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Statistical energy analysis, energy distribution models and system modes

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Abstract

Expressions for the energy influence coefficients of a built-up structure are found in terms of the modes of the whole structure. These coefficients relate the time and frequency average energies of the subsystems to the subsystem input powers. Rain-on-the-roof excitation over a frequency band Ω is assumed. It is then seen that the system can be described by an SEA model only if a particular condition involving the mode shapes of the system is satisfied. Broadly, the condition holds if the mode shapes of the modes in the frequency band of excitation are, on average, typical enough of all the modes of the system in terms of the distribution of energy throughout the system. If this condition is satisfied then the system can be modelled using an "quasi-SEA" approach, irrespective of the level of damping or of the strength of coupling. However, the resulting model need not be of a proper SEA form, and in particular the indirect coupling loss factors may not be negligible.

1. Introduction

Energy-based modelling approaches are often used to describe the higher frequency vibrational behaviour of complex systems in some average or approximate way. The system is divided into subsystems and the response is described in terms of the total time average subsystem energies $\mathbf{E}$ and input powers $\mathbf{P}_{in}$. While these can in principle be discrete frequency responses, they are usually frequency averaged, typically over third octave bands, the model then relating the time and frequency average subsystem energies and the input powers.

The most important of these methods is statistical energy analysis (SEA) [1]. In an SEA model the coupling power between two subsystems is assumed to be proportional to the difference in

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their modal energies, the constants of proportionality being related to the coupling loss factors. SEA involves a number of assumptions and approximations, however, whose validity and accuracy are usually unknown.

There are two main purposes to this paper. The first is to examine under what general conditions the coupling powers are indeed proportional to the modal energy differences and hence under what general conditions one might apply an SEA model with confidence. The second is to provide a basis from which the parameters of an SEA model (e.g., the coupling loss factors) can be estimated.

The analysis involves the modes of the system as a whole. These are first used to form an energy distribution (ED) model. The conditions under which an SEA model can then be found are next considered. A distinction is made between a "quasi-SEA" model and a "proper SEA" model. A "quasi-SEA" model is one for which the SEA parameters satisfy two necessary conditions, i.e., conservation of energy and consistency. A "proper SEA" model is one where, furthermore, all the indirect coupling loss factors are zero, so that the assumption of coupling power proportionality is valid. It is seen that, given some mild conditions (primarily that there are enough modes within the frequency band and that the mode shapes of these modes are, on average, typical of all the modes), an SEA-like model will exist, but the indirect coupling loss factors may be non-zero and hence it will not be a proper SEA model. The coupling loss factors can, however, be estimated in terms of the modes of the system.

A number of previous studies have been concerned with developing ED models from system modes. These include the work of Guyader et al. [2], who explicitly considered systems of coupled rectangular plates. Plate systems of arbitrary shape were considered in Refs. [3,4], which were concerned with numerical studies of SEA in the strong coupling region. A more general approach is described in Ref. [5].

The concept of indirect coupling loss factors has been introduced elsewhere [3, 6–9]. Numerical examples were considered in Refs. [3,8,9], while Refs. [6,7] contain general discussions. Some of the general conclusions of Refs. [6,7] were that SEA models can be formed if "subsystem energy" is defined appropriately, but there may be non-zero indirect coupling loss factors. The arguments were developed using system Green functions, which of course can be written in terms of system modes. Indeed, there are further similarities between the conclusions of Ref. [7] and those described here, except that here energy is described uniquely and a condition involving the system modal properties found for SEA to hold. Expressions are developed for the indirect coupling loss factors in terms of these modal properties.

In the next section ED and SEA models and the properties of SEA parameters are discussed. Following this, the means by which ED models can be formed from system modes is outlined. The existence of an SEA model for the system is discussed in Section 4 and numerical examples presented in Section 5. First, however, some aspects of terminology are discussed.

1.1. Terminology

In this paper the term *system mode* is used to describe a mode of the assembled system, while a *subsystem mode* is one of a subsystem when uncoupled from the remainder of the structure. Uncoupling can be achieved in any manner such as freeing or fixing the interfaces between the subsystem and the rest of the structure.

A *global mode* is a system mode which is global in the sense that the kinetic energy of the mode is spread out globally through the system. A *local mode*, however, is a system mode which is localized within a region of the system, so that the kinetic energy tends to be contained within one (or maybe a few) subsystems.

The terms *weak coupling* and *strong coupling* are used in an SEA sense, although there is no universal, commonly accepted definition of what this means as yet. From a wave perspective the "γ-parameter" [8,10,11] has been proposed, while the Smith criterion of weak coupling is that the coupling loss factors are much smaller than the damping loss factors. However, studies indicate that in the strong coupling regime the coupling loss factors become proportional to the damping loss factors. Thus, the Smith criterion should strictly involve the classical, asymptotic coupling loss factor for large damping.

In contrast to the SEA strength of coupling, two subsystems can be described as being *strongly* (or *weakly*) *connected* if energy can (or cannot) flow freely across the interface between them. Typically, the interface between strongly connected subsystems has a large transmission coefficient or there is a small impedance mismatch across it. Generally, weak and strong connection are associated with there being predominantly local or global modes, respectively. A clear distinction is thus made between the SEA strength of coupling and the strength of connection.

2. Energy distribution and statistical energy analysis models

In this section, some general comments are made concerning ED and SEA models and the relationships between them.

The system is divided into subsystems which are excited by random, stationary, distributed forces. It is assumed that the system is linear and that the excitations applied to the different subsystems are uncorrelated. The response is described by the time-average input powers $\mathbf{P}_{in}$ and the subsystem energies $\mathbf{E}$ averaged over some frequency band Ω.

2.1. Energy distribution models and energy influence coefficients

In an ED model the energies and input powers are related by

$$\mathbf{E} = \mathbf{A}\mathbf{P}_{in}, \tag{1}$$

where $\mathbf{A}$ is a matrix of energy influence coefficients in the relevant frequency band. The element A_{rs} gives the (time and frequency average) energy in subsystem r per unit (time and frequency average) power input to subsystem s. $\mathbf{A}$ is not symmetric, although symmetry exists regarding the modal energies in some circumstances.

In the next section expressions for the energy influence coefficients are derived in terms of the modes of the system. These coefficients can also be measured (e.g., using the power injection method).

2.2. Statistical energy analysis models

In SEA a power balance equation is written for each subsystem, so that, for subsystem r,

$$P_{in,r} = P_{diss,r} + P_{coup,r}. \tag{2}$$

Here $P_{diss,r} = \omega\eta_r E_r$ is the dissipated power[1] in subsystem r, ω being the band centre frequency and η_r the loss factor of subsystem r, and $P_{coup,r} = \sum_s P_{rs}$ is the net coupling power for subsystem r, with P_{rs} being the coupling power from subsystem r to subsystem s.

Central to SEA is the assumption that the coupling power between two subsystems is proportional to the difference in their energies per (subsystem) mode. This is the so-called coupling power proportionality (CPP) assumption. In terms of the modal powers E/n (i.e., energies per mode of vibration per unit frequency)

$$P_{rs} = \omega n_r \eta_{rs}\left(\frac{E_r}{n_r} - \frac{E_s}{n_s}\right), \tag{3}$$

where η_{rs} is the coupling loss factor between subsystems r and s and n_r is the asymptotic modal density of subsystem r. This can alternatively be written as

$$P_{rs} = \omega\eta_{rs}E_r - \omega\eta_{sr}E_s \tag{4}$$

so that, if CPP holds, the coupling loss factors must be related by the *consistency relation*

$$n_r\eta_{rs} = n_s\eta_{sr}. \tag{5}$$

Assembling these SEA equations gives

$$\mathbf{P}_{in} = \mathbf{LE}, \tag{6}$$

where $\mathbf{L}$ is a matrix of damping and coupling loss factors given by

$$\mathbf{L} = \omega\,\mathrm{diag}\left(\eta_j\right) + \omega\begin{bmatrix} \eta_{12}+\eta_{13}+\cdots & -\eta_{21} & -\eta_{31} & \cdots \\ -\eta_{12} & \eta_{21}+\eta_{23}+\cdots & -\eta_{32} & \cdots \\ -\eta_{13} & -\eta_{23} & \eta_{31}+\eta_{32}+\cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{7}$$

where $\mathrm{diag}(\cdot)$ is a diagonal matrix. The columns of the matrix of coupling loss factors sum to zero.

Strictly, CPP and the SEA equations are assumed to hold in an ensemble average sense, i.e. when powers and energies are averaged over an ensemble of similar, but slightly different,

[1] It is conventional in SEA to assume structural damping and to write the SEA equations in terms of damping loss factors. In Section 3, where the response is written in terms of the system modes, a viscous damping model is assumed for two reasons. Firstly, the dissipated power is then proportional to the kinetic energy, which is the quantity most easily measured. Secondly, the relation here is approximate and is only exact for the resonant response, when the frequency average kinetic and potential energies are equal. For light damping, one can move from one model to another by substituting $\eta = 2\zeta$ where ζ is the viscous damping factor. In practice, negligible errors are introduced for the types of applications envisaged — if these errors are not negligible, the analyst will be in some difficulties in applying SEA anyway.

systems. When applied to an individual system, they are assumed to be a good approximation only when averaged over a suitably wide frequency range.

2.2.1. "Quasi-SEA" and "proper SEA" models and matrices

An ED model requires the assumptions of linearity and uncorrelated excitations to be valid, whereas SEA involves a number of further assumptions and approximations, some of them quite sweeping. Only if these are valid does the CPP relation of Eq. (3) hold, and the coupling power between two coupled subsystems then depends only on their modal energies and the coupling loss factors. Indirect coupling loss factors are zero. The system can then be described by a "proper SEA" model, which has great advantages to the analyst, since it is easy to relate the parameters of the SEA model to physical damping and coupling components and to predict the effects of modifications.

The parameters of an SEA model satisfy two necessary conditions if the behaviour of the system is to be described by SEA. First, the sum of the rth column of $\mathbf{L}$ must equal $\omega\eta_r$ to satisfy conservation of energy. Secondly, the off-diagonal terms must satisfy the consistency relation (5).

There are also a number of *desirable* conditions that are only satisfied if all the assumptions of SEA are valid (i.e., those leading to the CPP equation (3)). First, and most important, the coupling loss factor between subsystems which are not physically coupled should be zero: there must be no non-zero "indirect coupling loss factors". The further conditions concern the coupling loss factors. All the coupling loss factors should be positive. They should be independent of the damping loss factors, at least in the classical approach to SEA. Finally, the coupling loss factor η_{rs} should depend only on 'local' properties of the junction between subsystems r and s and those of the subsystems connected to that junction, and not on the properties of any other, distant subsystems. The coupling loss factors for two subsystems in isolation are therefore the same when they form part of a larger structure, so that the analysis reduces to many, small analyses. In contrast, the energy influence coefficients in an ED model depend implicitly on the properties of the whole structure, making their explicit calculation unrealistic for large structures. However, no approximations are involved.

If the two necessary conditions are satisfied and the indirect coupling loss factors are all zero, $\mathbf{L}$ is said here to be a "proper-SEA" matrix, and SEA in its classical form can be used to model the system. If only the necessary conditions are satisfied, $\mathbf{L}$ is said to be a "quasi-SEA" matrix: an SEA-like analysis can, in principle, be adopted since the necessary conditions are satisfied, but indirect coupling loss factors would be involved.

It is worth emphasizing here that the existence of a non-zero indirect coupling loss factor η_{ri} does *not* imply that energy flows between subsystems r and i (which are not physically coupled). Instead, it means that the coupling power P_{rs} between two subsystems r and s which are physically coupled depends also on the energy of the third subsystem i. Hence, the coupling power P_{rs} is not given by the coupling power proportionality relation of Eq. (3). The existence of indirect coupling loss factors thus indicates that some of the assumptions of SEA break down: the system may still be modelled using an SEA-like approach, but indirect coupling loss factors are required to accurately model the system's response.

(Finally, it should be noted that even in a proper-SEA model, the coupling loss factors might not have the ideal properties described above, for example they might depend on the damping loss factor.)

2.3. SEA matrices from energy distribution models

If an ED model is formed, then Eq. (1) can be inverted to give

$$\mathbf{P}_{in} = \mathbf{X}\mathbf{E}, \quad \mathbf{X} = \mathbf{A}^{-1}. \tag{8}$$

Ideally, **X** is a proper SEA matrix and its elements satisfy both the necessary and the desirable conditions described above. However, this need not be the case for two reasons. Firstly, **X** describes the response of a single system over a specific frequency band and thus normally differs from the ensemble average (e.g., finite number of modes, specific details of those modes, etc.). Secondly, the SEA assumptions and approximations may be invalid or inaccurate (i.e., CPP does not hold). The result is that the elements of **X** may not satisfy all the necessary or desirable conditions of a proper-SEA or an SEA-like model even when ensemble averaged. For example, **X** may not satisfy the consistency relations, in which case **X** is not "SEA-like".

In this paper, expressions for the ED matrix **A** are found from the modal properties of the system. This requires only the assumptions of linearity and uncorrelated excitations, although various other simplifying assumptions are made for convenience. The conditions under which an SEA model can be formed are then examined. It is seen that, under some mild conditions (primarily that there are enough, 'typical' modes in the band), the matrix **X** is an SEA-like matrix, but it need not be a proper-SEA matrix: it satisfies all the necessary conditions of an SEA matrix but need not satisfy the desirable conditions.

3. Energy distribution models from system modes

In this section, expressions for the energy influence coefficients are derived in terms of the modes of the system. Proportional, viscous damping is assumed, although the modes may have different bandwidths.

3.1. Discrete frequency response

A time harmonic point force acts at a point $x = x_1$ so that the applied excitation is $f(x,t) = F\exp(\mathrm{i}\omega t)\delta(x - x_1)$. Here x may be a vector for two or three-dimensional structures. The amplitude of the response at point x_2 is given in terms of the system modes by

$$W(\omega, x_1, x_2) = \sum_j \alpha_j(\omega)\phi_j(x_1)\phi_j(x_2)F, \tag{9}$$

where $\phi_j(x)$ is the mode shape of the jth system mode and where

$$\alpha_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + \mathrm{i}\Delta_j\omega} \tag{10}$$

is the modal receptance of the jth mode, ω_j being the jth natural frequency. Here, viscous damping has been assumed, with

$$\Delta_j = 2\zeta_j\omega_j = \eta_j\omega_j \tag{11}$$

being the half-power modal bandwidth and ζ_j the damping factor. Proportional damping is assumed, although ζ may vary from mode to mode, and thus in effect be frequency dependent. Alternatively structural damping with a loss factor η may be assumed, although some of the following relations are then very good approximations rather than being exact. The system modes are assumed to be mass normalized so that

$$\int \rho(x)\phi_j(x)\phi_k(x)\mathrm{d}x = \delta_{jk}, \tag{12}$$

where $\rho(x)$ is the mass density.

3.1.1. Kinetic energy density and input power

The time-average kinetic energy density at x_2 is

$$\begin{aligned} D_T(\omega, x_1, x_2) &= \tfrac{1}{2}\mathrm{Re}\left\{\tfrac{1}{2}\rho(x_2)\omega^2 WW^*\right\} \\ &= \sum_j \sum_k \left(\tfrac{1}{4}\omega^2\beta_{jk}(\omega)\right)\left(\phi_j(x_1)\phi_k(x_1)\left|F^2\right|\right)\left(\rho(x_2)\phi_j(x_2)\phi_k(x_2)\right), \end{aligned} \tag{13}$$

where

$$\beta_{jk}(\omega) = \mathrm{Re}\left\{\alpha_j(\omega)\alpha_k^*(\omega)\right\} \tag{14}$$

and where * denotes the complex conjugate. In Eq. (13) the right-hand side has been written as the product of three terms which depend, respectively, on, frequency, the excitation and the response location.

The time average input power is

$$P_{in} = \tfrac{1}{2}\mathrm{Re}\left\{\mathrm{i}\omega W(\omega, x_1, x_1)F^*\right\} = \sum_j \tfrac{1}{2}\omega^2 \Delta_j \beta_{jj}(\omega)\phi_j^2(x_1)\left|F^2\right|, \tag{15}$$

where

$$\beta_{jj} = \frac{1}{\left[\left(\omega_j^2 - \omega^2\right)^2 + \left(\Delta_j\omega\right)^2\right]}. \tag{16}$$

3.1.2. Conservation of energy

The total kinetic energy can be found by integrating the kinetic energy density (Eq. (13)) over the whole system. From the orthogonality relations (12), it follows that, for each mode j,

$$P_{in,j}(\omega) = 2\Delta_j T_j, \tag{17}$$

where T_j is the kinetic energy in mode j. This is of course merely a statement of conservation of energy, mode by mode. Subsequently, the total energy will be approximated by twice the kinetic energy, and the terms Δ_j, $2\zeta_j\omega_j$ and $\eta_j\omega_j$ will be used interchangeably. (If a structural damping model is assumed Eq. (17) becomes $P_{in,j}(\omega) = 2\Delta_j V_j$, where V_j is the potential energy of mode j. Of course, for broadband excitation, the kinetic and potential energies of modes in the excitation band are equal.)

3.2. Energy influence coefficients

In this section, expressions for the energy influence coefficients are found in terms of the modal properties of the system. It is assumed that the excitations applied to different subsystems are uncorrelated, so that the excitations can be considered one at a time.

Spatially distributed "rain-on-the-roof" excitation is applied to subsystem s and acts over a frequency band Ω. The power input to subsystem s and the energy in subsystem r (twice the kinetic energy if averaged over a sufficiently broad frequency band, to a very good approximation) are found by integrating Eq. (15) and (13) over frequency, over the excited subsystem s and over the responding subsystem r.

3.2.1. "Rain-on-the-roof"

"Rain" is here defined to be random excitation whose spatial distribution is delta-correlated and whose amplitude is proportional to the local mass density $\rho(x)$. Thus $S'_f(x,\omega) = S_f(\omega)\rho(x)$ while its spatial cross-spectral density is zero. Such excitation applies equal modal forces to all the subsystem modes and injects energy into the direct wavefield equally at all points of the excited subsystem. Under these circumstances, the contribution to the mean square excitation $\overline{f^2}$ in a frequency band $\delta\omega$ is

$$\delta\overline{f^2} = \tfrac{1}{2}\left|F^2\right| = S_f(\omega)\rho(x_1)\delta\omega, \tag{18}$$

where the spectral density $S_f(\omega)$ is potentially frequency dependent, but is henceforth assumed to be constant for convenience. The time and frequency average kinetic energy of subsystem r is given by

$$T^{(r)} = \frac{1}{\Omega}\int_{\omega\in\Omega;x_1\in s;x_2\in r} D_T(\omega,x_1,x_2)\,\mathrm{d}\omega\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{19}$$

For rain excitation as defined above this, and the power input to subsystem s, become

$$\begin{aligned} T^{(r)} &= 2S_f \textstyle\sum_j \sum_k \Gamma_{jk}\psi^{(s)}_{jk}\psi^{(r)}_{jk}, \\ P^{(s)}_{in} &= 2S_f \textstyle\sum_j 2\Delta_j\Gamma_{jj}\psi^{(s)}_{jj}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Gamma_{jk} &= \frac{1}{\Omega}\textstyle\int_{\omega\in\Omega}\frac{1}{4}\omega^2\beta_{jk}(\omega)\,\mathrm{d}\omega, \\ \psi^{(r)}_{jk} &= \textstyle\int_{x\in r}\rho(x)\phi_j(x)\phi_k(x)\,\mathrm{d}x. \end{aligned} \tag{21}$$

The input power is given by a sum of powers input to each mode, while the kinetic energy is the sum of cross-modal terms involving modes taken two at a time. The energy influence coefficient is thus given by

$$A_{rs} = \frac{E^{(r)}}{P^{(s)}_{in}} = \frac{2T^{(r)}}{P^{(s)}_{in}} = \frac{\sum_j\sum_k \Gamma_{jk}\psi^{(s)}_{jk}\psi^{(r)}_{jk}}{\sum_j \Delta_j\Gamma_{jj}\psi^{(s)}_{jj}}. \tag{22}$$

3.3. Discussion

The expressions above involve two types of terms, whose behaviours are described in this subsection. The first is the cross-modal power mobility Γ_{jk}, which depends on the natural frequencies and bandwidths of the modes. The second is the cross-mode participation factor $\psi_{jk}^{(r)}$, which depends on the spatial correlation between the jth and kth mode shapes in the rth subsystem. Large contributions to the input power come from modes j for which the participation factor $\psi_{jj}^{(s)}$ is large, that is, from those modes which respond well (i.e., for which the modal displacement is large) in the excited subsystem. Mode pairs for which both $\psi_{jk}^{(s)}$ and $\psi_{jk}^{(r)}$ are large tend to give large contributions to the kinetic energy in subsystem r. These are mode pairs that are both well-excited and which respond well. The input power depends on how many modes are excited, while the kinetic energy depends also on how well these modes overlap.

3.3.1. Frequency integrals Γ_{jk}

The cross-modal term Γ_{jk} is a frequency integral whose magnitude depends primarily on the natural frequencies of the two modes and how close are these natural frequencies. Cross-modal terms for which Γ_{jk} is large may contribute strongly to the total response. Generally, Γ_{jk} is small unless both modes j and k are resonant, i.e. unless their natural frequencies lie in Ω ($\omega_j \in \Omega; \omega_k \in \Omega$). The term is particularly large if the modes *overlap*, i.e., they lie within each others half-power bandwidths such that $|\omega_j - \omega_k| \le (\Delta_j + \Delta_k)/2$. The self-term Γ_{jj} is always large if mode j is resonant.

Analytic expression for the integrals exist (e.g. Ref. [5]), but these will not be repeated here. The largest terms arise from resonant mode-pairs (especially overlapping pairs). For these, the limits of the integration can be extended to $(0, \infty)$ to a good approximation.

3.3.2. Small damping and constant damping approximations for Γ_{jk}

If the damping is small, then for resonant modes the frequency integrals can be approximated by

$$\begin{aligned}\Gamma_{jj} &= \frac{1}{\Omega}\frac{\pi}{8\Delta_j}\\ \Gamma_{jk} &= \frac{1}{\Omega}\frac{\pi(\omega_j+\omega_k)^2(\Delta_j+\Delta_k)}{16\left[\left(\omega_j^2-\omega_k^2\right)^2+(\Delta_j\omega_j+\Delta_k\omega_k)^2\right]}.\end{aligned} \tag{23}$$

These expressions are exact if the limits of integration are $(0, \infty)$, whatever the level of damping.

Finally, if in addition the bandwidths of modes j and k are equal then

$$\begin{aligned}\Gamma_{jk} &= \frac{1}{\Omega}\frac{\pi}{8\Delta}\frac{1}{1+((\omega_j-\omega_k)/\Delta)^2} = \Gamma_{jj}\frac{1}{1+((\omega_j-\omega_k)/\Delta)^2},\\ \Gamma_{jj} &= \frac{1}{\Omega}\frac{\pi}{8\Delta}\end{aligned} \tag{24}$$

and where all the Γ_{jj} for resonant modes are equal. This can be written as

$$\begin{aligned} \Gamma_{jk} &= \Gamma_{jj}\mu_{jk}, \\ \mu_{jk} &= \frac{1}{\left(1+S_{jk}^2\right)} = \frac{M_{jk}^2}{\left(1+M_{jk}^2\right)}, \\ S_{jk} &= \frac{\left|\omega_j-\omega_k\right|}{\Delta}, \quad M_{jk} = \frac{1}{S_{jk}} = \frac{\Delta}{\left|\omega_j-\omega_k\right|}, \end{aligned} \tag{25}$$

where M_{jk} is the modal overlap of the two modes (i.e., the ratio of the bandwidth to the modal spacing) and S_{jk} is the modal separation of the modes. The term μ_{jk} acts as a filter that determines which mode pairs contribute significantly to the response: μ_{jk} is close to unity if the modes overlap, but is small if they do not overlap. It can thus be approximated as

$$\mu_{j,k} = \begin{cases} 1, & \left|\omega_j-\omega_k\right| < \Delta, \\ 0, & \left|\omega_j-\omega_k\right| > \Delta. \end{cases} \tag{26}$$

3.3.3. *Participation factors* $\psi_{jk}^{(r)}$

The cross-mode participation factor $\psi_{jk}^{(r)}$ indicates the correlation of the (j–k)th mode pair within subsystem r. The self-term $\psi_{jj}^{(r)}$ gives the proportion of the modal kinetic energy stored in subsystem r and indicates the degree to which the jth mode is localized within that subsystem. Global modes of the system are those for which $\psi_{jj}^{(r)}$ is substantial for all (or at least many) of the subsystems, while local modes are those for which $\psi_{jj}^{(r)}$ is non-negligible in only one (or a few) of the subsystems.

Every system mode is orthogonal over the whole system, and therefore if there are N_s subsystems

$$\sum_{s=1}^{N_s} \psi_{jk}^{(s)} = \delta_{jk}. \tag{27}$$

The self-terms $\psi_{jj}^{(r)}$ are necessarily positive, while the cross-terms may be positive or negative.

3.3.4. *Modal density*

Suppose that the damping is light and the bandwidth is large enough so that the response is dominated by resonant modes. The power input to subsystem s is, from Eqs. (20) and (23),

$$P_{in}^{(s)} = S_f \frac{\pi}{2\Omega} \sum_j \psi_{jj}^{(s)}. \tag{28}$$

Now suppose that subsystem s is isolated from the rest of the system. It is known that the input power is proportional to the modal density when averaged over a wide enough bandwidth (e.g., Ref. [12]). In the notation of this paper

$$\mathrm{E}\left[P_{in}^{(s)}\right] = S_f \frac{\pi}{2} n_s(\omega), \tag{29}$$

where E[.] represents the expectation, or the asymptotic average, over many modes. This equation also follows from Eq. (28) by noting that of the system were to comprise just a single subsystem, then $\psi_{jj}^{(s)} = 1$—although the mode shapes will of course be the subsystems modes and are different from those when s forms part of the larger system—and the sum becomes the expected number of modes in the band.

In Ref. [12] it is also noted that Eq. (29) holds irrespective of the boundary conditions, and hence holds when s forms part of the larger system. By taking the expectation of Eq. (28), it follows that, asymptotically,

$$\mathrm{E}\left[\psi_{jj}^{(s)}\right] = v_s = \frac{n_s}{n_{tot}}, \tag{30}$$

where n_{tot} is the total modal density of the system (the sum of the modal densities of the individual subsystems, e.g. Ref. [13], Section 6] and where v_s is the fractional modal density of subsystem s, i.e., the ratio of the modal density of subsystem s to the total modal density of the system.

3.4. Some properties of the energy influence coefficients

Suppose, for convenience, that the damping is the same for all modes. The energy influence coefficients then satisfy

$$A_{rs} \geq 0; \qquad \sum_r A_{rs} = \frac{1}{\Delta}. \tag{31}$$

The latter result, that the column sums of $\mathbf{A}$ equal $1/\Delta$, is a consequence of conservation of energy. If the modes have different damping levels then similar relations can be developed for the dissipated powers rather than the subsystem energies directly.

If the damping is light, so that the contributions of the off-resonant out-of-band modes can be neglected, then

$$A_{rs} = \frac{1}{\Delta} \frac{\sum_j \sum_k \mu_{jk} \psi_{jk}^{(r)} \psi_{jk}^{(s)}}{\sum_j \psi_{jj}^{(s)}}, \tag{32}$$

where the sums now run over all modes with natural frequencies in the band Ω.

4. Statistical energy analysis and energy flow models

In the previous section expressions for the matrix $\mathbf{A}$ of energy influence coefficients were developed. The system was assumed to be linear and the excitations uncorrelated. ‘‘Rain’’ excitation with a spectral density independent of frequency was then assumed to act on each subsystem. Proportional damping was assumed, and further expressions were developed for the simple and convenient case where all modes in the excitation band have light and equal damping.

This section concerns the inverse matrix $\mathbf{X} = \mathbf{A}^{-1}$ and under what conditions it is a ‘‘quasi-SEA’’ matrix $\mathbf{L}$, as discussed in Sections 2.2.1 and 2.3. This indicates under what conditions an SEA-like model can be used to describe the behaviour of the system. Reasons why the response of a single system might differ from this SEA average are also discussed. Whether or not $\mathbf{L}$ is a proper-SEA matrix is considered in more detail in Ref. [14].

4.1. The existence of quasi-SEA matrices

For $\mathbf{L}$ to be an SEA-like matrix the rth column must sum to $\omega\eta_r$ and the off-diagonal elements must satisfy the consistency relation. $\mathbf{A}$ can be written as

$$\mathbf{A} = \frac{1}{\omega\eta}(\mathbf{I} - \boldsymbol{\alpha}), \tag{33}$$

where

$$\alpha_{rs} = \delta_{rs} - \frac{\sum_j \sum_k \mu_{jk} \psi_{jk}^{(r)} \psi_{jk}^{(s)}}{\sum_j \psi_{jj}^{(s)}}. \tag{34}$$

(Damping is now expressed in terms of loss factor for consistency with conventional SEA equations.) The columns of $\boldsymbol{\alpha}$ sum to zero (Eq. (27)). The inverse matrix $\mathbf{X} = \mathbf{A}^{-1}$ can therefore be written as

$$\begin{gathered} \mathbf{X} = \omega\eta\mathbf{I} + \omega\mathbf{C}, \\ \mathbf{C} = \eta\boldsymbol{\alpha}(\mathbf{I} - \boldsymbol{\alpha})^{-1} = \eta\left(\boldsymbol{\alpha} + \boldsymbol{\alpha}^2 + \boldsymbol{\alpha}^3 + \cdots\right). \end{gathered} \tag{35}$$

Firstly, since the columns of $\boldsymbol{\alpha}$ sum to zero then so do the columns of $\mathbf{C}$, as can be shown by premultiplying by the row vector $\left[*20c1 \quad 1 \quad \ldots \right]$. Secondly, $\boldsymbol{\alpha}$ can be written as

$$\boldsymbol{\alpha} = \boldsymbol{\beta}\ \mathrm{diag}\left(1/\sum_j \psi_{jj}^{(s)}\right), \qquad \beta_{rs} = \sum_j \psi_{jj}^{(s)} \delta_{rs} - \sum_j \sum_k \mu_{jk} \psi_{jk}^{(r)} \psi_{jk}^{(s)}, \tag{36}$$

where $\boldsymbol{\beta}$ is a symmetric matrix. It follows that $\boldsymbol{\alpha}^2, \boldsymbol{\alpha}^3, \ldots$ and hence $\mathbf{C}$ in Eq. (35) are also products of a symmetric matrix and the diagonal matrix in Eq. (36). If $\mathbf{X}$ is to be SEA-like, then $\mathbf{C}$ must be a matrix of coupling loss factors (Eq. (7)) with the off-diagonal element $C_{rs} = -\eta_{sr}$. The elements of $\mathbf{C}$ will only satisfy the SEA consistency relation (Eq. (5)), i.e.,

$$n_r C_{sr} = n_s C_{rs} \tag{37}$$

if the ratio $\sum_j \psi_{jj}^{(r)}/n_r$ is a constant for all subsystems, i.e., if the modes in the band are such that

$$\overline{\psi_{jj}^{(r)}} = \mathrm{E}\left[\psi_{jj}^{(r)}\right] = v_r, \tag{38}$$

where $\overline{\psi_{jj}^{(r)}}$ is the average value of $\psi_{jj}^{(r)}$ for all modes in the frequency band. Thus the consistency relation is satisfied for the band Ω if the average value $\overline{\psi_{jj}^{(r)}}$ for all modes in the band approximates the average $\mathrm{E}\left[\psi_{jj}^{(r)}\right]$ of Eq. (30) sufficiently well: there must be enough modes in the band and their mode shapes must be, on average, ‘typical enough’ of all the modes of the system, in terms of the distribution of kinetic energy throughout the system. Furthermore, since $\overline{\psi_{jj}^{(r)}}$ depends only on the mode shapes of the modes in Ω, then the consistency relation holds irrespective of the level of damping, and hence irrespective of the strength of coupling, whether it be strong or weak.

Thus $\mathbf{X}$ always satisfies conservation of energy, as one would expect, and also satisfies the consistency relation if the condition in Eq. (38) holds. These are the *necessary* conditions of an SEA-matrix and hence $\mathbf{X}$ is a “quasi-SEA” matrix if Eq. (38) holds. However, $\mathbf{X}$ does not necessarily satisfy all the *desirable* properties of SEA, so that it may not be a proper SEA matrix (indeed, in general it will not be). In particular:

- coupling loss factors can be negative;
- there may be indirect coupling loss factors, since C_{rs} need not be zero if subsystems r and s are not physically coupled;
- the coupling loss factors η_{rs} and η_{sr} generally depend on the *global* properties of the system rather than the *local* properties of the subsystems r and s (especially if the damping is light enough);
- the coupling loss factors generally depend on the damping loss factor η, especially if the damping is light enough.

Thus coupling power proportionality holds, and the system can be modelled using a quasi-SEA approach, but "indirect" coupling powers must be included. The above, however, gives a means for calculating the indirect coupling loss factors for the system.

To summarize, if certain rather mild conditions are satisfied, and particularly Eq. (38), then a quasi-SEA model of a system can be made, with the coupling loss factors being given in terms of the system modes. However, indirect coupling loss factors must in general be included.

4.2. SEA and the response of a single system

The subsystem energies $\hat{\mathbf{E}} = \hat{\mathbf{A}}\hat{\mathbf{P}}_{in}$ for a specific system under specific excitation over a specific frequency band may differ from the expressions derived above, for which various approximations and assumptions were made. Furthermore, the frequency band may not be large enough that the response can be described by a quasi-SEA model. This is the cause of variability in SEA predictions. The reasons behind this are briefly described here.

Firstly, the excitation may be frequency dependent, nor may it have the spatial dependence assumed above for "rain". Various modes may be excited preferentially, for example. Different modes may have different loss factors, so some may respond preferentially. Thirdly, the damping may be large enough, or the bandwidth narrow enough, such that non-resonant modes contribute substantially to the response. Finally, the damping may not be proportional, so that the response should strictly be described in terms of complex modes. These are all minor causes of variability and can, in principle, be accommodated.

A more fundamental cause of variability in the responses of individual systems arises from finite frequency band excitation and the statistics of those modes which lie in the band. The subsystem energies depend on $\psi_{jk}^{(s)}\psi_{jk}^{(r)}$ while the input power depends on $\psi_{jj}^{(s)}$ and hence the energy influence coefficients depend on both. Strictly, an ensemble of systems which differ in detail should be defined, with the ensemble average response being defined by $\mathbf{A}$. For a specific realization $\hat{\mathbf{A}} = \mathbf{A}(\mathbf{I} + \varepsilon)$, with $\mathrm{E}[\varepsilon] = 0$. The elements of $\boldsymbol{\varepsilon}$ are likely to be strongly correlated and non-Gaussian, especially if the bandwidth is small. If $\mathbf{A}^{-1} = \mathbf{L}$ is a quasi-SEA matrix then $\hat{\mathbf{A}}^{-1} = \mathbf{X}^{-1} = (\mathbf{I} + \boldsymbol{\varepsilon})^{-1}\mathbf{A}^{-1} = \left(\mathbf{I} - \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^2 ...\right)\mathbf{L}$. Ensemble averaging gives $\mathrm{E}\left[\hat{\mathbf{X}}\right] = \left(\mathbf{I} + \mathrm{E}\left[\boldsymbol{\varepsilon}^2\right] + \cdots\right)\mathbf{L}$. Note that this is biased, which raises some doubts over the accuracy of estimating the (ensemble average) coupling loss factor by averaging estimates found from individual systems. Further issues surrounding the variability of SEA estimates are the subject of further research.

5. Numerical examples

In this section some numerical examples are presented to illustrate the foregoing text. The system considered is shown in Fig. 1 and comprises four coupled rods undergoing axial vibrations. The rods are uniform and identical except for their lengths and cross-sectional areas and the lengths are chosen so that the ratio of the lengths of any two rods is irrational. The values used are shown in Table 1, with the lengths of the rods given to four decimal places. The total modal density of the system $n_{tot} = 1$, so that the mode count is approximately equal to the frequency ω. The bandwidth $\omega\eta$ is thus numerically equal to the modal overlap. The system modes are found using a component modal approach as described in Ref. [15].

Fig. 2 shows the modal energy participation factors $\psi_{jj}^{(r)}$ for the first 100 modes for all subsystems (i.e., $j = 1, \ldots, 100, r = 1, \ldots, 4$). Also shown are the fractional modal densities ν_r. While there is clear variation from mode to mode, most system modes tend to be somewhat global, in that the response is significant in many subsystems. There is more variation for subsystem 1, which has both the lowest modal density and the largest cross-sectional area and hence the largest characteristic impedance.

Closer study shows that $\psi_{jj}^{(r)}$ are more-or-less uniformly distributed, while there is a small and generally negative correlation between $\psi_{j+1,\,j+1}^{(r)}$ and $\psi_{jj}^{(r)}$. Together these imply that the average $\overline{\psi_{jj}^{(r)}}$ will converge at least fairly quickly towards the expected value.

Fig. 3 shows the frequency average $\overline{\psi_{jj}^{(r)}}$ over a frequency band centred on the 50th natural frequency as a function of the number of modes contained in the band. This illustrates how the averages asymptote to the fractional modal densities as the number of modes increases. This convergence also indicates, according to the criterion of Eq. (38), whether one is able to model the system using a quasi-SEA approach — for this centre frequency, one might expect a quasi-SEA approach matrices to exist for bandwidths including more than, say, 10 system modes.

The remaining results show direct and indirect coupling loss factors for a third octave band centred on the 50th global natural frequency. This contains 13 system modes. Over this frequency band the bandwidth $\Delta = \omega\eta$ of the modes is held constant, so that the modal overlap does not depend on frequency. The calculated results are approximate, in that non-resonant (i.e. out-of-band) modes are ignored, while the approximate expressions of Eq. (24) are used for the terms Γ_{jk} rather than the exact results [5]. Errors introduced by this approximation are only significant for the higher values of $\omega\eta$, where the modal overlap becomes large. For a one-dimensional system such as this, large modal overlap implies a rapid spatial decay of response across the system, with the subsystems no longer being reverberant. Nevertheless, a quasi-SEA approach can still be adopted.

Figs. 4 and 5 show $\omega n_r \eta_{sr}$ as a function of $\omega\eta$, some of the indirect coupling loss factors being negative. In all cases $\omega n_r \eta_{sr}$ and $\omega n_s \eta_{sr}$ are equal, to a very good approximation, indicating that

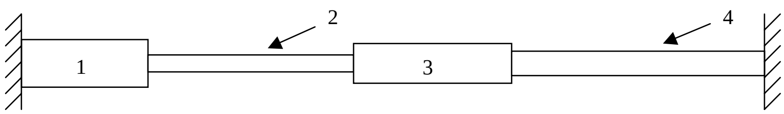

Fig. 1. System comprising four coupled rods.

Table 1
Rod properties (arbitrary units)

Rod	Area	Length	ν
1	1	0.5171	0.1646
2	0.4	0.8885	0.2828
3	0.9	0.7072	0.2251
4	0.6	1.0288	0.3275

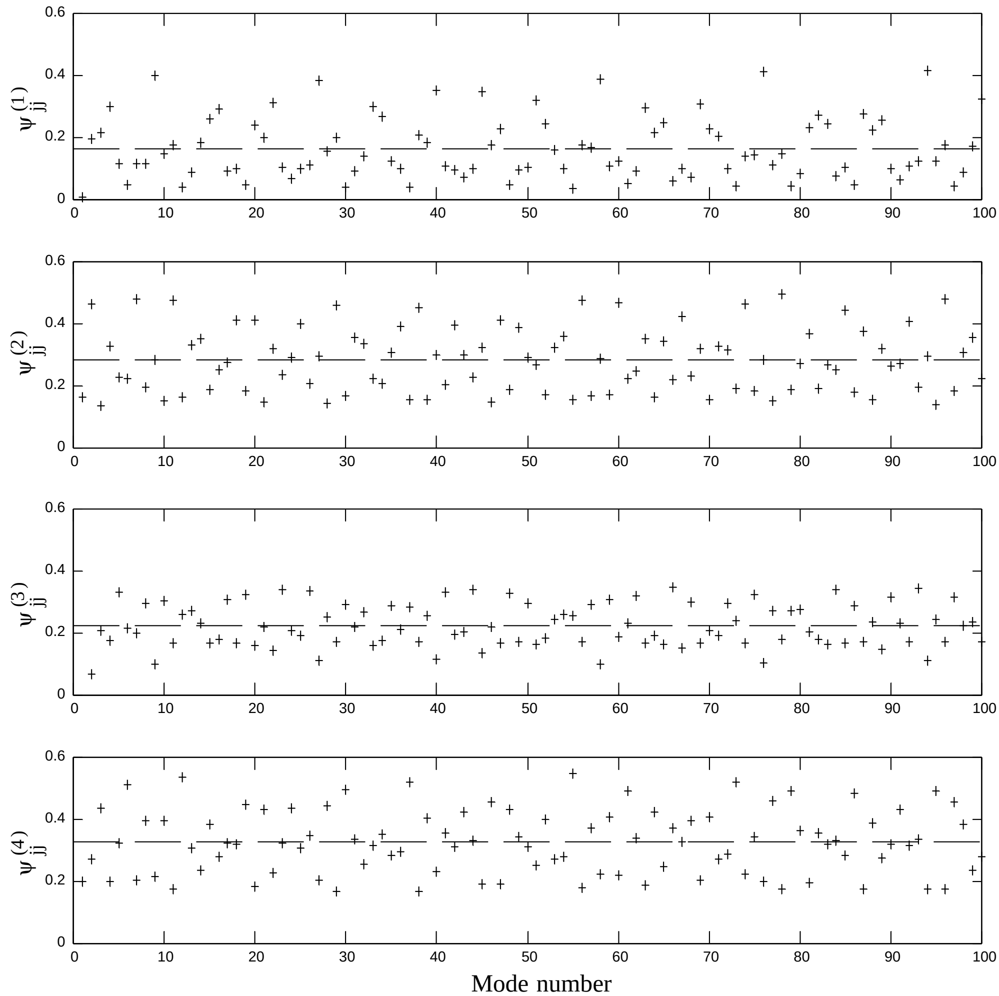

Fig. 2. Participation factors $\psi_{jj}^{(r)}$ for the rods, first 100 modes, and (- - -) fractional modal densities.

the SEA consistency relation of Eq. (5) holds, even for the indirect coupling loss factors. This is expected from Fig. 3 and Eq. (38). For small damping the coupling loss factors are proportional to η, because the sums in the numerator of Eq. (32) are dominated by the 'self' terms $\psi_{jj}^{(r)}\psi_{jj}^{(s)}$, the

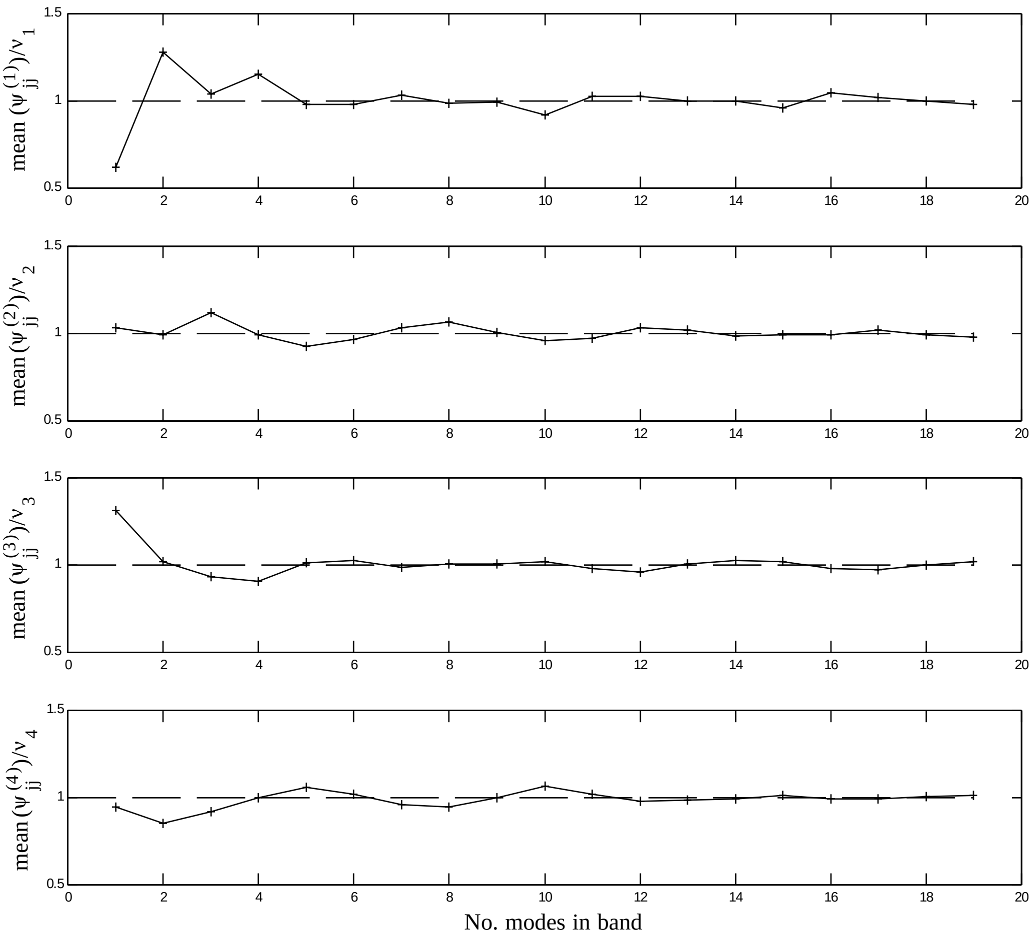

Fig. 3. Ratio of frequency average participation factors $\overline{\psi_{jj}^{(r)}}$ to fractional modal density ν_r for a frequency band centred on the 50th natural frequency as a function of the number of modes in the band and (- - -) fractional modal densities.

cross terms being negligible since the modal bandwidth is small and hence so, too, is Γ_{jk}. Thus $\mathbf{A}$ is proportional to $1/\omega\eta$ and hence $\mathbf{X} = \mathbf{A}^{-1}$ and the coupling loss factor matrix $\mathbf{C}$ are proportional to η. Similar behaviour has been observed before with analyses proceeding along wave [10,11] or FE/modal lines [3,4]. As damping increases the coupling loss factors reach a peak and tend towards a value close to the asymptotic wave expression [1] (differences arise because the contributions from out-of-band modes become substantial as modal overlap becomes large). This indicates the transition from strong to weak coupling, as characterized by the parameter γ of Ref. [10], as $\omega\eta$ increases. The dependence of the coupling loss factors on modal overlap is considered further in Ref. [14].

The indirect coupling loss factors are smaller than the direct coupling loss factors, but are by no means negligible for low damping (and hence strong coupling). For moderate and high damping the 'more distant' indirect coupling loss factors η_{14} and η_{41} are smaller than and asymptote to zero substantially more rapidly than the 'neighbouring' indirect coupling loss factors η_{13}, η_{24}, etc. This

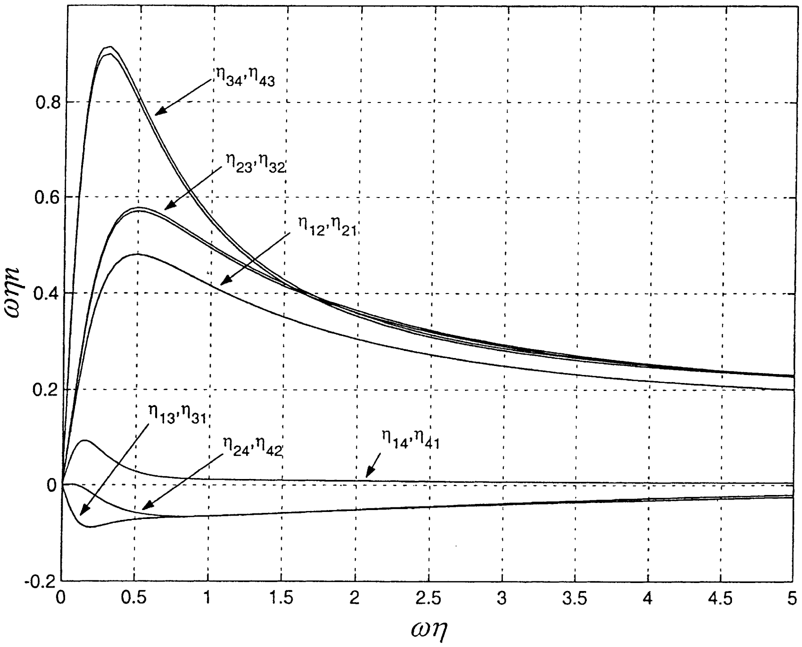

Fig. 4. Direct and indirect coupling loss factors: $\omega n_r \eta_{rs}$ as a function of $\omega\eta$.

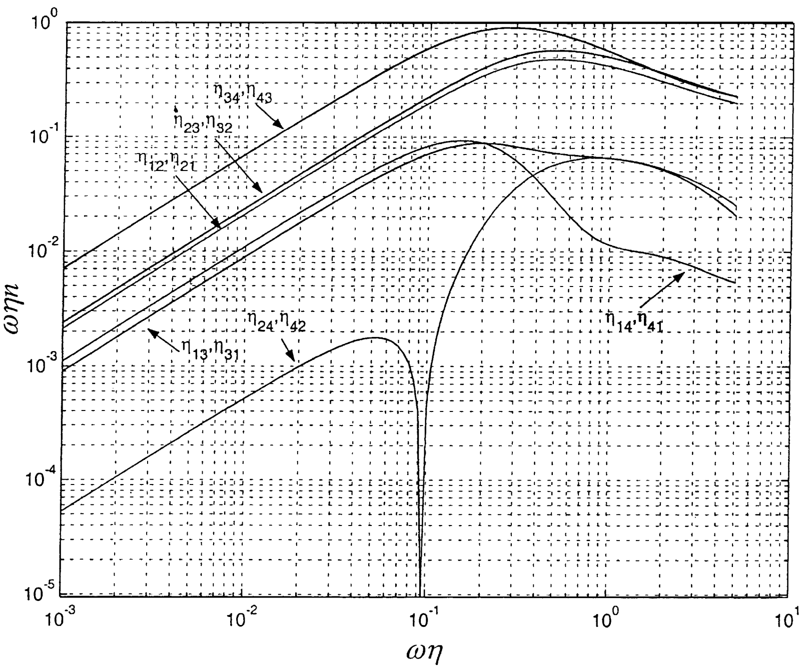

Fig. 5. Magnitudes of direct and indirect coupling loss factors: $\omega n_r \eta_{rs}$ as a function of $\omega\eta$.

behaviour might be expected since the correlation between mode shapes in two subsystems would be expected to decrease as the separation of the subsystems increases.

6. Concluding remarks

In this paper, expressions were derived for the energy influence coefficients in terms of the modes of the system. From this it was seen that an SEA-like model can be formed if the condition given in Eq. (38) is satisfied to acceptable accuracy. In a sense this accords with the practical observation that SEA requires there to be a sufficient mode count in the analysis band. However, if Eq. (38) is satisfied, then it holds irrespective of the level of damping (and hence the strength of coupling) in the system: the system can be modelled by SEA irrespective of the strength of coupling. However, the resulting model need not be of proper-SEA form: the indirect coupling loss factors may not be negligible, especially if the damping is small enough such that the coupling is strong.

In the foregoing it was assumed that the system is linear and that the excitations applied to the different subsystems are uncorrelated. These assumptions are necessary.

Various other assumptions were made, primarily for convenience—these can be relaxed if required, but the resulting expressions become more complicated. The damping was assumed to be light, proportional and viscous. In many of the equations the response was assumed to be dominated by resonant modes. For example, the total energy was assumed to equal twice the kinetic energy. Simple expressions were developed for resonant contributions by extending the frequency limits of integration to $(0, \infty)$ and assuming all modes have the same bandwidth. Finally "rain-on-the-roof" excitation was assumed to act, and was defined to be frequency independent, spatially delta-correlated and proportional to the local mass density.

The modal formulation provides a method by which direct and indirect coupling loss factors can be calculated and by which the dependence of these parameters on the modal overlap can be investigated [14]. Some numerical examples were given.

The system modes can be used to determine the direct and indirect coupling loss factors. This requires knowledge of those modes, however, which will often make the approach intractable in a practical application. However, at least the existence of an SEA model can be inferred. It also begs the question as to what extent "fairly local" parameters (e.g. direct and next-neighbour indirect coupling loss factors) depend on "distant" system properties—this is currently under investigation.

Finally, the analysis here is deterministic: no ensemble averaging is involved. One might hope, however, that the statistics of the modes of similar systems might be similar, even if the details of those modes differ.

References

[1] R.H. Lyon, R.G. DeJong, Theory and Application of Statistical Energy Analysis, 2nd Edition, Butterworth-Heinemann, Boston, 1995.

[2] J.L. Guyader, C. Boisson, C. Lesueur, Energy transmission in finite coupled plates, Part I: theory, Journal of Sound and Vibration 81 (1982) 81–92.

[3] F.F. Yap, J. Woodhouse, Investigation of damping effects on statistical energy analysis of coupled structures, Journal of Sound and Vibration 197 (1996) 351–371.
[4] B.R. Mace, J. Rosenberg, The SEA of two coupled plates: an investigation into the effects of subsystem irregularity, Journal of Sound and Vibration 212 (1998) 395–415.
[5] B.R. Mace, P.J. Shorter, Energy flow models from finite element analysis, Journal of Sound and Vibration 233 (2000) 369–389.
[6] R.S. Langley, A general derivation of the statistical energy analysis equations for coupled dynamic systems, Journal of Sound and Vibration 135 (1989) 499–508.
[7] R.S. Langley, A derivation of the coupling loss factors used in statistical energy analysis, Journal of Sound and Vibration 141 (1990) 207–219.
[8] S. Finnveden, Ensemble averaged vibration energy flows in a three-element structure, Journal of Sound and Vibration 187 (1995) 495–529.
[9] C.R. Fredo, Statistical Energy Analysis and the Individual Case, Ph.D. Thesis, Chalmers University of Technology, 1995.
[10] B.R. Mace, The statistical energy analysis of two continuous one-dimensional subsystems, Journal of Sound and Vibration 166 (1993) 429–461.
[11] E.C.N. Wester, B.R. Mace, Statistical energy analysis of two, edge-coupled rectangular plates — ensemble averages, Journal of Sound and Vibration 193 (1996) 793–822.
[12] L. Cremer, M. Heckl, E.E. Ungar, Structure-Borne Sound, 2nd Edition, Springer, Berlin, 1988.
[13] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol. 1. Interscience, New York, 1953.
[14] B.R. Mace, The dependence of SEA parameters on modal overlap, Proceedings of ISMA 2002, International Conference on Noise and Vibration, Leuven, Belgium, 2002.
[15] B.R. Mace, P.J. Shorter, A local modal/perturbational method for estimating frequency response statistics of built-up structures with uncertain parameters, Journal of Sound and Vibration 242 (2001) 793–811.