\lambda}_1 = \bar{\lambda}_2$ for different h have been shown. Curves 1 are related to plates with $h = 1$, curves 2 to plates with $h = 0.25$ (the solid lines correspond to stiffly supported plate, whereas the dashed lines correspond to rolling supported plate). The results have been obtained using the MVI for $N = 12$.

As it can be seen from Fig. 6, decreasing the transversal stiffness for both supports, results in the fundamental frequencies approaching each other independent of the thickness. For $\bar{\lambda} = 0.25$ they differ by 10% for $h = 0.25$ and by 1% for $h = 1$. Therefore, one can conclude that for thick

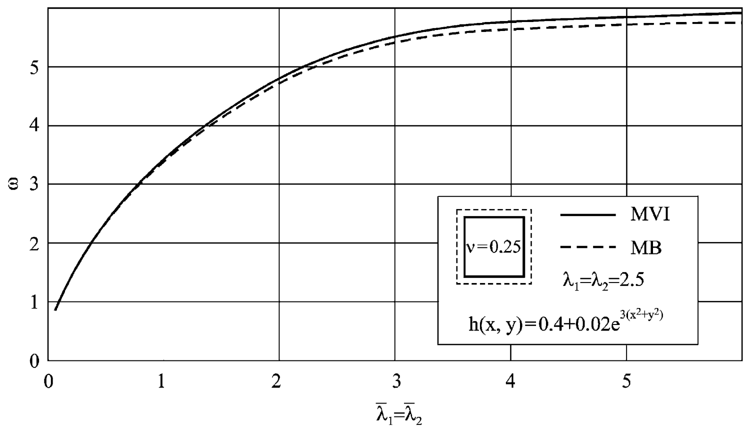

Fig. 5. Frequency of vibrations of a roller supported plate and exponentially changeable thickness h using the MVI and MB methods.

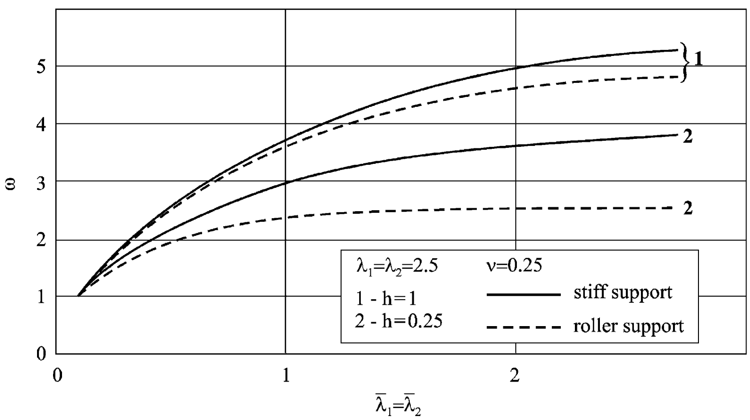

Fig. 6. Frequency of vibrations of a plate with different thickness h defined by the MVI method for $N = 12$.

plates with low transverse stiffness the support conditions influence only slightly free vibration frequencies obtained using shells with the transverse deformation and rotary inertia model.

(iii) *Investigation of plate thickness influence for different transversal stiffness:* In Fig. 2, contrary to the previous one, an example of approaching tune frequencies of the fundamental modes for different thickness and low transverse stiffness has been presented. This effect is analyzed in detail. In Fig. 7, the dependences ω versus h for different $\bar{\lambda}_1 = \bar{\lambda}_2$ and for $\lambda_1 = \lambda_2 = 2.5$, $v = 0.25$ have been shown. This case corresponds to the rolling support of the plate. Analogous dependences are shown in Fig. 8 for a stiffly supported plate.

Analysis of the diagrams leads to the conclusion that for plates with low transverse stiffness $\bar{\lambda}_1 = \bar{\lambda}_2 \leqslant 0.25$ the fundamental mode frequency weakly depends on a plate thickness. For instance, for $\bar{\lambda}_1 = \bar{\lambda}_2 = 0.25$ the frequencies for $h = 2$ and 0.5 for plates with a rolling support differ by 4%, whereas for plates with a stiff support they differ by 2%.

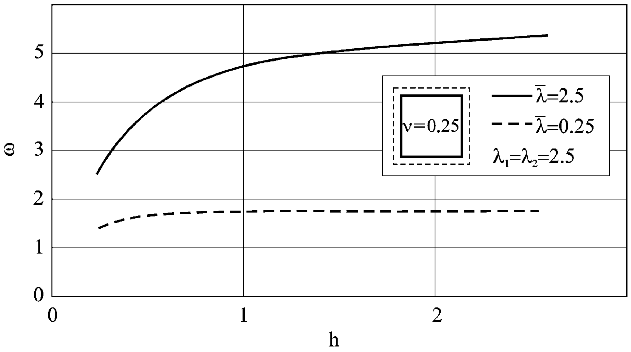

Fig. 7. Frequency of a roller supported plate versus its thickness h.

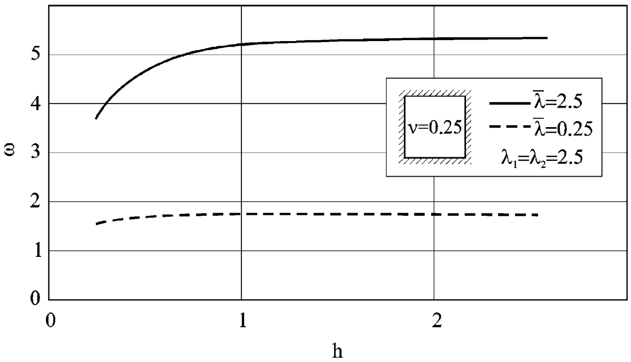

Fig. 8. Frequency of a stiffly supported plate versus its thickness h.

Finally, the finite-dimensional approximations of the rational design of plates and shells is also formally discussed.

Appendix A

Theorem A.1. *If conditions* (6) *and* (8) *are satisfied, then the G operator is symmetric*, *continuous and positively defined in* $H_0^2(\Omega)$.

Proof. Because G is a linear operator, its continuity results directly from its boundness on $H_0^2(\Omega)$. Symmetry of G is obvious, whereas a positive definition results from Ref. [34] using assumptions (6) and (8). □

According to Theorem 1 G is the homoeomorphism transformation from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$. The inverse operator G^{-1} is continuous and symmetric.

Let us introduce a functional-vector space

$$V_0 = \left\{ \vec{u} = \left(w, \gamma_x, \gamma_y\right) \in \left(H^1(\Omega)\right)^3 \middle| \, w_{(S_1+S_2)} = 0,\ \gamma_{\tau(S_2+S_3)} = 0,\ \gamma_{n(S_3)} = 0 \right\}.$$

If $\vec{u} \in V_0$, then $w \in H^1(\Omega)$. Therefore, for bounded in Ω curvatures k_x and k_y we have

$$\nabla^2_{,k} w = \frac{\partial^2}{\partial y^2}(k_x w) + \frac{\partial^2}{\partial x^2}\left(k_y w\right) \in H^{-2}(\Omega).$$

Therefore, a solution (because of F) of Eq. (10) exists which reads

$$F = G^{-1}\left(\nabla^2_{,k} w\right), \qquad F \in H^2_0(\Omega).$$

Observe that Eqs. (1) can be presented in the following operator form:

$$Z[h]\vec{u} - \omega^2 M[h]\vec{u} = 0, \tag{A.1}$$

$$G(F) = -\nabla^2_{,k} w, \tag{A.2}$$

where $Z[h]$ and $M[h]$ are linear differential operators defined by the expressions

$$Z[h] = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}, \tag{A.3}$$

$$M[h] = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix}, \tag{A.4}$$

$$\begin{aligned}
L_{11} &= -\tfrac{2}{3}\left[\frac{\partial}{\partial x}\left(A_{1313}h\frac{\partial(\bullet)}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{2323}h\frac{\partial(\bullet)}{\partial y}\right)\right] + \nabla^2_k G^{-1}\left(\nabla^2_{,k}(\bullet)\right),\\
L_{12} &= -\tfrac{2}{3}\frac{\partial}{\partial x}(A_{1313}h\bullet), \quad L_{13} = -\tfrac{2}{3}\frac{\partial}{\partial y}(A_{2323}h\bullet), \quad L_{21} = \tfrac{2}{3}A_{1313}h\frac{\partial(\bullet)}{\partial x},\\
L_{22} &= -\tfrac{2}{3}\left[\frac{\partial}{\partial x}\left(A_{1111}h^3\frac{\partial(\bullet)}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{1212}h^3\frac{\partial(\bullet)}{\partial y}\right)\right] + \tfrac{2}{3}A_{1313}h(\bullet),\\
L_{23} &= -\tfrac{2}{3}\left[\frac{\partial}{\partial x}\left(A_{1122}h^3\frac{\partial(\bullet)}{\partial y}\right) + \frac{\partial}{\partial y}\left(A_{1212}h^3\frac{\partial(\bullet)}{\partial x}\right)\right], \quad L_{31} = \tfrac{2}{3}A_{2323}h\frac{\partial(\bullet)}{\partial y},\\
L_{32} &= -\tfrac{2}{3}\left[\frac{\partial}{\partial x}\left(A_{1212}h^3\frac{\partial(\bullet)}{\partial y}\right) + \frac{\partial}{\partial y}\left(A_{2211}h^3\frac{\partial(\bullet)}{\partial x}\right)\right],\\
L_{33} &= -\tfrac{2}{3}\left[\frac{\partial}{\partial x}\left(A_{1212}h^3\frac{\partial(\bullet)}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{2222}h^3\frac{\partial(\bullet)}{\partial y}\right)\right] + \tfrac{2}{3}A_{2323}h(\bullet).
\end{aligned}$$

For $m_{ii}(i=1,2,3)$ we get $m_{11}=h, m_{22}=m_{33}=2h^3/3$. Let us introduce a bilinear form defining a work on the virtual displacement $\vec{v}=\left(\tilde{w},\ \tilde{\gamma}_x,\ \tilde{\gamma}_y\right)$

$$a_h(\vec{u},\vec{v}) = \int_\Omega \left\{ \frac{2}{3}\left[A_{1111}h^3 \frac{\partial\gamma_x}{\partial x}\frac{\partial\tilde{\gamma}_x}{\partial x} + A_{1122}h^3\frac{\partial\gamma_y}{\partial y}\frac{\partial\tilde{\gamma}_x}{\partial x} + A_{1122}h^3\frac{\partial\gamma_x}{\partial x}\frac{\partial\tilde{\gamma}_y}{\partial y} + A_{2222}h^3\frac{\partial\gamma_y}{\partial y}\frac{\partial\tilde{\gamma}_y}{\partial y} + A_{1212}h^3\left(\frac{\partial\gamma_y}{\partial x}+\frac{\partial\gamma_x}{\partial y}\right)\left(\frac{\partial\tilde{\gamma}_y}{\partial x}+\frac{\partial\tilde{\gamma}_x}{\partial y}\right)\right] + \frac{2}{3}\left[A_{1313}h\left(\gamma_x+\frac{\partial w}{\partial x}\right)\left(\tilde{\gamma}_x+\frac{\partial\tilde{w}}{\partial x}\right) + A_{2323}h\left(\gamma_y+\frac{\partial w}{\partial y}\right)\left(\tilde{\gamma}_y+\frac{\partial\tilde{w}}{\partial y}\right)\right] + \nabla_k^2 G^{-1}\left(\nabla_{,k}^2 w\right)\tilde{w}\right\}\,\mathrm{d}\Omega \tag{A.5}$$

The following results related to $a_h(\vec{u},\vec{v})$ hold.

Theorem A.2. *Let assumptions* (A.6)–(A.9) *be satisfied and let k_x, k_y be bounded in Ω. Then, the linear form* (A.5) *is continuous, symmetric and positive in the $V_0\times V_0$ space. It means that the following relations hold*:

$$|\,a_h(\vec{u},\vec{v})\,|\leqslant c_2|\vec{u}|_{V_0}|\vec{v}|_{V_0}, \qquad \forall\vec{u},\vec{v}\in V_0, \tag{A.6}$$

$$a_h(\vec{u},\vec{v}) = a_h(\vec{v},\vec{u}) \qquad \forall\vec{u},\vec{v}\in V_0, \tag{A.7}$$

$$a_h(\vec{u},\vec{u})\geqslant 0 \qquad \forall\vec{u}\in V_0, \quad c_2 = const > 0. \tag{A.8}$$

Proof. (A) *Continuity*: From Eq. (A.5) one gets

$$|a_h(\vec{u},\vec{v})|\leqslant \tfrac{2}{3}A_B h_B^3\left(\left|\frac{\partial\gamma_x}{\partial x}\right|_{0,\Omega}\left|\frac{\partial\tilde{\gamma}_x}{\partial x}\right|_{0,\Omega} + \left|\frac{\partial\gamma_y}{\partial y}\right|_{0,\Omega}\left|\frac{\partial\tilde{\gamma}_x}{\partial x}\right|_{0,\Omega} + \left|\frac{\partial\tilde{\gamma}_y}{\partial y}\right|_{0,\Omega}\left|\frac{\partial\gamma_x}{\partial x}\right|_{0,\Omega} + \left|\frac{\partial\gamma_y}{\partial y}\right|_{0,\Omega}\left|\frac{\partial\tilde{\gamma}_y}{\partial y}\right|_{0,\Omega}\right) + \left|\left(G^{-1}\left(\nabla_{,k}^2 w\right),\ \nabla_{,k}^2\tilde{w}\right)\right| + \tfrac{2}{3}A_B h_B\left(\left|\gamma_x+\frac{\partial w}{\partial x}\right|_{0,\Omega}\left|\tilde{\gamma}_x+\frac{\partial\tilde{w}}{\partial x}\right|_{0,\Omega} + \left|\gamma_y+\frac{\partial w}{\partial y}\right|_{0,\Omega}\left|\tilde{\gamma}_y+\frac{\partial\tilde{w}}{\partial y}\right|_{0,\Omega}\right) + \tfrac{2}{3}A_B h_B^3\left|\frac{\partial\gamma_y}{\partial x}+\frac{\partial\gamma_x}{\partial y}\right|_{0,\Omega}\left|\frac{\partial\tilde{\gamma}_y}{\partial x}+\frac{\partial\tilde{\gamma}_x}{\partial y}\right|_{0,\Omega}. \tag{A.9}$$

The last factor of Eq. (A.9) may be transformed into the following form:

$$\left|\left(G^{-1}\left(\nabla_{,k}^2 w\right),\ \nabla_{,k}^2\tilde{w}\right)\right| = \left|-\left(F,\ \nabla_{,k}^2\tilde{w}\right)\right| = \left|\left(\nabla_k^2 F,\ \tilde{w}\right)\right|.$$

Therefore

$$\left|\left(G^{-1}\left(\nabla_{,k}^2 w\right),\ \nabla_{,k}^2\tilde{w}\right)\right|\leqslant\left|\nabla_k^2 F\right|_{0,\Omega}\cdot|\tilde{w}|_{0,\Omega},$$

whereas

$$\left|\nabla_k^2 F\right|_{0,\Omega} \leqslant \text{const}|F|_{0,\Omega}.$$

Because the operator G^{-1} is bounded in $H^{-2}(\Omega)$

$$|F|_{2,\Omega} \leqslant \text{const.}\left|\nabla_{,k}^2 w\right|_{-2,\Omega}.$$

Taking into account that $\left|\nabla_{,k}^2 w\right|_{-2,\Omega} \leqslant \text{const}|w|_{0,\Omega}$, for limited k_x, k_y, we get

$$\left|\left(G^{-1}\left(\nabla_{,k}^2 w\right), \nabla_{,k}^2 \tilde{w}\right)\right| \leqslant \text{const}|w|_{0,\Omega} \cdot |\tilde{w}|_{0,\Omega}. \tag{A.10}$$

Then, from Eqs. (A.9) and (A.10) we get Eq. (A.6). This proves boundness. Because $a_h(\vec{u}, \vec{v})$ is bilinear, this property is equivalent to the continuity.

(B) *Symmetry*: The first seven terms of Eq. (A.5) are symmetric. Consider the last expression

$$\int_\Omega \nabla_k^2 G^{-1}\left(\nabla_{,k}^2 w\right)\tilde{w}\, \mathrm{d}\Omega = \int_\Omega G^{-1}\left(\nabla_{,k}^2 w\right)\nabla_{,k}^2 \tilde{w}\, \mathrm{d}\Omega = -\int_\Omega F\nabla_{,k}^2 \tilde{w}\, \mathrm{d}\Omega.$$

Eq. (A.2) yields

$$\int_\Omega F\nabla_{,k}^2 \tilde{w}\, \mathrm{d}\Omega = -\int_\Omega FG\left(\tilde{F}\right) \mathrm{d}\Omega = -\int_\Omega G(F)\tilde{F}\, \mathrm{d}\Omega,$$

and consequently one gets

$$\int_\Omega \nabla_k^2 G^{-1}\left(\nabla_{,k}^2 w\right)\tilde{w}\, \mathrm{d}\Omega = \int_\Omega G(F)\tilde{F}\, \mathrm{d}\Omega = \int_\Omega \nabla_{,k}^2 w G^{-1}\left(\nabla_{,k}^2 \tilde{w}\right) \mathrm{d}\Omega = \int_\Omega w\nabla_k^2 G^{-1}\left(\nabla_{,k}^2 \tilde{w}\right) \mathrm{d}\Omega.$$

This proves the symmetry of the last term of Eq. (A.5).

(C) *Positiveness*: Let $\vec{u} = \vec{v}$. Then $a_h(\vec{u}, \vec{u})$ presents a deformation energy

$$\begin{aligned} a_h(\vec{u},\vec{u}) = &\frac{2}{3}\int_\Omega \Bigg\{ h^3\left[A_{1111}\left(\frac{\partial\gamma_x}{\partial x}\right)^2 + 2A_{1122}\frac{\partial\gamma_x \partial\gamma_y}{\partial x\, \partial y} + A_{2222}\left(\frac{\partial\gamma_y}{\partial y}\right)^2\right] \\ &+ A_{1212}h^3\left(\frac{\partial\gamma_y}{\partial x} + \frac{\partial\gamma_x}{\partial y}\right)^2 + A_{1313}h\left(\gamma_x + \frac{\partial w}{\partial x}\right)^2 + A_{2323}h\left(\gamma_y + \frac{\partial w}{\partial y}\right)^2\Bigg\} \mathrm{d}\Omega \\ &+ \int_\Omega \nabla_k^2 G^{-1}\left(\nabla_{,k}^2 w\right) w\, \mathrm{d}\Omega. \end{aligned}$$

However, the last term is equal to $\int_\Omega G(F)F\, \mathrm{d}\Omega$ and is positive according to Eq. (8). The positiveness of the other terms results from Eqs. (6) and (8). □

Appendix B

Theorem B.1. *The eigenvalue problem* (12) *has a discrete series of non-negative eigenvalues* $\{\lambda_i\}_{i=1}^{\infty}$. *The eigenfunctions (modes) related to different eigenvalues are orthogonal.*

Proof. The discreteness of spectra of (12) results from the symmetry of the bilinear forms $a_h(\vec{u},\vec{v})$ and $b_h(\vec{u},\vec{v})$, and also from the existence of an inversed operator to $M[h]$. It is an analogous to the proof given on page 266 in Ref. [25].

Let us prove the positiveness of $\lambda_i, i = 1, 2, \ldots$. Let λ_i be a certain eigenvalue and let $\vec{u}_i$ be the corresponding eigenfunction. Then, taking $\vec{v} = \vec{u}_i$, we obtain

$$a_h(\vec{u}_i, \vec{u}_i) - \lambda_i b_h(\vec{u}_i, \vec{u}_i) = 0,$$

or

$$\lambda_i = \frac{a_h(\vec{u}_i, \vec{u}_i)}{b_h(\vec{u}_i, \vec{u}_i)}.$$

Because $a_h(\vec{u}_i, \vec{u}_i) \geqslant 0$ and $b_h(\vec{u}_i, \vec{u}_i) > 0$, $\lambda_i \geqslant 0$ for all $i = 1, 2, \ldots$. Let $\vec{u}_i$ and $\vec{u}_j$ be the eigenfunctions corresponding to λ_i and λ_j, and $\lambda_i \neq \lambda_j$. Then we get

$$a_h\left(\vec{u}_i, \vec{u}_j\right) - \lambda_i b_h\left(\vec{u}_i, \vec{u}_j\right) = 0, \tag{B.1}$$

$$a_h\left(\vec{u}_j, \vec{u}_i\right) - \lambda_j b_h\left(\vec{u}_j, \vec{u}_i\right) = 0. \tag{B.2}$$

Finding Eq. (B.1) from Eq. (B.2) and taking into account the symmetry of the bilinear forms we get

$$\left(\lambda_j - \lambda_i\right) b_h\left(\vec{u}_i, \vec{u}_j\right) = 0.$$

Because $\lambda_i \neq \lambda_j$,

$$b_h\left(\vec{u}_i, \vec{u}_j\right) = 0. \tag{B.3}$$

Observe that for the eigenvalue problem (12) the following Rayleigh's formula holds:

$$\lambda_i = \min_{\vec{u} \in V_0 \vec{u} \perp u_j} \frac{a_h(\vec{u}, \vec{u})}{b_h(\vec{u}, \vec{u})}, \quad i = 1,\ 2, \ldots;\ j = 1,\ 2, \ldots, i-1. \tag{B.4}$$

The sign $\perp$ denotes the generalized orthogonal operation in the sense of Eq. (B.3), i.e.,

$$\vec{u} \perp \vec{u}_j \Leftrightarrow b_h\left(\vec{u}, \vec{u}_j\right) = 0.$$

Eq. (B.4) yields the lowest eigenvalue λ_1 (or the lowest vibration frequency $\omega_1^2 = \lambda_1$) defined by

$$\omega_1^2 = \min_{\vec{u} \in V_0} \frac{a_h(\vec{u}, \vec{u})}{b_h(\vec{u}, \vec{u})}, \qquad \vec{u} \equiv 0. \tag{B.5}$$

The values of F corresponding to a certain vibration mode are defined by Eq. (A.2) and have the form

$$F_i = -G^{-1}\left(\nabla^2_{,k} w_i\right),$$

where w_i is the first component of the displacement vector $\vec{u}_i$ corresponding to $\lambda_i = \omega_i^2$. □

Appendix C

Theorem C.1. *Let $Z[h]$ and $M[h]$ be positively defined operators $\forall h \in U_\partial$ in the corresponding 'energy' space $H(\Omega)$ $(D(M) \subset D(Z) \subset H(\Omega))$. Then, the frequency series $\omega_{11}^{(p,N)}$ do not increase in relation to p for an arbitrarily taken number N. It is bounded from below. The exact value of the first shell free vibration frequency is equal to frequency ω_{11}^T (a case of higher frequencies is not considered here).*

Proof. Consider Eq. (22) for $k = m = 1$. Let us multiply it by $\vec{u}_{j1}^{2(p)}(y)$, then make a sum because of j and integrate it because of y in the interval $[0, l_2]$. We can arbitrarily take [24]:

$$\int_0^{l_1}\int_0^{l_2} M[h]\vec{u}_{11}^{(p,N)}\vec{u}_{11}^{(p,N)}\,\mathrm{d}x\,\mathrm{d}y = 1, \tag{C.1}$$

and we get

$$\left(\omega_{11}^{(p,N)}\right)^2 = \int_0^{l_1}\int_0^{l_2} Z[h]\vec{u}_{11}^{(p,N)}\vec{u}_{11}^{(p,N)}\,\mathrm{d}x\,\mathrm{d}y.$$

According to the minimalization of the fundamental frequency we have

$$\left(\omega_{11}^{T}\right)^2 \leqslant \left(\omega_{11}^{(p,N)}\right)^2 \leqslant \int_0^{l_1}\int_0^{l_2} Z[h]\vec{v}(x,y)\vec{v}(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

for an arbitrarily taken $\vec{v}(x,y)$ with shape (21). It holds only if for $\vec{v}(x,y)$ the norm condition (C.1) is satisfied. In particular, when we take $\vec{v}(x,y) = \vec{u}_{11}^{(p-1,N)}(x,y)$,

$$\left(\omega_{11}^{(p,N)}\right)^2 \leqslant \int_0^{l_1}\int_0^{l_2} Z[h]\vec{u}_{11}^{(p-1,N)}\vec{u}_{11}^{(p-1,N)}\,\mathrm{d}x\,\mathrm{d}y = \left(\omega_{11}^{(p-1,N)}\right)^2$$

or

$$\left(\omega_{11}^{T}\right)^2 \leqslant \left(\omega_{11}^{(p,N)}\right)^2 \leqslant \left(\omega_{11}^{(p-1,N)}\right)^2. \tag{C.2}$$

This proves the theorem. □

Corollary 1. *If each element of the base space system $H(\Omega)$ has the form $\vec{\Theta}_i(x,y) = \vec{\varphi}_i(x)\otimes\vec{\psi}_i(y)$, where $\vec{\varphi}_i \in H([0,l_1])$ and $\vec{\psi}_i \in H([0,l_2])$, and $\{\vec{\varphi}_i(x)\}$ or $\{\vec{\psi}_i(y)\}$ are taken as the initial functions, then for an arbitrary number of MVI steps the following inequality is satisfied:*

$$\left(\omega_{11}^{(p,N)}\right)^2 \leqslant \left(\omega_{11}^{(\sigma,N)}\right)^2, \tag{C.3}$$

where $\omega_{11}^{(\sigma,N)}$ is the fundamental frequency, obtained using the MB method during a projection on the basis system $\{\vec{\Theta}_i(x,y)\}_{i=1}^{N}$

Proof. According to Eq. (C.2) it is sufficient to prove Eq. (C.3) for $p = 1$. Let us take

$$\vec{u}_{11}^{(1,N)}(x,y) = \sum_{i=1}^{N}\vec{\varphi}_i(x)\otimes\vec{u}_{i1}^{2}(y).$$

Then, in order to define $\omega_{11}^{(1,N)}$ and $\vec{u}_{i1}^{2}(y)$ the following equations hold:

$$\sum_{i=1}^{N}\left\{\int_0^{l_1} Z[h]\vec{\varphi}_i(x)\otimes\vec{u}_{i1}^{2}(y)\cdot\vec{\varphi}_j(x)\,\mathrm{d}x - \left(\omega_{11}^{(1,N)}\right)^2\int_0^{l_1} M[h]\vec{\varphi}_i(x)\otimes\vec{u}_{i1}^{2}(y)\cdot\vec{\varphi}_j(x)\,\mathrm{d}x\right\} = 0,$$
$$(j = 1,\ 2, \ldots, N).$$

From the above we get

$$\left(\omega_{11}^{(1,N)}\right)^2 \leqslant \int_0^{l_1}\int_0^{l_2} Z[h]\sum_{i=1}^{N} c_i\vec{\varphi}_i(x)\otimes\vec{\psi}_i(y)\sum_{j=1}^{N} c_j\vec{\varphi}_j(x)\otimes\vec{\psi}_j(y)\,\mathrm{d}x\,\mathrm{d}y,$$

for the arbitrary series $\{c_i\}_{i=1}^N$ whose combination $\sum_{i=1}^N c_i\vec{\varphi}_i(x)\otimes\vec{\psi}_i(y)$ satisfies Eq. (C.1). Let $c_i = c_i^\sigma$ be the coefficients defined by the MB method. Then we get

$$\left(\omega_{11}^{(1,N)}\right)^2 \leqslant \left(\omega_{11}^{(\sigma,N)}\right)^2,$$

and the inequality (C.3) is proved. □

Corollary C.1. *If MB convergence is proved for problem* (13), *then for arbitrary p values MVI method also converges with* $N\to\infty$.

Proof. Linking inequalities (C.3) and (C.2) we get

$$\left(\omega_{11}^{T}\right)^2 \leqslant \left(\omega_{11}^{(p,N)}\right)^2 \leqslant \left(\omega_{11}^{(\sigma,N)}\right)^2 \qquad \forall p = 1,\ 2,\ldots. \tag{C.4}$$

If the convergence of the MB method is proved, then $\omega_{11}^{(\sigma,N)}\to\omega_{11}^T$ for $N\to\infty$ and, independently of the iteration number, one obtains

$$\omega_{11}^{(p,N)} \underset{N\to\infty}{\to} \omega_{11}^T. \qquad \square$$

Appendix D

In order to apply FEM, we divide the [0, 1] interval into N finite elements (see Fig. 1). On each element ('e' denotes the element number) the functions w_i, φ_i, ψ_i and F_i ($i = 1,2$) are presented in the forms

$$w_i^e(x_i) = \sum_{k=1}^{3} w_{ik}^e \xi_k^e(x_i),, \tag{D.1}$$

$$\varphi_i^e(x_i) = \sum_{k=1}^{3} \varphi_{ik}^e \xi_k^e(x_i), \tag{D.2}$$

$$\psi_i^e(x_i) = \sum_{k=1}^{3} \psi_{ik}^e \xi_k^e(x_i), \tag{D.3}$$

$$F_i^e(x_i) = \sum_{k=1}^{3} F_{ik}^e \xi_k^e(x_i), \tag{D.4}$$

where $x_i = x$ for $i = 1$ and $x_i = y$ for $i = 2$.

Above $\xi_k^e(x_i)$ denote modes defined by the following relations:

$$\xi_1^e(x_i) = \left(1-\frac{2x_i}{\Delta_e}\right)\left(1-\frac{x_i}{\Delta_e}\right),$$

$$\xi_2^e(x_i) = \frac{4x_i}{\Delta_e}\left(1-\frac{x_i}{\Delta_e}\right), \qquad \xi_3^e(x_i) = -\frac{x_i}{\Delta_e}\left(1-\frac{2x_i}{\Delta_e}\right),$$

where Δ_e is the interval length with e, the element number.

The first function corresponds to the interval beginning, the second one to the middle of the interval, whereas the third one corresponds to the interval end.

In expressions (D.1)–(D.4) by w_{ik}^e, φ_{ik}^e, ψ_{ik}^e, F_{ik}^e the functions w_i, φ_i, ψ_i and F_i being sought are denoted in k node on the eth interval. Substituting Eqs. (D.1)–(D.4) into Eqs. (41)–(48) and making a projection on $\xi_m^e(x_i)$, the following algebraic equations system with regard to w_{ik}^e, φ_{ik}^e, ψ_{ik}^e, F_{ik}^e for the finite element e are defined:

$$\begin{aligned}
&\tfrac{2}{3}\bar{\lambda}_2 w_{1k}^e \int_{\Delta e} A(h, w_2, w_2, 1, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}y + \tfrac{2}{3}\bar{\lambda}_1 w_{1k}^e \int_{\Delta e} A(h, w_2, w_2, 0, 0)\xi_{k,y}^e \xi_{m,y}^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{2}{3}\bar{\lambda}_1 \varphi_{1k}^e \int_{\Delta e} A(h, \varphi_2, w_2, 0, 0)\xi_k^e \xi_{m,y}^{e'}\,\mathrm{d}y + \tfrac{2}{3}\bar{\lambda}_2 \psi_{1k}^e \int_{\Delta e} A(h, \psi_2, w_2, 0, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad - F_{1k}^e \int_{\Delta e} A(k_y, F_2, w_2, 0, 0)\xi_{k,yy}^e \xi_m^{e'}\,\mathrm{d}y - F_{1k}^e \int_{\Delta e} A(k_x, F_2, w_2, 2, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad - \omega^2 w_{1k}^e \int_{\Delta e} A(h, w_2, w_2, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y = 0,
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
&\tfrac{2}{3}\bar{\lambda}_1 w_{1k}^e \int_{\Delta e} A(h, w_2, \varphi_2, 0, 0)\xi_{k,y}^e \xi_m^{e'}\,\mathrm{d}y + \tfrac{1}{12}A_{1212}\varphi_{1k}^e \int_{\Delta e} A(h^3, \varphi_2, \varphi_2, 1, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{2}{3}\bar{\lambda}_1 \varphi_{1k}^e \int_{\Delta e} A(h, \varphi_2, \varphi_2, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{1}{12}\lambda^{-2} A_{1111}\varphi_{1k}^e \int_{\Delta e} A(h^3, \varphi_2, \varphi_2, 0, 0)\xi_{k,y}^e \xi_{m,y}^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{1}{12}A_{1122}\psi_{1k}^e \int_{\Delta e} A(h^3, \psi_2, \varphi_2, 1, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{1}{12}A_{1212}\psi_{1k}^e \int_{\Delta e} A(h^3, \psi_2, \varphi_2, 0, 1)\xi_{k,y}^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad - \tfrac{1}{12}\lambda_1^{-2}\omega^2 \varphi_{1k}^e \int_{\Delta e} A(h^3, \varphi_2, \varphi_2, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y = 0,
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
&\tfrac{2}{3}\bar{\lambda}_2 w_{1k}^e \int_{\Delta e} A(h, w_2, \psi_2, 1, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y + \tfrac{1}{12}A_{1122}\varphi_{1k}^e \int_{\Delta e} A(h^3, \varphi_2, \psi_2, 0, 1)\xi_{k,y}^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{1}{12}A_{1212}\varphi_{1k}^e \int_{\Delta e} A(h^3, \varphi_2, \psi_2, 1, 0)\xi_k^e \xi_{m,y}^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{1}{12}\lambda^2 A_{2222}\psi_{1k}^e \int_{\Delta e} A(h^3, \psi_2, \psi_2, 1, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
&\quad + \tfrac{2}{3}\bar{\lambda}_2 \psi_{1k}^e \int_{\Delta e} A(h, \psi_2, \psi_2, 0, 0)\xi_k^e \xi_m^{e'}\,dy \\
&\quad + \tfrac{1}{12}A_{1212}\psi_{1k}^e \int_{\Delta e} A(h^3, \psi_2, \psi_2, 0, 0)\xi_{k,y}^e \xi_{m,y}^{e'}\,\mathrm{d}y \\
&\quad - \tfrac{1}{12}\lambda_2^{-2}\omega^2 \psi_{1k}^e \int_{\Delta e} A(h^3, \psi_2, \psi_2, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}y = 0,
\end{aligned} \tag{D.7}$$

$$
\begin{aligned}
& w_{1k}^{e} \int_{\Delta e} A(k_x, w_2, F_2, 0, 2)\xi_k^e \xi_m^{e'}\,\mathrm{d}y + w_{1k}^{e} \int_{\Delta e} A(k_y, w_2, F_2, 0, 0)\xi_k^e \xi_{m,yy}^{e'}\,\mathrm{d}y \\
& \quad + \lambda^{-4} a_{1111} F_{1k}^{e} \int_{\Delta e} A(h^{-1}, F_2, F_2, 0, 0)\xi_{k,yy}^e \xi_{m,yy}^{e'}\,\mathrm{d}y \\
& \quad + a_{1122} F_{1k}^{e} \int_{\Delta e} A(h^{-1}, F_2, F_2, 2, 0)\xi_k^e \xi_{m,yy}^{e'}\,\mathrm{d}y \\
& \quad + a_{1122} F_{1k}^{e} \int_{\Delta e} A(h^{-1}, F_2, F_2, 0, 2)\xi_{k,yy}^e \xi_m^{e'}\,\mathrm{d}y \\
& \quad + \lambda^{4} a_{2222} F_{1k}^{e} \int_{\Delta e} A(h^{-1}, F_2, F_2, 2, 2)\xi_k^e \xi_m^{e'}\,\mathrm{d}y \\
& \quad - a_{1212} F_{1k}^{e} \int_{\Delta e} A(h^{-1}, F_2, F_2, 1, 1)\xi_{k,y}^e \xi_{m,y}^{e'}\,\mathrm{d}y = 0,
\end{aligned} \tag{D.8}
$$

$$
\begin{aligned}
& \tfrac{2}{3}\bar{\lambda}_1 w_{2k}^{e} \int_{\Delta e} A(h, w_1, w_1, 1, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}x + \tfrac{2}{3}\bar{\lambda}_2 w_{2k}^{e} \int_{\Delta e} A(h, w_1, w_1, 0, 0)\xi_{k,x}^e \xi_{m,x}^{e'}\,\mathrm{d}x \\
& \quad + \tfrac{2}{3}\bar{\lambda}_1 \varphi_{2k}^{e} \int_{\Delta e} A(h, \varphi_1, w_1, 0, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}x + \tfrac{2}{3}\bar{\lambda}_2 \psi_{2k}^{e} \int_{\Delta e} A(h, \psi_1, w_1, 0, 0)\xi_k^e \xi_{m,x}^{e'}\,\mathrm{d}x \\
& \quad - F_{2k}^{e} \int_{\Delta e} A(k_x, F_1, w_1, 0, 0)\xi_{k,xx}^e \xi_m^{e'}\,\mathrm{d}x - F_{2k}^{e} \int_{\Delta e} A(k_y, F_1, w_1, 2, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}x \\
& \quad - \omega^2 w_{2k}^{e} \int_{\Delta e} A(h, w_1, w_1, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}x = 0,
\end{aligned} \tag{D.9}
$$

$$
\begin{aligned}
& \tfrac{2}{3}\bar{\lambda}_1 w_{2k}^{e} \int_{\Delta e} A(h, w_1, \varphi_1, 1, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}x + \tfrac{1}{12}\lambda^{-2} A_{1111} \varphi_{2k}^{e} \int_{\Delta e} A(h^3, \varphi_1, \varphi_1, 1, 1)\xi_k^e \xi_m^{e'}\,\mathrm{d}x \\
& \quad + \tfrac{1}{12} A_{1212} \varphi_{2k}^{e} \int_{\Delta e} A(h^3, \varphi_1, \varphi_1, 0, 0)\xi_{k,x}^e \xi_{m,x}^{e'}\,\mathrm{d}x + \tfrac{2}{3}\bar{\lambda}_1 \varphi_{2k}^{e} \int_{\Delta e} A(h, \varphi_1, \varphi_1, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}x \\
& \quad + \tfrac{1}{12} A_{1212} \psi_{2k}^{e} \int_{\Delta e} A(h^3, \psi_1, \varphi_1, 1, 0)\xi_k^e \xi_{m,x}^{e'}\,\mathrm{d}x \\
& \quad + \tfrac{1}{12} A_{1122} \psi_{2k}^{e} \int_{\Delta e} A(h^3, \psi_1, \varphi_1, 0, 1)\xi_{k,x}^e \xi_m^{e'}\,\mathrm{d}x \\
& \quad - \tfrac{1}{12}\lambda_1^{-2} \omega^2 \varphi_{2k}^{e} \int_{\Delta e} A(h^3, \varphi_1, \varphi_1, 0, 0)\xi_k^e \xi_m^{e'}\,\mathrm{d}x = 0,
\end{aligned} \tag{D.10}
$$

$$
\begin{aligned}
& \tfrac{2}{3}\bar{\lambda}_2 w_{2k}^{e} \int_{\Delta e} A(h, w_1, \psi_1, 0, 0)\xi_{k,x}^e \xi_m^{e'}\,\mathrm{d}x + \tfrac{1}{12} A_{1122} \varphi_{2k}^{e} \int_{\Delta e} A(h^3, \varphi_1, \psi_1, 1, 0)\xi_k^e \xi_{m,x}^{e'}\,\mathrm{d}x \\
& \quad + \tfrac{1}{12} A_{1212} \varphi_{2k}^{e} \int_{\Delta e} A(h^3, \varphi_1, \psi_1, 0, 1)\xi_{k,x}^e \xi_m^{e'}\,\mathrm{d}x \\
& \quad + \tfrac{1}{12}\lambda^2 A_{2222} \psi_{2k}^{e} \int_{\Delta e} A(h^3, \psi_1, \psi_1, 0, 0)\xi_{k,x}^e \xi_m^{e'}\,\mathrm{d}x
\end{aligned}
$$

$$+\tfrac{1}{12}A_{1212}\psi_{2k}^{e}\int_{\Delta e}A(h^{3},\psi_{1},\psi_{1},1,1)\xi_{k}^{e}\xi_{m}^{e'}\,\mathrm{d}x+\tfrac{2}{3}\bar{\lambda}_{2}\psi_{2k}^{e}\int_{\Delta e}A(h,\psi_{1},\psi_{1},0,0)\xi_{k}^{e}\xi_{m}^{e'}\,\mathrm{d}x$$
$$-\tfrac{1}{12}\lambda_{2}^{-2}\omega^{2}\psi_{2k}^{e}\int_{\Delta e}A(h^{3},\psi_{1},\psi_{1},0,0)\xi_{k}^{e}\xi_{m}^{e'}\,\mathrm{d}x=0, \tag{D.11}$$

$$w_{2k}^{e}\int_{\Delta e}A(k_{x},w_{1},F_{1},0,0)\xi_{k}^{e}\xi_{m,xx}^{e'}\,\mathrm{d}x+w_{2k}^{e}\int_{\Delta e}A(k_{y},w_{1},F_{1},0,2)\xi_{k}^{e}\xi_{m}^{e'}\,\mathrm{d}x$$
$$+\lambda^{-4}a_{1111}F_{2k}^{e}\int_{\Delta e}A(h^{-1},F_{1},F_{1},2,2)\xi_{k}^{e}\xi_{m}^{e'}\,\mathrm{d}x+a_{1122}F_{2k}^{e}\int_{\Delta e}A(h^{-1},F_{1},F_{1},0,2)\xi_{k,xx}^{e}\xi_{m}^{e'}\,\mathrm{d}x$$
$$+a_{1122}F_{2k}^{e}\int_{\Delta e}A(h^{-1},F_{1},F_{1},0,0)\xi_{k,xx}^{e}\xi_{m,xx}^{e'}\,\mathrm{d}x$$
$$-a_{1212}F_{2k}^{e}\int_{\Delta e}A(h^{-1},F_{1},F_{1},1,1)\xi_{k,x}^{e}\xi_{m,x}^{e'}\,\mathrm{d}x$$
$$+\lambda^{4}a_{2222}F_{2k}^{e}\int_{\Delta e}A(h^{-1},F_{1},F_{1},0,0)\xi_{k,xx}^{e}\xi_{m,xx}^{e'}\,\mathrm{d}x=0. \tag{D.12}$$

During considerations of Eqs. (D.5)–(D.12) a summation with repeated symbols has been used. In addition, the $k,m=1,2,3\ldots$, $e=e_1,e_2,\ldots,e_N$, are the numbers of those finite elements, which touch the m node of the e finite element. Let us introduce a vector of the unknown local variables on the e finite element

$$\vec{u}_i^e=\{w_{i1}^e,w_{i2}^e,w_{i3}^e,\varphi_{i1}^e,\varphi_{i2}^e,\varphi_{i3}^e,\psi_{i1}^e,\psi_{i2}^e,\psi_{i3}^e,F_{i1}^e,F_{i2}^e,F_{i3}^e\}.$$

Then, Eqs. (D.5)–(D.12) can be presented in the form

$$C_i^e\vec{u}_i^e-\omega^2 D_i^e\vec{u}_i^e=0\quad(i=1,\ 2). \tag{D.13}$$

For $i=1$ Eq. (D.13) 'models' Eqs. (D.5)–(D.8), whereas for $i=2$ it models Eqs. (D.9)–(D.12). The matrices C_i^e and D_i^e may be presented as the blocks

$$C_i^e=\begin{pmatrix} Cww_i^e & Cw\varphi_i^e & Cw\psi_i^e & CwF_i^e\\ C\varphi w_i^e & C\varphi\varphi_i^e & C\varphi\psi_i^e & 0\\ C\psi w_i^e & C\psi\varphi_i^e & C\psi\psi_i^e & 0\\ CFw_i^e & 0 & 0 & CFF_i^e\end{pmatrix}, \tag{D.14}$$

$$D_i^e=\begin{pmatrix} Dww_i^e & 0 & 0 & 0\\ 0 & D\varphi\varphi_i^e & 0 & 0\\ 0 & 0 & D\psi\psi_i^e & 0\\ 0 & 0 & 0 & DFF_i^e\end{pmatrix}. \tag{D.15}$$

The sub-matrices of C_i^e and D_i^e are defined below ($j=1$ if $i=2$, and $j=2$ if $i=1$):

$$Cww^e_{ikm} = ww1i\int_{\Delta e} A\big(h, w_j, w_j, 1, 1\big)\xi^e_{k,x_j}\xi^{e'}_m\,\mathrm{d}x_j + ww2i\int_{\Delta e} A\big(h, w_j, w_j, 0, 0\big)\xi^e_k\xi^{e'}_{m,x_j}\,\mathrm{d}x_j,$$

$$Cw\varphi^e_{ikm} = w\varphi 1i\int_{\Delta e} A(h, \varphi_j, w_j, 0, 0)\xi^e_k\xi^{e'}_{m,x_j}\,\mathrm{d}x_j + w\varphi 2i\int_{\Delta e} A(h, \varphi_j, w_j, 0, 1)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j,$$

$$Cw\psi^e_{ikm} = w\psi 1i\int_{\Delta e} A(h, \psi_j, w_j, 0, 1)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j + w\psi 2i\int_{\Delta e} A(h, \psi_j, w_j, 0, 0)\xi^e_k\xi^{e'}_{m,x_j}\,\mathrm{d}x_j,$$

$$CwF^e_{ikm} = \int_{\Delta e} A\big(k_i, F_j, w_j, 2, 0\big)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j + \int_{\Delta e} A\big(k_j, F_j, w_j, 0, 0\big)\xi^e_{k,x_j}\xi^{e'}_m\,\mathrm{d}x_j,$$

$$\begin{aligned} C\varphi w^e_{ikm} &= \varphi w1i\int_{\Delta e} A(h, w_j, \varphi_j, 0, 0)\xi^e_{k,x_j}\xi^{e'}_m\,\mathrm{d}x_j \\ &\quad + \varphi w2i\int_{\Delta e} A(h, w_j, \varphi_j, 1, 0)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j, \\ C\varphi\varphi^e_{ikm} &= \varphi\varphi 1i\int_{\Delta e} A(h, \varphi_j, \varphi_j, 0, 0)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j \\ &\quad + \varphi\varphi 2i\int_{\Delta e} A(h^3, \varphi_j, \varphi_j, 1, 1)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j \\ &\quad + \varphi\varphi 3i\int_{\Delta e} A(h^3, \varphi_j, \varphi_j, 0, 0)\xi^e_{k,x_j}\xi^{e'}_{m,x_j}\,\mathrm{d}x_j, \\ C\varphi\psi^e_{ikm} &= \varphi\psi 1i\int_{\Delta e} A(h^3, \psi_j, \varphi_j, 1, 0)\xi^e_k\xi^{e'}_{m,x_j}\,\mathrm{d}x_j \\ &\quad + \varphi\psi 2i\int_{\Delta e} A(h^3, \psi_j, \varphi_j, 0, 1)\xi^e_{k,x_j}\xi^{e'}_m\,\mathrm{d}x_j, \end{aligned} \tag{D.16}$$

$$C\psi w^e_{ikm} = \psi w1i\int_{\Delta e} A\big(h, w_j, w_j, 1, 0\big)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j + \psi w2i\int_{\Delta e} A\big(h, w_j, w_j, 0, 0\big)\xi^e_{k,x_j}\xi^{e'}_m\,\mathrm{d}x_j,$$

$$C\psi\varphi^e_{ikm} = \psi\varphi 1i\int_{\Delta e} A(h^3, \varphi_j, \psi_j, 0, 1)\xi^e_{k,x_j}\xi^{e'}_m\,\mathrm{d}x_j + \psi\varphi 2i\int_{\Delta e} A(h^3, \varphi_j, \psi_j, 1, 0)\xi^e_k\xi^{e'}_{m,x_j}\,\mathrm{d}x_j,$$

$$\begin{aligned} C\psi\psi^e_{ikm} &= \psi\psi 1i\int_{\Delta e} A(h, \psi_j, \psi_j, 0, 0)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j \\ &\quad + \psi\psi 2i\int_{\Delta e} A(h^3, \psi_j, \psi_j, 1, 1)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j + \psi\psi 3i\int_{\Delta e} A(h^3, \psi_j, \psi_j, 0, 0)\xi^e_{k,x_j}\xi^{e'}_{m,x_j}\,\mathrm{d}x_j, \end{aligned}$$

$$CFw^e_{ikm} = \int_{\Delta e} A\big(k_i, w_j, F_j, 0, 2\big)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j + \int_{\Delta e} A\big(k_j, w_j, F_j, 0, 0\big)\xi^e_k\xi^{e'}_{m,x_jx_j}\,\mathrm{d}x_j,$$

$$\begin{aligned} CFF^e_{ikm} &= FF1i\int_{\Delta e} A\big(h^{-1}, F_j, F_j, 2, 2\big)\xi^e_k\xi^{e'}_m\,\mathrm{d}x_j \\ &\quad + FF2i\int_{\Delta e} A\big(h^{-1}, F_j, F_j, 0, 2\big)\xi^e_{k,x_jx_j}\xi^{e'}_m\,\mathrm{d}x_j + FF3i\int_{\Delta e} A\big(h^{-1}, F_j, F_j, 2, 0\big)\xi^e_k\xi^{e'}_{m,x_jx_j}\,\mathrm{d}x_j \\ &\quad + FF4i\int_{\Delta e} A\big(h^{-1}, F_j, F_j, 1, 1\big)\xi^e_{k,x_j}\xi^{e'}_{m,x_j}\,\mathrm{d}x_j + FF5i\int_{\Delta e} A\big(h^{-1}, F_j, F_j, 0, 0\big)\xi^e_{k,x_jx_j}\xi^{e'}_{m,x_jx_j}\,\mathrm{d}x_j, \end{aligned}$$

$$Dww^e_{ikm} = \int_{\Delta e} A\big(h, w_j, w_j, 0, 0\big)\xi^e_k \xi^{e'}_m \, \mathrm{d}x_j,$$

$$D\varphi\varphi^e_{ikm} = \frac{1}{12}\lambda_1^{-2} \int_{\Delta e} A(h^3, \varphi_j, \varphi_j, 0, 0)\xi^e_k \xi^{e'}_m \, \mathrm{d}x_j,$$

$$D\psi\psi^e_{ikm} = \frac{1}{12}\lambda_2^{-2} \int_{\Delta e} A(h^3, \psi_j, \psi_j, 0, 0)\xi^e_k \xi^{e'}_m \, \mathrm{d}x_j.$$

The coefficients used ($ww1i$, $ww2i$ and others) are defined by

$$ww1i = \frac{2}{3}\begin{cases} \bar{\lambda}_2, & i = 1, \\ \bar{\lambda}_1, & i = 2, \end{cases} \qquad ww2i = \frac{2}{3}\begin{cases} \bar{\lambda}_1, & i = 1, \\ \bar{\lambda}_2, & i = 2, \end{cases}$$

$$\varphi w1i = w\varphi 1i, \qquad \varphi w2i = w\varphi 2i,$$

$$w\varphi 1i = \frac{2}{3}\begin{cases} \bar{\lambda}_1, & i = 1, \\ 0, & i = 2, \end{cases} \qquad w\varphi 2i = \frac{2}{3}\begin{cases} 0, & i = 1, \\ \bar{\lambda}_2, & i = 2, \end{cases}$$

$$\varphi\varphi 1i = \frac{2}{3}\begin{cases} \bar{\lambda}_1, & i = 1, \\ \bar{\lambda}_2, & i = 2, \end{cases} \qquad \varphi\varphi 2i = \frac{1}{12}\begin{cases} A_{1212}, & i = 1, \\ \lambda^{-2}A_{1111}, & i = 2, \end{cases}$$

$$\varphi\varphi 3i = \frac{1}{12}\begin{cases} \lambda^{-2}A_{1111}, & i = 1, \\ A_{1212}, & i = 2, \end{cases} \qquad \varphi\psi 1i = \frac{1}{12}\begin{cases} A_{1122}, & i = 1, \\ A_{1212}, & i = 2, \end{cases}$$

$$\varphi\psi 2i = \frac{1}{12}\begin{cases} A_{1212}, & i = 1, \\ A_{1122}, & i = 2, \end{cases} \qquad \psi w1i = \frac{2}{3}\begin{cases} \bar{\lambda}_2, & i = 1, \\ 0, & i = 2, \end{cases}$$

$$\psi w2i = \frac{2}{3}\begin{cases} 0, & i = 1, \\ \bar{\lambda}_2, & i = 2, \end{cases} \qquad \psi\varphi 1i = \frac{1}{12}\begin{cases} A_{1122}, & i = 1, \\ A_{1212}, & i = 2, \end{cases}$$

$$\psi\varphi 2i = \frac{1}{12}\begin{cases} A_{1212}, & i = 1, \\ A_{1122}, & i = 2, \end{cases} \qquad \psi\psi 1i = \frac{2}{3}\begin{cases} \bar{\lambda}_2, & i = 1, \\ \bar{\lambda}_1, & i = 2, \end{cases}$$

$$\psi\psi 2i = \frac{1}{12}\begin{cases} A_{1212}, & i = 1, \\ \lambda^2 A_{2222}, & i = 2, \end{cases} \qquad \psi\psi 3i = \frac{1}{12}\begin{cases} \lambda^2 A_{2222}, & i = 1, \\ A_{1212}, & i = 2, \end{cases}$$

$$FF1i = \begin{cases} \lambda^4 a_{2222}, & i = 1, \\ \lambda^{-4} a_{1111}, & i = 2, \end{cases} \qquad FF2i = \begin{cases} a_{1122}, & i = 1, \\ a_{1122}, & i = 2, \end{cases}$$

$$FF3i = \begin{cases} a_{1122}, & i = 1, \\ a_{1122}, & i = 2, \end{cases} \qquad FF4i = \begin{cases} -a_{1212}, & i = 1, \\ -a_{1212}, & i = 2, \end{cases}$$

$$FF5i = \begin{cases} \lambda^{-4} a_{1111}, & i = 1, \\ \lambda^4 a_{2222}, & i = 2. \end{cases}$$

Thus, in order to get C_i^e and D_i^e matrices for different $i = 1, 2$ it is sufficient to change the coefficients according to blocks (D.16).

Appendix E. Nomenclature

x, y, z	Descartes co-ordinates
α, β, γ	curvilinear orthogonal co-ordinates
t	time
Ω	set of points in R^2 (of shallow shell)
$\partial\Omega$	boundary of Ω
A_H, A_B, a_H, a_B, h_H, h_B	below (H) and upper (B) bounds
C_0, C_1	constants
$G(\cdot)$, $G^{-1}(\cdot)$	differential and integral operators, respectively
$A(\alpha,\beta)$, $B(\alpha,\beta)$	coefficient of first square forms related to a surface
$K_1(\alpha,\beta)$, $K_2(\alpha,\beta)$	main curvatures of a surface
$V = \{V \in L^2(\Omega) \mid V \in P\}$	set of functions from the space $L^2(\Omega)$, possessing P property
$\in$	belonging
$\to$	tends to
$\Leftrightarrow$	equivalently
$L[h]$, $M[h]$, $A[h]$, $B[h]$	differential operators depending on h
$\tilde{V}$	quantity conjugate to V
$\forall V \in A$	for any V belonging to A
UV	vector with components (U, V_1, U_2, V_2, $\ldots$,$U_n V_n$) in R^N
$\{V_i\}_{i=1}^N$	set of functions for $i=1, 2,\ldots,N$
det	matrix determinant
$V\|_s$	function value on a boundary
$[\cdot],_x$ or $(\cdot),_x$	derivative with respect to x
$0(h^\alpha)$	rest of order h^α
$V' = \dfrac{dV}{dt}$	
F	stress function
W	normal displacement of the middle of the surface in Z direction
γ_x, γ_y	angles of rotation of a normal to the middle of the surface in the planes xz and yz, respectively
$h(x,y)$ or $2h(x,y)$	shell thickness in the point (x,y)
a, b or $2a$, $2b$	shell (plate) dimensions
$K=1/R_x$, $K=1/R_y$	curvatures
ρ	shell material weight density
E_1, E_2	Young's modulus in directions x and y ($E_1=E_2=E$ for an isotropic material)
$m = \rho h$	mass of a unit surface
g	acceleration due to gravity
v_{12}, v_{21}	The Poisson coefficients of an orthotropic material (for isotropic material $v_{12} = v_{21} = v$)

G_{12}, G_{13}, G_{23} — shear moduli in planes xy, xz, yz

$$A_{1111} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad a_{1111} = \frac{1}{E_1} \ (1 \leftrightarrows 2), \quad A_{1122} = \frac{E_1 \nu_{21}}{1 - \nu_{12}\nu_{21}},$$

$$a_{1122} = -\frac{\nu_{21}}{E_2}, \quad A_{1212} = G_{12}, \quad a_{1212} = G_{12}^{-1} \ (1, \overset{\leftrightarrows}{2}, 3),$$

ε_{ij}	tensor deformation components for a middle of the shell surface
T_{11}, T_{12}, T_{22}	stresses in the middle of the surface
Q_1, Q_2	lateral forces
M_{11}, M_{12}, M_{22}	moments
q	lateral loads
$\omega_n \ (n = 1, 2\ldots)$	frequencies of vibrations measured in H_z
$D = \dfrac{Eh^3}{12(1 - \nu^2)}$	bending stiffness of isotropic plate
R^n	Euclidean space of n dimension
$C^m(\Omega)$	space of functions m times continuously differentiable into subset Ω in R^n
$L^p(\Omega)$	space of functions integrable with power p
$(U, V) = \int_\Omega UV \, d\Omega$	scalar product in $L^2\Omega$
$a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$	bilinear forms depending on h
$(\vec{U}, \vec{V}) = \int \vec{U}\vec{V} \, d\Omega$	scalar product in $\left(L^2(\Omega)\right)^n$
$H_0^m(\Omega)$	closure $C^\infty(\Omega)$ with a compact carrier in $H^m(\Omega)$
$\|\|V\|\|_{v_0}$	norm in space V_0

$$\|V\|_{m,\Omega} = \left(\sum_{|\alpha| \le m} \int_\Omega \left| \frac{\partial^\alpha V}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|^2 d\Omega \right)^{1/2}, \quad \alpha = \alpha_1 + \alpha_2$$

$$\|\vec{V}\|_{m,\Omega} = \left(\sum_{i=1} \|V_i\|_{m,\Omega}^2 \right)^{1/2} \text{ for } \vec{V} = (V_1, \ V_2, \ldots V_n) \in (H^m(\Omega))^n$$

$\|a\|$	modulus of number a
$L^2(O, T; H^m(\Omega))$	space of functions V such that $\int_O^T \|\|V\|\|_{m,\Omega}^2 \, dt$ is bounded
$\nabla_k^2 V = K_y \dfrac{\partial^2 V}{\partial x^2} + K_x \dfrac{\partial^2 V}{\partial y^2}$	
$U_{\partial i}$	set of admissible control.

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