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On the propagation of acceleration waves in incompressible hyperelastic solids

T. Gültop

Department of Civil Engineering, Gazi University, Maltepe, Ankara 06570, Turkey

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Abstract

The conditions for the propagation of acceleration waves (sound waves) in incompressible elastic media undergoing finite deformation are investigated. The incompressible hyperelastic solid media is considered in accordance with the general constitutive theory of materials subject to internal mechanical constraints. The equation of motion of acceleration waves is obtained using the theory of singular surfaces. A general comparison is made between the magnitudes of the propagation speeds of waves in incompressible and unconstrained solid media by the use of Mandel's inequalities. The magnitudes of the speeds of propagation of acceleration waves in the incompressible hyperelastic material classes of neo-Hookean, Mooney–Rivlin, and St. Venant–Kirchhoff solids are determined. Comparisons are made of the specific results concerning the magnitudes of wave propagation speeds making use of the corresponding material parameters.

1. Introduction

The mathematical modelling of acceleration waves which are commonly known as sound waves is often made by the propagating singular surfaces of second order [1–3]. The second time derivative $\ddot{\mathbf{x}}$ of the displacement vector $\mathbf{x}$, that is, acceleration, and higher order time derivatives suffer discontinuities in the form of finite jumps across an acceleration wave front. In order to obtain explicit results concerning wave motion, the theory of singular surfaces has to be considered in conjunction with the constitutive theory of materials in which the propagation of waves is examined. In the present study, the constitutive equations of neo–Hookean, Mooney–Rivlin and St. Venant–Kirchhoff materials are considered as specific examples of the general class of incompressible hyperelastic materials.

E-mail address: tekin@mmf.gazi.edu.tr (T. Gültop).

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Various aspects of acceleration wave motion in elastic media were investigated, for example, by Şuhubi [4], Ogden [5], Chen [6], Truesdell and Toupin [3], and Scott [7]. Acceleration waves in elastic bodies subject to purely kinematical constraints were considered by Scott [8], and acceleration waves in thermoelastic materials subject to arbitrary thermomechanical constraints were examined by Reddy [9]. Ericksen investigated the propagation of waves in isotropic incompressible elastic materials [10]. Similar research on this subject was carried out by Truesdell and Noll [11] and Scott and Hayes [12].

In this study, the physical variables of temperature and entropy are neglected; hence, attention is confined to the purely mechanical theory of incompressible hyperelastic solids. The basic results of the differential geometry of singular surfaces concerning acceleration waves is reviewed and the constitutive equations of incompressible hyperelastic solids are presented. The magnitudes of the speeds of propagation of acceleration waves in incompressible and unconstrained materials are compared on a general basis by using the method of Mandel [13, 14]. A similar comparison of wave speeds in elastic and elastic–plastic solids by the use of Mandel's type of inequalities was made by Reddy and Gültop [15]. The magnitudes of the speeds of propagation of acceleration waves were obtained in specific classes of incompressible hyperelastic solids by using the strain energy functions of neo-Hookean, Mooney–Rivlin and St. Venant–Kirchhoff materials. The results are then compared with each other making use of the specific material parameters.

Both indicial notation and vector–tensor notation will be adopted whichever is appropriate to use. Indices of upper case letters will be used to denote the variables in the reference configuration and indices of lower case letters will be used to denote the variables in the current configuration throughout the text. Summation convention will be in use throughout the present study. For example, in an equation such as $a_{\mathrm{j}} = A_{jk} b_k$ summation is implied over index k, but not over j.

2. Singular surfaces and acceleration waves

In this section the basic concepts of the theory of singular surfaces within the context of acceleration waves is reviewed. The material reviewed here is covered more extensively by, for example, Eringen and Şuhubi [1], McCarthy [2], and Wang and Truesdell [16].

A moving surface in $\boldsymbol{R}^3$ is described as a one-parameter family of surfaces and will be denoted by $S(t)$. In the reference co-ordinates the position of the surface at time t is described by the equation

$$\Sigma(\mathbf{X}, t) = 0. \tag{1}$$

The function $\Sigma(\mathbf{X},t)$ is assumed to be at least continuously differentiable, but of arbitrary shape. Here $\mathbf{X}$ denotes the position vector of a point on the surface in the reference co-ordinates.

The velocity $\mathbf{u}$ of a material point on the surface, and the normal velocity U of the surface, or its speed of displacement, are defined by

$$\mathbf{u} = \frac{\partial \mathbf{x}}{\partial t}, \quad U = \mathbf{u} \bullet \mathbf{N}, \tag{2}$$

where $\mathbf{N}$ is the unit normal vector to the surface in the reference co-ordinates. While the position of the surface $S(t)$ is being changed in $\boldsymbol{R}^3$ due to the motion, the unit normal vector $\mathbf{n}$ to the surface in the current co-ordinates will change. The relation between the unit normal $\mathbf{N}$ in the reference co-ordinates to the surface $\Sigma(\mathbf{X},t)$ and the unit normal $\mathbf{n}$ in the current co-ordinates to the surface $S(t\ )$ may be stated by

$$\mathbf{N} = \frac{\mathbf{F}^{\mathrm{T}}\mathbf{n}}{\left|\mathbf{F}^{\mathrm{T}}\mathbf{n}\right|} \quad \Leftrightarrow \quad \mathbf{n} = \frac{\mathbf{F}^{-\mathrm{T}}\mathbf{N}}{\left|\mathbf{F}^{-\mathrm{T}}\mathbf{N}\right|}, \tag{3}$$

where $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient tensor.

Singular surfaces Σ in the reference configuration, and S in the current configuration with their corresponding unit normals $\mathbf{N}$ and $\mathbf{n}$ are illustrated in Fig. 1. The vectors $\mathbf{X}$ and $\mathbf{x}$ in this figure represent, respectively, the positions of a point in the reference and in the current configurations of a geometrically non-linear continuum.

A propagating smooth surface divides the body Ω into two regions, forming a common boundary between them. The unit normal $\mathbf{N}$ to the surface $S(t)$ is considered to be in the direction in which $S(t)$ propagates. The region ahead of the surface is denoted by Ω^+ and the region behind the surface is denoted by Ω^-. Let $f(\mathbf{X},t)$ be an arbitrary scalar-, vector- or tensor-valued function which is continuous in both Ω^+ and Ω^-. This function has definite limits f^+ and f^- at a point on $S(t)$, as the point is approached from Ω^+ and Ω^-. The jump of f at $\mathbf{X} \in S(t)$ is defined by

$$[f(\mathrm{X})] \equiv f^+ - f^-. \tag{4}$$

The surface $S\ (t)$ is called a singular surface with respect to f at time t if $[f] \neq 0$. A singular surface that has a non-zero normal velocity U is called a wave.

Attention will be focused henceforth on an acceleration wave, which is defined as a propagating singular surface across which the motion $\mathbf{x}(\mathbf{X},t)$, velocity $\dot{\mathbf{x}}(\mathrm{X},t)$ and, hence the deformation gradient $\mathbf{F}(\mathbf{X},t)$ are continuous, but the quantities involving the second derivatives of the motion such as acceleration $\ddot{\mathbf{x}}$ and the time rate of the deformation gradient $\dot{\mathbf{F}}$ are discontinuous. An acceleration wave is thus a second order singular surface. It is assumed that all functions which suffer discontinuities across a singular surface are continuous elsewhere in the material body. The jumps in acceleration and the deformation gradient rate will be denoted by

$$[\ddot{\mathbf{x}}] = \mathbf{s}, \quad [\dot{\mathbf{F}}] = -U^{-1}\mathbf{s} \otimes \mathbf{N}. \tag{5}$$

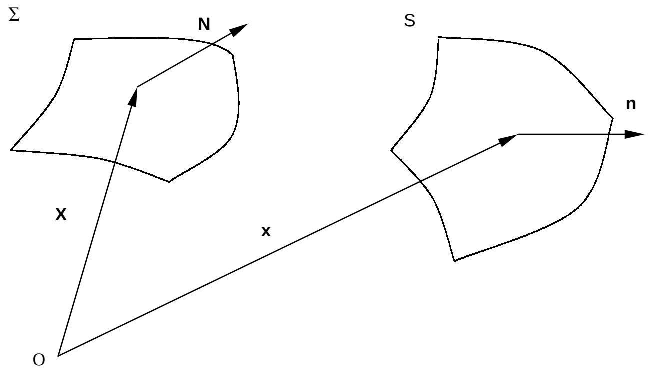

Fig. 1. Position vectors $\mathbf{X}$ and $\mathbf{x}$, singular surfaces Σ and S, unit normals $\mathbf{N}$ and $\mathbf{n}$, respectively, in reference and current configurations.

Here, the components of the dyadic product of vectors $\mathbf{s}$ and $\mathbf{N}$ may be defined as $\mathbf{s}\otimes\mathbf{N} = s_j N_K$. Hence, $(\mathbf{u}\otimes\mathbf{v})\mathbf{a} = \mathbf{u}(\mathbf{v}\cdot\mathbf{a})$ holds for any three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{a}$. Differentiating the identity $\mathbf{F}^{-1}\mathbf{F}\equiv\mathbf{I}$ with respect to time t and using relations (3) and (5) the following results are obtained:

$$\left[\frac{\partial \mathbf{F}^{-1}}{\partial t}\right] = \left|\mathbf{F}^{-\mathrm{T}}\mathbf{N}\right| U^{-1}(\mathbf{F}^{-1}\mathbf{s})\otimes\mathbf{n}, \quad \left[\frac{\partial \mathbf{F}^{-\mathrm{T}}}{\partial t}\right] = \left|\mathbf{F}^{-\mathrm{T}}\mathbf{N}\right| U^{-1}\mathbf{n}\otimes(\mathbf{s}\,\mathbf{F}^{-\mathrm{T}}). \tag{6}$$

The local form of the law of the balance of linear momentum written in the reference configuration is

$$\mathrm{Div}\,\mathbf{T} + \rho_0\mathbf{b} = \rho_0\ddot{\mathbf{x}}. \tag{7}$$

where $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor, $\mathbf{b}$ the specific body force and ρ_0 the mass density in the reference configuration. Body forces are assumed to be continuous at all times, so that the jump of the equation of the balance of linear momentum (7) across the singular surface is

$$[\mathrm{Div}\,\mathbf{T}] = \rho_0[\ddot{\mathbf{x}}]. \tag{8}$$

By using Eq. (5) and the following relation:

$$[\mathrm{Grad}\,\mathbf{A}] = -U^{-1}[\dot{\mathbf{A}}]\otimes\mathbf{N} \tag{9}$$

between the jumps in spatial and temporal derivatives of a second order tensor $\mathbf{A}$, Eq. (8) becomes

$$[\dot{\mathbf{T}}]\,\mathbf{N} = -\rho_0 U\,\mathbf{s}. \tag{10}$$

3. Constitutive equations of incompressible hyperelastic solids

Hyperelastic materials have the ability to undergo finite deformation and their corresponding strain energy function W may be expressed in terms of the deformation gradient tensor $\mathbf{F}$. The first Piola–Kirchhoff stress tensor $\mathbf{T}$ is obtained by differentiating the corresponding strain energy function, i.e.,

$$W = W(\mathbf{F}) \Rightarrow \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}. \tag{11}$$

According to the physical description of incompressibility constraint, the incompressible materials may undergo only isochoric motion so that their total volume, and mass density remain constant. The mathematical definition of incompressibility which follows the physical description is

$$\det\mathbf{F} = \frac{\rho_0}{\rho} \equiv 1, \tag{12}$$

where ρ is the mass density in the current configuration, hence the mass density of an incompressible material in the reference configuration is equal to its mass density in the current configuration at all times.

The theory of incompressible materials that will be considered here conforms to the general theories of constrained elastic materials proposed by Trapp [17] and Gurtin and Podio-Guidugli [18]. According to this theory the strain energy function W of a material subject to a purely mechanical constraint is expressed by

$$W(\mathbf{F}) = W_0(\mathbf{F}) + k\phi(\mathbf{F}). \tag{13}$$

Here $W_0(\mathbf{F})$ is the unconstrained counterpart of the strain energy, k is an arbitrary scalar field, and $\phi(\mathbf{F})$ is the mechanical constraint function such that

$$\phi(\mathbf{F}) = 0. \tag{14}$$

Eq. (14) indicates that the constraint function does not make any contribution to the strain energy (13), but its present form of representation is necessary in the context of hyperelastic solids. Stress is the derivative of strain energy function with respect to $\mathbf{F}$ as described by Eq. (11) and a contribution of constraint exists on stress, since the derivative of the constraint function ϕ with respect to $\mathbf{F}$ is not equal to zero. For an incompressible material the mechanical constraint function (14) becomes

$$\phi(\mathbf{F}) = \det \mathbf{F} - 1, \tag{15}$$

and the scalar k is identified by

$$k = -p, \tag{16}$$

where p is the hydrostatic pressure. In view of Eqs. (15), (12) and (13) it is obvious that the incompressibility constraint does not contribute to the strain energy, as explained by Spencer [19]. However, the incompressibility constraint contributes to stress and the substitution of Eqs. (13), (15) and (16) in (11) yields

$$\mathbf{T} = \mathbf{T}^0 - p\mathbf{F}^{-\mathrm{T}} = \frac{\partial W_0}{\partial \mathbf{F}} - p\mathbf{F}^{-\mathrm{T}}, \tag{17}$$

where $\mathbf{T}^0$ is the unconstrained counterpart of stress.

A strain energy function for unconstrained hyperelastic solids was proposed by Ciarlet and Geymonat [20], which is of the form

$$W(\mathbf{F}) = a\|\mathbf{F}\|^2 + b\|\mathrm{Cof}\,\mathbf{F}\|^2 + c(\det \mathbf{F})^2 - d\,\mathrm{Log}\,(\det \mathbf{F}) + e \tag{18}$$

with $a>0$, $b>0$, $c>0$, $d>0$, $e\in\boldsymbol{R}$, and where

$$\|\mathbf{F}\|^2 = \mathrm{tr}\,\mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \|\mathrm{Cof}\,\mathbf{F}\|^2 = \tfrac{1}{2}(\mathrm{tr}\,\mathbf{F}^{\mathrm{T}}\mathbf{F})^2 - \tfrac{1}{2}\mathrm{tr}\,(\mathbf{F}^{\mathrm{T}}\mathbf{F})^2.$$

Here tr defines the trace of a second rank tensor, that is, $\mathrm{tr}\mathbf{A} = A_{jj} = A_{11} + A_{22} + A_{33}$ for any second rank tensor $\mathbf{A}$.

According to Ciarlet [21], the strain energy function of incompressible Mooney–Rivlin materials may be recovered from Eq. (18) by setting $c=d=e=0$. Further simplification of Eq. (18) by the assumption that $b=0$ yields the result obtained by Blatz [22], which is the strain energy function of neo-Hookean materials. These results will be embedded in Eq. (13) as it provides a more powerful tool in view of Eq. (17). The first Piola–Kirchhoff stress tensors $\mathbf{T}^{MR}$ for Mooney–Rivlin materials, and $\mathbf{T}^{H}$ for neo-Hookean materials are obtained by substituting the corresponding cases of Eq. (18) in Eq. (17), which are

$$\mathbf{T}^{MR} = 2a\mathbf{F} + 2b\,(\mathrm{tr}\,\mathbf{F}^{\mathrm{T}}\mathbf{F})\,\mathbf{F} - 2b\,\mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{F} - p\,\mathbf{F}^{-\mathrm{T}}, \tag{19}$$

$$\mathbf{T}^{H} = 2a\mathbf{F} - p\mathbf{F}^{-\mathrm{T}}. \tag{20}$$

St. Venant–Kirchhoff materials are another class of compressible hyperelastic materials whose unconstrained counterpart of strain energy function W_0^{VK} is defined in Ref. [21] as

$$W_0^{VK} = \frac{\lambda}{2}(\mathrm{tr}\,\mathbf{E})^2 - \mu\,\mathrm{tr}\,\mathbf{E}^2. \tag{21}$$

Here λ and μ are the Lame constants, and $\mathbf{E}$ is the Green–St. Venant strain tensor which is defined by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}).$$

The strain energy function W^{VK} of incompressible St. Venant–Kirchhoff materials is obtained by the substitution of Eq. (20) in Eq. (13). The corresponding first Piola–Kirchhoff stress tensor $\mathbf{T}^{\mathrm{VK}}$ is obtained from Eq. (17) which is of the form

$$\mathrm{T}^{VK} = \lambda\,(\mathrm{tr}\,\mathbf{E})\,\mathbf{F} + 2\mu\mathbf{F}\mathbf{E} - p\mathbf{F}^{-\mathrm{T}}. \tag{22}$$

4. Propagation of acceleration waves in incompressible elastic media

The differentiation of the constraint equation $\phi(\mathbf{F})$ in Eq. (15) with respect to time yields

$$\frac{\partial\phi}{\partial t} = \frac{\partial \det\mathbf{F}}{\partial\mathbf{F}}\bullet\frac{\partial\mathbf{F}}{\partial t} = (\det\mathbf{F})\mathbf{F}^{-\mathrm{T}}\bullet\dot{\mathbf{F}} = 0. \tag{23}$$

By using Eqs. (3), (5a) and (6) with the fact that, det $\mathbf{F}=1$ in an incompressible medium, result (23) is simplified to the following:

$$\mathbf{s}\bullet\mathbf{n} = 0, \tag{24}$$

which is a condition imposed by the incompressibility constraint on the propagation of acceleration waves. Eq. (24) indicates that the amplitude vector $\mathbf{s}$ and the unit normal vector $\mathbf{n}$ to the wave front are perpendicular to each other, that is, acceleration waves in incompressible media can propagate only in the form of transverse waves. Hence, propagation of a longitudinal wave with $\mathbf{s}=\mathbf{n}$ is impossible in an incompressible solid.

The substitution of the derivative of the stress $\mathbf{T}$ in Eq. (17) for incompressible media should be substituted in Eq. (10) in order to obtain the relation between the jumps in the time derivative of unconstrained part of stress $\dot{\mathbf{T}}^0$, time derivative of pressure $\dot{p}$, and the amplitude vector $\mathbf{s}$. The result is the following:

$$[\dot{\mathbf{T}}^0]\,\mathbf{N} - U^{-1}p\left|\mathbf{F}^{-\mathrm{T}}\mathbf{N}\right|^2(\mathbf{n}\otimes\mathbf{n})\,\mathbf{s} - \left|\mathbf{F}^{-\mathrm{T}}\mathbf{N}\right|[\dot{p}]\,\mathbf{n} = -\rho_0\,U\,\mathbf{s}. \tag{25}$$

The jump in the time rate of pressure is obtained from Eq. (25) by using Eq. (24) as follows:

$$[\dot{p}] = \left|\mathbf{F}^{-\mathrm{T}}\mathbf{N}\right|^{-1}\mathbf{n}\bullet[\dot{\mathbf{T}}^0]\,\mathbf{N}. \tag{26}$$

The substitution of Eq. (26) in Eq. (25) yields

$$(\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\,[\dot{\mathbf{T}}^0]\,\mathbf{N} = -\rho_0\,U\,\mathbf{s}. \tag{27}$$

The components of the time rate $\dot{\mathbf{T}}^0$ of stress may be obtained by using Eq. (11) as follows:

$$\frac{\partial \dot{T}^0_{iK}}{\partial t} = \frac{\partial^2 W_0}{\partial F_{iK} \partial F_{jL}} \frac{\partial F_{jL}}{\partial t}. \tag{28}$$

By using Eq. (5), the jump of Eq. (28) across the wave front becomes

$$[\dot{T}^0_{iK}] = -U^{-1} \xi_{iKjL}\, s_j N_L, \tag{29}$$

where the components of the fourth order elasticity tensor ξ are defined by

$$\xi_{iKjL} = \frac{\partial^2 W_0}{\partial F_{iK} \partial F_{jL}} = \frac{\partial T_{iK}}{\partial F_{jL}}. \tag{30}$$

The substitution of (29) in Eq. (27) yields the equation of motion of acceleration waves in incompressible elastic media, which is

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\, \mathbf{Q}\mathbf{s} = \rho_0 U^2 \mathbf{s}. \tag{31}$$

Here $\mathbf{Q}$ is the second order acoustic tensor for the unconstrained elastic media, whose components are defined by

$$Q_{ij} = \xi_{iKjL} N_K N_L. \tag{32}$$

The corresponding acoustic tensor $\mathbf{Q}^*$ for the incompressible elastic media is as follows:

$$\mathbf{Q}^* = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\, \mathbf{Q}. \tag{33}$$

The acoustic tensor $\mathbf{Q}$ of the unconstrained elastic media is symmetric, whereas the acoustic tensor $\mathbf{Q}^*$ of the incompressible elastic media is not symmetric. If $\mathbf{Q}$ possesses the property of positive definiteness in addition to the symmetry it has three eigenvalues $\lambda_j = \rho_0 U_j^{\,2}$ corresponding to three possible wave speeds, and an orthonormal triad $\{\mathbf{s}_j\}$of wave amplitudes that correspond to the eigenvectors. Scott and Hayes [12] showed that, despite the lack of symmetry of the acoustic tensor in constrained materials, the eigenvalues are real. Ericksen [10] first showed that one of the eigenvalues of the acoustic tensor in incompressible isotropic materials is zero, and hence only two non-zero eigenvalues exist. Truesdell and Noll [11] generalized the results of Ericksen for anisotropic materials. Therefore, in an incompressible elastic material one of the speeds of propagation is zero: $U = \lambda = 0$, which has no physical significance. The remaining two speeds of propagation may be obtained from the roots of the characteristic polynomial equation which is obtained from Eq. (31) as follows:

$$\mathrm{Det}\,(\mathbf{Q}^* - \lambda \mathbf{I}) = \lambda^2 - \mathbf{I}^*_{\mathbf{Q}} \lambda + \mathbf{II}^*_{\mathbf{Q}} = 0. \tag{34}$$

Here $\mathbf{I}^*_{\mathbf{Q}}$ and $\mathbf{II}^*_{\mathbf{Q}}$ are the first and the second invariants of the $\mathbf{Q}^*$ which is a singular matrix, for which, $\mathbf{III}^*_{\mathbf{Q}} = \mathrm{Det}\,\mathbf{Q}^* = 0$. The roots of Eq. (34) are obtained by the following:

$$\lambda_{1,2} = \frac{\mathbf{I}^*_{\mathbf{Q}} \pm \sqrt{\mathbf{I}^{*2}_{\mathbf{Q}} - 4\mathbf{II}^*_{\mathbf{Q}}}}{2}. \tag{35}$$

Mandel compared first in Refs. [13,14] the speeds of propagation of acceleration waves in elastic and plastic materials by the use of certain inequalities. These inequalities indicate that the magnitudes of wave speeds are greater in elastic materials than the magnitudes of wave speeds in plastic materials. A similar comparison will be made in the context of incompressible elastic solids by using Mandel's type of inequalities.

The characteristic equation (34) may be rewritten in the form

$$f(\lambda) = \det \mathbf{A} \det \{\mathbf{I} - \mathbf{A}^{-1}(\mathbf{n}\otimes\mathbf{n})\mathbf{Q}\}, \tag{36}$$

where $\mathbf{A} = \mathbf{Q} - \lambda\mathbf{I}$. Using the identity det $(\mathbf{I} - \mathbf{a}\otimes\mathbf{b}) \equiv 1 - \mathbf{a}\cdot\mathbf{b}$, for any two vectors **a** and **b** and using the symmetry of **Q**, Eq. (36) is transformed as follows:

$$f(\lambda) = \det\mathbf{A}(1 - \mathbf{A}^{-1}\mathbf{n}\bullet\mathbf{Q}\mathbf{n}) = 0. \tag{37}$$

Coinciding the co-ordinate axes in use with the principal axes of **A** and **Q**, (37) can be written in the form

$$\begin{aligned} f(\lambda) = &(q_1 - \lambda)(q_2 - \lambda)(q_3 - \lambda) - (q_2 - \lambda)(q_3 - \lambda)\, q_1 n_1^2 \\ &- (q_1 - \lambda)(q_3 - \lambda)\, q_2 n_2^2 - (q_1 - \lambda)(q_2 - \lambda)\, q_3 n_3^2 = 0, \end{aligned} \tag{38}$$

where $\{q_j\}$ are the eigenvalues of **Q** and $\{n_j\}$ are the components of **n**. One of the roots of Eq. (38) is zero, that is $\lambda_3 = 0$, which is compatible with the results of Ericksen [10]. The corresponding eigenvalues are ordered by

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 = 0. \tag{39}$$

The eigenvalues of **Q** may be ordered similarly:

$$q_1 \geqslant q_2 \geqslant q_3. \tag{40}$$

If $\{q_j\}$ are substituted in Eq. (38) the following inequalities are obtained:

$$f(q_1) \leqslant 0, \quad f(q_2) \geqslant 0, \quad f(q_3) \leqslant 0, \quad f(-\infty) \geqslant 0. \tag{41}$$

Inequalities (41) yield the following Mandel's type of inequalities:

$$q_1 \geqslant \lambda_1 \geqslant q_2 \geqslant \lambda_2 \geqslant q_3 \geqslant 0. \tag{42}$$

Since $\lambda_j = \rho_0 U_j^2$, the following result is obtained from Eq. (42):

$$U_1^c \geqslant U_1 \geqslant U_2^c \geqslant U_2 \geqslant U_3^c \geqslant 0. \tag{43}$$

Here $\{U_j\}$ are the speeds of propagation of acceleration waves in incompressible media, and $\{U_j^c\}$ are the corresponding speeds in compressible media. According to inequalities (43), the magnitudes of the speeds of propagation of acceleration waves in compressible media in each direction constitute the upper bounds for the speeds of propagation of acceleration waves in incompressible media.

5. Determination of wave speeds in specific material classes

In this section the speeds of propagation of acceleration waves in the incompressible hyperelastic material classes of Mooney–Rivlin, neo-Hookean and St. Venant–Kirchhoff materials will be obtained. Acceleration waves will be assumed to be propagating in a region which is in a state of an isochoric deformation whose deformation gradient tensor **F** is described by the following matrix $\{\mathbf{F}\}$ of its components:

$$\{\mathbf{F}\} = \begin{Bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^{-1} \end{Bmatrix}, \tag{44}$$

where $\alpha > 0$ and $\beta > 0$.

The components of normal vectors $\mathbf{N}$ and $\mathbf{n}$ to the wave front defined by Eq. (3) will be assumed to be of the form

$$\{\mathbf{N}\} = \{\mathbf{n}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}. \tag{45}$$

The acoustic tensor $\mathbf{Q}^*_{MR}$ for Mooney–Rivlin materials is obtained using Eqs. (19), (22), (30) and (33):

$$\begin{aligned} \mathbf{Q}^*_{MR} = {} & (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\,\{2a\mathbf{I} + 2b\,(\mathbf{FN}\otimes\mathbf{FN}) + 2b\,(\operatorname{tr}\mathbf{F}^{\mathrm{T}}\mathbf{F})\,\mathbf{I} - 2b(\mathbf{FN}\bullet\mathbf{FN})\mathbf{I} \\ & - 2b(\mathbf{n}\otimes\mathbf{n})\,(\mathbf{FN}\otimes\mathbf{FN}) - 2b\,\mathbf{F}^{\mathrm{T}}\mathbf{F} - 2b\,(\mathbf{n}\otimes\mathbf{n})\}. \end{aligned} \tag{46}$$

Finally the substitution of (44) and (45) in Eq. (46) yields the specific form of $\mathbf{Q}^*_{MR}$ for the given state of isochoric deformation, whose element matrix is

$$\{\mathbf{Q}^*_{MR}\} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{Bmatrix}, \tag{47}$$

where

$$Q_{22} = 2a + 2b\alpha^{-2}\beta^{-2}, \quad Q_{33} = 2a + 2b\beta^2. \tag{48}$$

The two non-zero and real speeds of propagation U_1^{MR}, U_2^{MR} of acceleration waves in Mooney–Rivlin materials are obtained from the eigenvalues of the obviously singular tensor defined in Eq. (47):

$$U_1^{MR} = \sqrt{\frac{2a + 2b\alpha^{-2}\beta^{-2}}{\rho_0}}, \quad U_2^{MR} = \sqrt{\frac{2a + 2b\beta^2}{\rho_0}}. \tag{49}$$

The eigenvectors $\mathbf{s}^1$ and $\mathbf{s}^2$ of Eq. (46) corresponding to the wave speeds given in Eq. (49) in normalized form become

$$\{\mathbf{s}^1\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \{\mathbf{s}^2\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{50}$$

The eigenvectors $\mathbf{s}^1$ and $\mathbf{s}^2$ of $\mathbf{Q}^*$ and the corresponding eigenvectors of the unconstrained acoustic tensor $\mathbf{Q}$ coincide.

The modified forms of the results in Eq. (49) are illustrated graphically: $(U_1^{MR})^2(\rho_0/b)$ in Fig. 2, and $(U_2^{MR})^2(\rho_0/b)$ in Fig. 3 for varying α in the special deformation case with $\alpha = \beta$, for certain values of the dimensionless parameter (a/b).

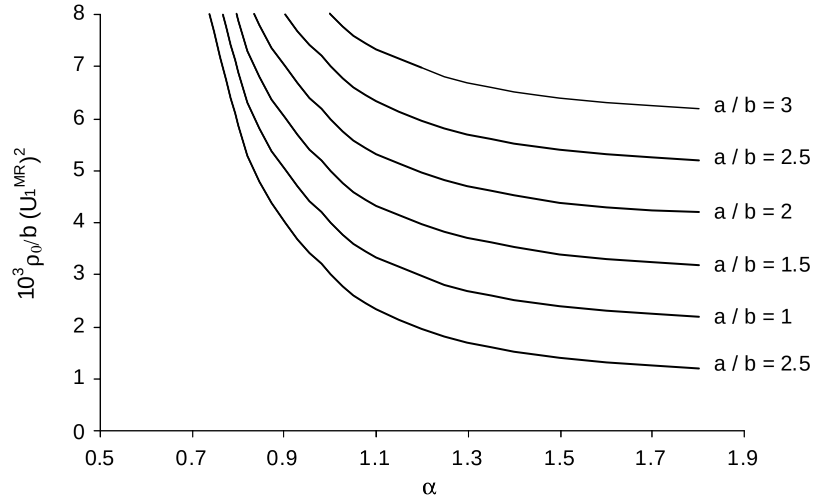

Fig. 2. Variation of $(\rho_0/b)\left(U_1^{MR}\right)^2$ as a function of stretch α, for certain values of the dimensionless parameter a/b.

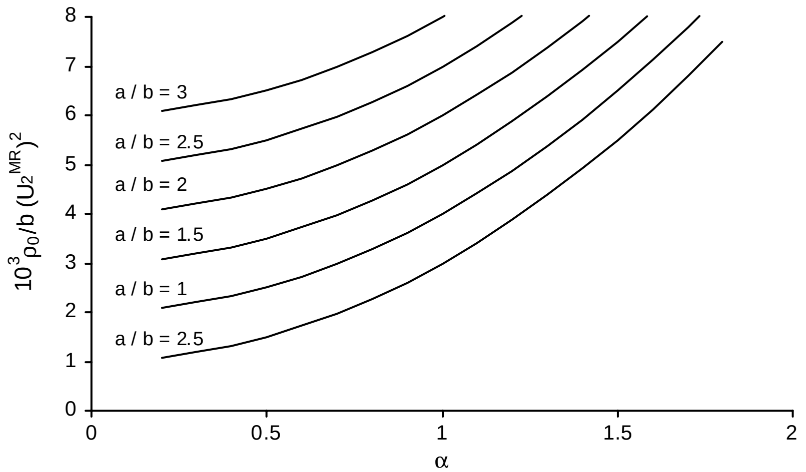

Fig. 3. Variation of $(\rho_0/b)\left(U_2^{MR}\right)^2$ as a function of the stretch α, for certain values of the dimensionless parameter a/b.

In an undeformed state ahead of the wave front $\mathbf{F}=\mathbf{I}$, so that $\alpha=\beta=1$ and the non-zero propagation speeds in Eq. (49) are reduced to

$$U^{MR} = U_1^{MR} = U_2^{MR} = \sqrt{\frac{2a+2b}{\rho_0}}. \tag{51}$$

Two non-zero identical wave propagation speeds are obtained in neo-Hookean materials by setting $b=0$ in Eq. (49) which are

$$U^{H} = U_1^{H} = U_2^{H} = \sqrt{\frac{2a}{\rho_0}}. \tag{52}$$

Eq. (52) indicates that wave propagation speeds in neo-Hookean materials are independent of the state of deformation ahead of the wave front.

The acoustic tensor $\mathbf{Q}^*_{VK}$ of St. Venant–Kirchhoff materials is obtained by the substitution of Eqs. (21), (30) and (32) in Eq. (33):

$$\mathbf{Q}^*_{VK} = (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\,\{\lambda(\mathrm{tr}\,\mathbf{E})\,\mathbf{I} + (\lambda + 2\mu)\,(\mathbf{FN}\otimes\mathbf{FN}) + 2\mu(\mathbf{EN}\bullet\mathbf{N})\,\mathbf{I} + 2\mu\mathbf{F}^{\mathrm{T}}\mathbf{F}\}. \tag{53}$$

For the deformation described by Eq. (44) the matrix of the components of (53) becomes

$$\{\mathbf{Q}^*_{VK}\} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{Bmatrix}, \tag{54}$$

where

$$Q_{22} = \frac{\lambda}{2}\left(\alpha^2 + \beta^2 + \alpha^{-2}\beta^{-2} - 3\right) + \mu\left(\alpha^2 + 2\beta^2 - 1\right), \tag{55}$$

$$Q_{33} = \frac{\lambda}{2}\left(\alpha^2 + \beta^2 + \alpha^{-2}\beta^{-2} - 3\right) + \mu\left(\alpha^2 + 2\alpha^{-2}\beta^{-2} - 1\right). \tag{56}$$

The non-zero eigenvalues of Eq. (54) yield the two speeds of propagation U_1^{VK} and U_2^{VK} of acceleration waves in St. Venant–Kirchhoff materials:

$$U_1^{VK} = \sqrt{\frac{\frac{\lambda}{2}\left(\alpha^2 + \beta^2 + \alpha^{-2}\beta^{-2} - 3\right) + \mu\left(\alpha^2 + 2\beta^2 - 1\right)}{\rho_0}}, \tag{57}$$

$$U_2^{VK} = \sqrt{\frac{\frac{\lambda}{2}\left(\alpha^2 + \beta^2 + \alpha^{-2}\beta^{-2} - 3\right) + \mu\left(\alpha^2 + 2\alpha^{-2}\beta^{-2} - 1\right)}{\rho_0}}. \tag{58}$$

The modified forms of the results in Eqs. (57) and (58) are illustrated graphically: $(U_1^{VK})^2(\rho_0/\mu)$ in Fig. 4, and $(U_2^{VK})^2(\rho_0/\mu)$ in Fig. 5 for varying α in the special deformation case with $\alpha = \beta$ for certain values of the dimensionless ratio (λ/μ) of the Lame constants.

Substitution of $\alpha = \beta = 1$, for on undeformed St. Venant–Kirchhoff material in Eq. (58) yields the corresponding speed U^{VK}:

$$U^{VK} = U_1^{VK} = U_2^{VK} = \sqrt{\frac{2\mu}{\rho_0}}. \tag{59}$$

According to Ciarlet [21] the following inequality holds for the material constant a in Eq. (18) and the Lame constant μ in Eq. (21)

$$a > \mu. \tag{60}$$

If the speeds of propagation of acceleration waves in undeformed material classes defined in Eqs. (51), (52) and (59) are compared in view of inequality (60) the following inequalities are derived:

$$U^{MR} > U^{H} > U^{VK}. \tag{61}$$

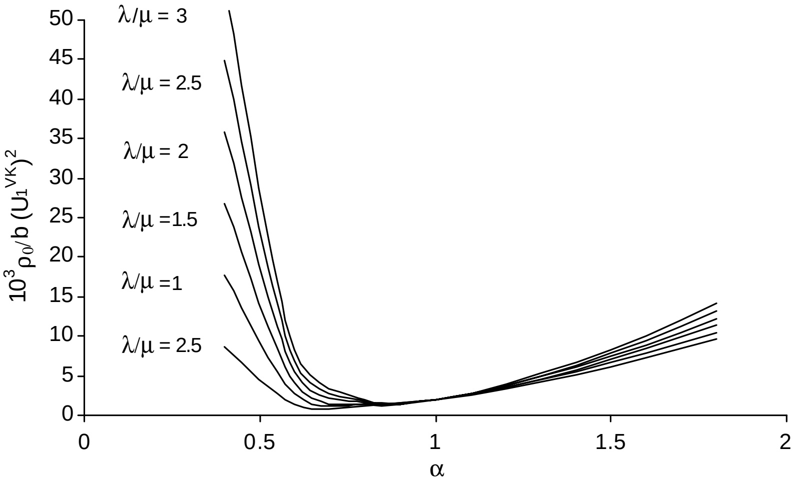

Fig. 4. Variation of $(\rho_0/\mu)\left(U_1^{VK}\right)^2$ as a function of the stretch α, for certain values of the dimensionless parameter λ/μ.

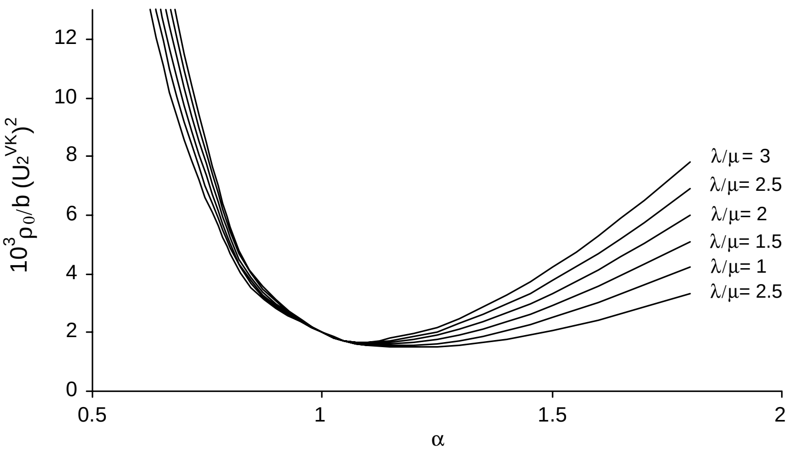

Fig. 5. Variation of $(\rho_0/\mu)\left(U_2^{VK}\right)^2$ as a function of the stretch α, for certain values of the dimensionless parameter λ/μ.

6. Conclusions

Incompressible hyperelastic solids are treated as a specific case of the general theory of hyperelastic materials with internal mechanical constraints, and the corresponding acoustic tensor is obtained using the theory of singular surfaces. Similar to the earlier investigations, for example in Refs. [10–12] the acoustic tensor is found to be non-symmetric and singular, one of whose eigenvalues is zero. The speeds of propagation of acceleration waves in incompressible and compressible media are compared by the use of Mandel's type of inequalities (43) which is one of the main purposes of the present study. It is concluded that the incompressibility constraint reduces the magnitudes of the propagation of acceleration waves in comparison to the corresponding speeds in unconstrained materials. Two non-zero speeds are obtained in

Mooney–Rivlin and St. Venant–Kirchhoff materials in a state of isochoric deformation (44). Finally the magnitudes of the speeds of propagation of acceleration waves in certain materials which represent a broad class of hyperelastic materials are ordered by the use of certain inequalities (61). Another comparison between results (49) and (51) indicate that the speeds of propagation of acceleration waves in deformed solids are greater than the corresponding speeds in solids that are in an undeformed state ahead of the wave font.

References

[1] A.C. Eringen, E.S. Șuhubi, Elastodynamics, Vol. I, Academic Press, New York, 1975.
[2] M.F. McCarthy, Singular surfaces and waves, in: A.C. Eringen (Ed.), Continuum Physics, Vol. II, Academic Press, New York, 1975, pp. 449–521.
[3] C. Truesdell, R.A. Toupin, The classical field theories, in: C. Truesdell (Ed.), Handbuch der Physik III/1, Springer, Berlin, 1960, pp. 491–529.
[4] E.S. Șuhubi, The growth of acceleration waves of arbitrary form in deformed hyperelastic materials, International Journal of Engineering Science 8 (1970) 699–710.
[5] R.W. Ogden, Growth and decay of acceleration waves in incompressible elastic solids, Quarterly Journal of Mechanics and Applied Mathematics 27 (1974) 451–464.
[6] P.J. Chen, Growth and decay of waves in solids, in: C. Truesdell (Ed.), Handbuch der Physik, Vol. VIa/3, Springer, Berlin, 1973, pp. 303–402.
[7] N.H. Scott, Acceleration waves in constrained elastic materials, Archive for Rational Mechanics and Analysis 58 (1975) 57–75.
[8] N.H. Scott, Acceleration waves in incompressible solids, Quarterly Journal of Mechanics and Applied Mathematics 29 (1976) 295–310.
[9] B.D. Reddy, The propagation and growth of acceleration waves in constrained thermoelastic materials, Journal of Elasticity 14 (1984) 387–402.
[10] J.L. Ericksen, On the propagation of waves in isotropic incompressible perfectly elastic materials, Journal of Rational Mechanics and Analysis 2 (1953) 329–337.
[11] C. Truesdell, W. Noll, The non-linear field theories of mechanics, in: S. Flügge, (Ed.), Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965, pp. 1–602.
[12] N.H. Scott, M. Hayes, A note on wave propagation in internally constrained hyperelastic materials, Wave Motion 7 (1985) 601–605.
[13] J. Mandel, Ondes plastiques dans un milieu indefini à trois dimensions, Journal de Mecanique 1 (1962) 3–30.
[14] J. Mandel, Thermodynamique et ondes dans les milieux viscoplastiques, Journal of the Mechanics and Physics of Solids 17 (1969) 125–140.
[15] B.D. Reddy, T. Gültop, Acceleration waves in finitely deformed elastic–plastic solids, European Journal of Mechanics A/Solids 14 (1995) 529–551.
[16] C.C. Wang, C. Truesdell, Introduction to Rational Elasticity, Noordhoff, Leiden, 1973.
[17] J.A. Trapp, Reinforced materials with thermomechanical constraints, International Journal of Engineering Science 9 (1971) 757–773.
[18] M.E. Gurtin, P. Podio-Guidugli, The thermodynamics of constrained materials, Archive for Rational Mechanics and Analysis 51 (1973) 192–208.
[19] A.J.M. Spencer, Continuum Mechanics, Longman, New York, 1980.
[20] P.G. Ciarlet, G. Geymonat, Sur les lois de comportement en elasticité non-lineaire compressible, Comptes Rendus de l'Académie des Sciences Paris, Serie II 295 (1982) 423–426.
[21] P.G. Ciarlet, Mathematical Elasticity, Three Dimensional Elasticity, Vol I, North-Holland, Amsterdam, 1988.
[22] P.J. Blatz, On the thermostatic behaviour of elastomers, in: A.J. Chompff, S. Newman (Eds.), Polymer Networks, Structure and Mechanical Properties, Plenum Press, New York, 1971, pp. 23–45.