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Letter to the Editor

Stability of a cracked Timoshenko beam column by modified Fourier series

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1. Introduction

Knowing the stability characteristics of a damaged structure is of significant importance in engineering. In civil engineering, structures like beam columns, bridges, piles, etc. will bear damages due to long-term service, collision, impact, etc. An important task of engineers is to determine the effect of these damages on the stability characteristics of these structures. In the past two decades, studies on dynamic/stability behaviour of damaged structures have attracted many researchers. Many relevant literatures have been published. Amongst them, the easily accessible are Refs. [1–9], etc.

Conceptually, the simulation of a cracked beam-column is analogous to that of a beam column with stepped changes of cross-sections and/or with intermediate point supports. Recently, the modified Fourier series (MFS) and the modified beam vibration functions (MBVF) were developed and have been successfully used in solving the vibration and stability problems of structures with stepped cross-sections and/or intermediate point supports [10–15].

Most recently, Zheng et al. [16–18,20] further developed the MFS and used it successfully in the vibration and stability analysis of cracked Euler beams [16,17,20] and vibration analysis of cracked Timoshenko beams [18].

In this paper, a new method is developed for computing the buckling load reduction of a Timoshenko beam column with an arbitrary number of transverse open cracks. The essence of this new method lies in the use of a kind of MFS that is developed specially for the analysis of a beam column with arbitrary number of transverse open cracks. Unlike the conventional Fourier series, the modified series is able to approach a function with internal geometrical discontinuities effectively. Based on the present MFS, one can treat the cracked beam column in the most usual way and thus reduce the problem to be a simple one. As can be seen from the stiffness matrix in stability equation (37), the extra effort needed is just to add the $\mathbf{K}_4$ matrix to the stiffness matrix of

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the beam column. In the present method, only standard linear eigenvalue equations, rather than non-linear algebraic equations, need to be solved. Since this new method falls within the frame of continuous methods, its capability of achieving higher accuracy is expected. Moreover, all the formulae are expressed in a unified way and in matrix form, which renders the computer coding quite straightforward. To demonstrate the effectiveness and accurateness of the present method, several numerical examples are shown.

2. Theory and formulation

2.1. Modified Fourier series $Y_m(y)$

Fig. 1 shows a beam column having $(Q-1)$ number of transverse open cracks located at $y = y_2, y_3, \ldots, y_Q$, N point-spring supports located at $y = s_1, s_2, \ldots, s_N$ and having a continuous elastic support $k_f(y)$, respectively. The beam column can have non-uniform cross-sectional areas $A(y)$ and various second moments of area $I(y)$ along the longitudinal y direction. The depths of the cracks are $\{a_i,\ i=1,2,\ldots,Q-1\}$, $a_i \geqslant 0$, and the translational-and-rotational stiffness of the point springs are $\{k_i, \chi_i, i = 1, 2, \ldots, N\}$. The point springs are introduced here for the purpose of modelling the boundary supports and the intermediate point supports, if any.

The transverse deflection and the rotation of cross-section of the Timoshenko beam column are denoted by $w(y)$ and $\psi(y)$, respectively, where y stands for the location. Considering the continuity

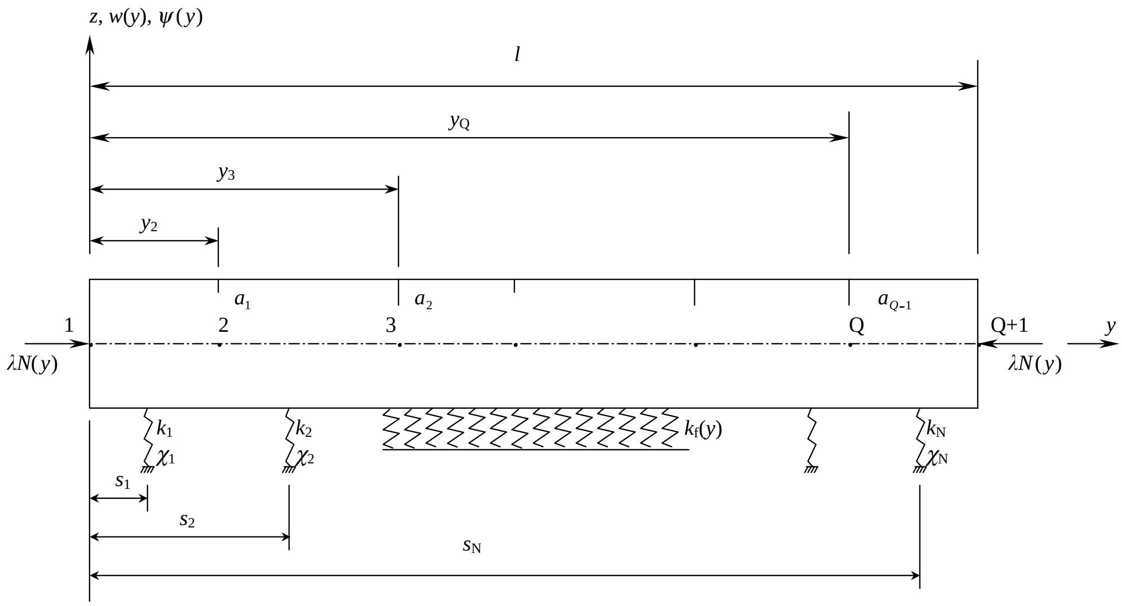

Fig. 1. An axially compressed Timoshenko beam having $(Q-1)$ number of cracks located at $y = y_2, y_3, \ldots, y_Q$, N spring supports located at $y = s_1, s_2, \ldots, s_N$ and a continuous elastic support $k_f(y)$.

of the function $w(y)$ and discontinuity of the function $\psi(y)$, we can express them as follows:

$$w(y) = \sum_{m=1}^{R} w_m \bar{Y}_m(y) = \bar{\mathbf{H}}(y)\mathbf{q}_1 \quad (R = 2r + 1), \tag{1}$$

$$\psi(y) = \sum_{m=1}^{R} \psi_m Y_m(y) = \mathbf{H}(y)\mathbf{q}_2 \quad (R = 2r + 1), \tag{2}$$

where

$$\bar{\mathbf{H}}(y) = [\,\bar{Y}_1(y) \quad \bar{Y}_2(y) \quad \ldots \quad \bar{Y}_R(y)], \quad \mathbf{H}(y) = [\,Y_1(y) \quad Y_2(y) \quad \ldots \quad Y_R(y)], \tag{3,4}$$

$$\mathbf{q}_1 = [\,w_1 \quad w_2 \quad \ldots \quad w_R\,]^{\mathrm{T}}, \quad \mathbf{q}_2 = [\,\psi_1 \quad \psi_2 \quad \ldots \quad \psi_R\,]^{\mathrm{T}}. \tag{5,6}$$

In the above equations, w_m and ψ_m are the generalized co-ordinates of deformation for the beam column; $\bar{Y}_m(y)$ is the Fourier series base function [19] and $Y_m(y) = \bar{Y}_m(y) + \tilde{Y}_m(y)$ ($\tilde{Y}_m(y)$ is the augmenting piece-wise constant function shown in Fig. 2) is the so-called modified Fourier function which is specifically constructed such that it can satisfy the following internal discontinuities [18]:

$$Y_m(y_j + 0) - Y_m(y_j - 0) = c_{j-1} EIY'_m(y \to y_j), \tag{7}$$

where c_{j-1} is the local flexibility coefficient of the crack having a depth of a_{j-1}. For rectangular cross-section and one-sided cracks, it can be expressed as [20]

$$\begin{aligned} C \approx {} & (h/12EI)\mathrm{e}^{1/(1-\zeta)}(-0.2314 \times 10^{-4}\zeta + 52.3790\zeta^2 - 130.2463\zeta^3 + 308.4111\zeta^4 \\ & - 602.1761\zeta^5 + 937.6805\zeta^6 - 1306.7397\zeta^7 + 1398.7523\zeta^8 - 1059.6215\zeta^9 + 388.1628\zeta^{10}) \\ & (\zeta = a/h,\ 0 \leqslant \zeta \leqslant 0.5,\ \text{error} < 0.038\%). \end{aligned} \tag{8}$$

For circular cross-section and one-sided cracks, it can be expressed as [20]

$$\begin{aligned} C \approx {} & (\pi D/64EI)\mathrm{e}^{1/(1-\zeta)}(0.1687 \times 10^{-3}\zeta^{0.4} - 0.9770 \times 10^{-2}\zeta^{0.8} + 0.2382\zeta^{1.2} \\ & - 3.2016\zeta^{1.6} + 25.5385\zeta^2 - 58.1428\zeta^{2.4} + 679.8828\zeta^{2.8} - 1350.4090\zeta^{3.2} \\ & + 794.0302\zeta^{3.6} - 11.3371\zeta^4 \\ & (\zeta = a/D,\ 0 \leqslant \zeta \leqslant 0.1,\ \text{error} < 0.045\%) \end{aligned} \tag{9a}$$

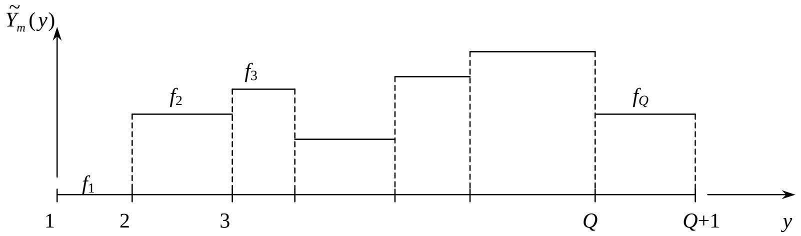

Fig. 2. Augmenting piece-wise constant function $\tilde{Y}_m(y)$.

and

$$C \approx (\pi D/64EI)e^{1/(1-\zeta)}(5.4931\zeta^{0.4} - 60.0706\zeta^{0.8} + 249.0679\zeta^{1.2} - 437.5001\zeta^{1.6} + 172.6435\zeta^{2} - 55.5990\zeta^{2.4} + 3036.1620\zeta^{2.8} - 7991.3829\zeta^{3.2} + 7992.1873\zeta^{3.6} - 2934.3483\zeta^{4} \quad (\zeta = a/D,\ 0.1 \leqslant \zeta \leqslant 0.5,\ \text{error} < 0.0064\%). \tag{9b}$$

2.2. Energy analysis

The potential energy of a cracked Timoshenko beam column under axial load can be expressed as the summation of the following five parts:

$$U = U_1 + U_2 + U_3 + U_4 + U_5 \tag{10}$$

in which U_1 and U_2 are the potential energies stored in the cracked beam column due to bending and shearing deformation of the beam column itself; U_3 is the potential energy stored in the distributed elastic support $k_f(y)$ and point springs which are used to model the boundary supports and also the intermediate supports (if any); U_4 is the potential energy stored in the massless rotational springs which are used to model the existence of cracks; U_5 is the potential energy of the external variable axial load $\lambda N(y)$.

Potential energy U_1:

$$U_1 = \sum_{i=1}^{Q} \frac{1}{2} \int_{y_i}^{y_{i+1}} EI(y)\psi_{,y}^2(y)\, dy. \tag{11}$$

Substituting Eq. (2) into Eq. (11), we have

$$U_1 = \tfrac{1}{2}\mathbf{q}_2^{\mathrm{T}}\mathbf{K}_1\mathbf{q}_2, \tag{12}$$

where $\mathbf{K}_1$ represents the stiffness matrix of the cracked beam column corresponding to the potential energy U_1 such that

$$K_1 = \sum_{i=1}^{Q} \int_{y_i}^{y_{i+1}} EI(y)\mathbf{H}_{,y}^{\mathrm{T}}(y)\mathbf{H}_{,y}(y)\, dy = \int_0^l EI(y)\bar{\mathbf{H}}_{,y}^{\mathrm{T}}(y)\bar{\mathbf{H}}_{,y}(y)\, dy. \tag{13}$$

Potential energy U_2:

$$U_2 = \frac{1}{2}\int_0^l k'GA(y)(w_{,y} - \psi)^2\, dy. \tag{14}$$

Substituting Eqs. (1) and (2) into Eq. (14), we have

$$U_2 = \tfrac{1}{2}\mathbf{q}_1^{\mathrm{T}}\mathbf{K}_{21}\mathbf{q}_1 - \mathbf{q}_1^{\mathrm{T}}\mathbf{K}_{22}\mathbf{q}_2 + \tfrac{1}{2}\mathbf{q}_2^{\mathrm{T}}\mathbf{K}_{23}\mathbf{q}_2, \tag{15}$$

where

$$\mathbf{K}_{21} = \int_0^l k'GA(y)\bar{\mathbf{H}}_{,y}^{\mathrm{T}}(y)\bar{\mathbf{H}}_{,y}(y)\, dy, \tag{16}$$

$$\mathbf{K}_{22} = \int_0^l k'GA(y)\bar{\mathbf{H}}_{,y}^{\mathrm{T}}(y)\mathbf{H}(y)\, dy, \tag{17}$$

$$\mathbf{K}_{23} = \int_0^l k' GA(y)\bar{\mathbf{H}}^{\mathrm{T}}(y)\mathbf{H}(y)\,\mathrm{d}y. \tag{18}$$

Potential energy U_3:

$$U_3 = \sum_{i=1}^{N} \tfrac{1}{2}[k_i w^2(s_i) + \chi_i \psi^2(s_i)] + \int_0^l \tfrac{1}{2} k_f(y) w^2(y)\,\mathrm{d}y. \tag{19}$$

Substituting Eqs. (1) and (2) into Eq. (19), we have

$$U_3 = \tfrac{1}{2}\mathbf{q}_1^{\mathrm{T}}\mathbf{K}_{31}\mathbf{q}_1 + \tfrac{1}{2}\mathbf{q}_2^{\mathrm{T}}\mathbf{K}_{32}\mathbf{q}_2, \tag{20}$$

where

$$\mathbf{K}_{31} = \sum_{i=1}^{N} k_i \bar{\mathbf{H}}^{\mathrm{T}}(s_i)\bar{\mathbf{H}}(s_i) + \int_0^l k_f(y)\bar{\mathbf{H}}^{\mathrm{T}}(y)\bar{\mathbf{H}}(y)\,\mathrm{d}y, \tag{21}$$

$$\mathbf{K}_{32} = \sum_{i=1}^{N} \chi_i \mathbf{H}^{\mathrm{T}}(s_i)\mathbf{H}(s_i). \tag{22}$$

Potential energy U_4:

$$U_4 = \sum_{j=2}^{Q} \frac{1}{2}\left[\frac{1}{c_{j-1}}\right][\psi(y_j + 0) - \psi(y_j - 0)]^2. \tag{23}$$

Substituting Eq. (2) into Eq. (23), we have

$$U_4 = \tfrac{1}{2}\mathbf{q}_2^{\mathrm{T}}\mathbf{K}_4\mathbf{q}_2, \tag{24}$$

where $\mathbf{K}_4$ represents the stiffness matrix of the cracked beam column corresponding to the potential energy U_4 such that

$$\mathbf{K}_4 = \sum_{j=2}^{Q} \{c_{j-1}[EI(y_j)]^2\}\bar{\mathbf{H}}_{,y}^{\mathrm{T}}(y_j)\bar{\mathbf{H}}_{,y}(y_j). \tag{25}$$

Potential energy U_5:

$$U_5 = -\tfrac{1}{2}\lambda \int_0^l N(y) w_{,y}^2(y)\,\mathrm{d}y. \tag{26}$$

Substituting Eq. (1) into Eq. (26), we have

$$U_5 = -\tfrac{1}{2}\mathbf{q}_1^{\mathrm{T}}\mathbf{K}_G\mathbf{q}_1, \tag{27}$$

where $\mathbf{K}_\mathrm{G}$ represents the stiffness matrix of the cracked beam column corresponding to the potential energy U_5 such that

$$\mathbf{K}_G = \int_0^l N(y)\bar{\mathbf{H}}_{,y}^{\mathrm{T}}\bar{\mathbf{H}}_{,y}\,\mathrm{d}y. \tag{28}$$

Finally, substituting Eqs. (12), (15), (20), (24) and (27) into Eq. (10), we obtain the total potential energy of the cracked-beam-column system,

$$U = \tfrac{1}{2}\mathbf{q}_1^{\mathrm{T}}(\mathbf{K}_{21} + \mathbf{K}_{31} + \mathbf{K}_f - \lambda\mathbf{K}_G)\mathbf{q}_1 + \tfrac{1}{2}\mathbf{q}_2^{\mathrm{T}}(\mathbf{K}_1 + \mathbf{K}_{23} + \mathbf{K}_{32} + \mathbf{K}_4)\mathbf{q}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{K}_{22}\mathbf{q}_2. \tag{29}$$

2.3. Stability equations

The equilibrium equations of the cracked Timoshenko beam-column are:

$$\frac{\partial U}{\partial \mathbf{q}_1} = \mathbf{0}, \tag{30}$$

$$\frac{\partial U}{\partial \mathbf{q}_2} = \mathbf{0}. \tag{31}$$

Substituting Eq. (29) into Eqs. (30) and (31), we have

$$(\mathbf{K}_{21} + \mathbf{K}_{31} - \lambda\mathbf{K}_G)\mathbf{q}_1 - \mathbf{K}_{22}\mathbf{q}_2 = \mathbf{0}, \tag{32}$$

$$(\mathbf{K}_1 + \mathbf{K}_{23} + \mathbf{K}_{32} + \mathbf{K}_4)\mathbf{q}_2 - \mathbf{K}_{22}^{\mathrm{T}}\mathbf{q}_1 = \mathbf{0}. \tag{33}$$

Eqs. (32) and (33) can be written into one matrix equation,

$$\mathbf{K}\mathbf{q} = \mathbf{0}, \tag{34}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{21} + \mathbf{K}_{31} - \lambda\mathbf{K}_G & -\mathbf{K}_{22} \\ -\mathbf{K}_{22}^{\mathrm{T}} & \mathbf{K}_1 + \mathbf{K}_{23} + \mathbf{K}_{32} + \mathbf{K}_4 \end{bmatrix}, \tag{35}$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}. \tag{36}$$

Eliminating $\mathbf{q}_2$ in Eq. (34), we have the following stability equation:

$$\lfloor \mathbf{K}_{21} + \mathbf{K}_{31} - \mathbf{K}_{22}(\mathbf{K}_1 + \mathbf{K}_{23} + \mathbf{K}_{32} + \mathbf{K}_4)^{-1}\mathbf{K}_{22}^{\mathrm{T}} \rfloor \mathbf{q}_1 = \lambda\mathbf{K}_G\mathbf{q}_1. \tag{37}$$

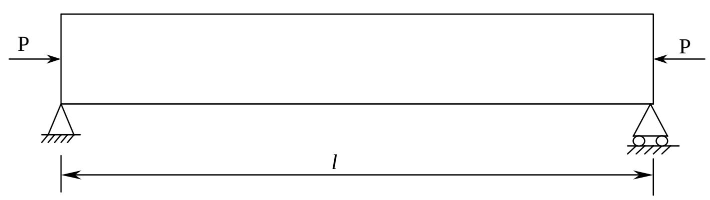

Fig. 3. A simply supported beam subjected to axial compression load.

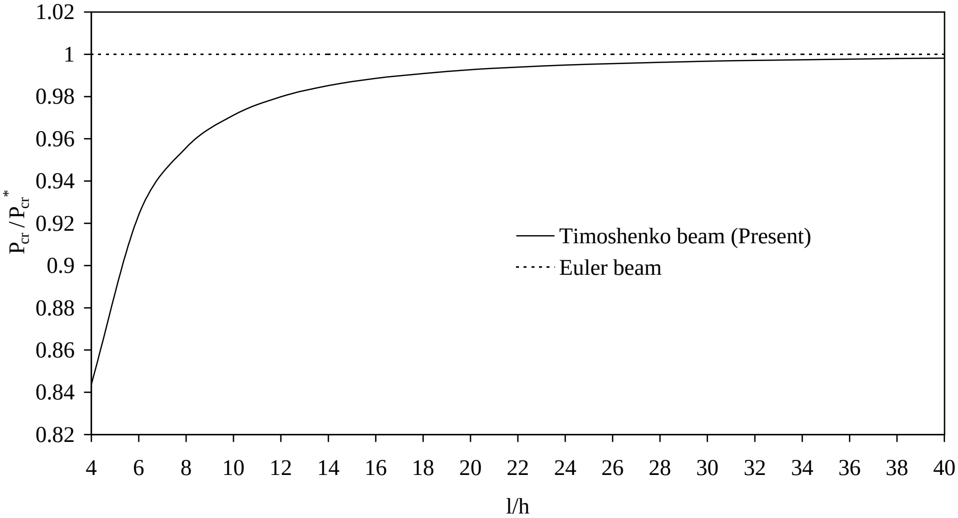

Fig. 4. Effect of the slenderness ratio on the buckling load of a simply supported beam without any cracks.

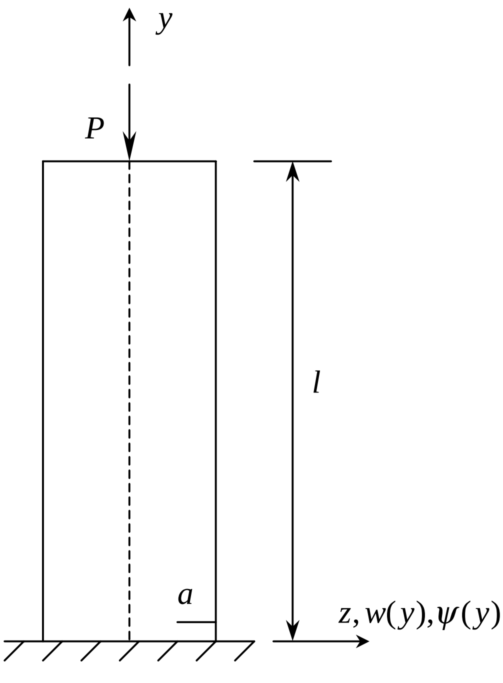

Fig. 5. A column with a crack at its bottom.

Eq. (37) is a standard generalized linear eigenvalue equation that can be solved by standard programs. It is also worth noting that the matrix $\mathbf{K}_4$ represents the cracks' effect on the stiffness of the beam column.

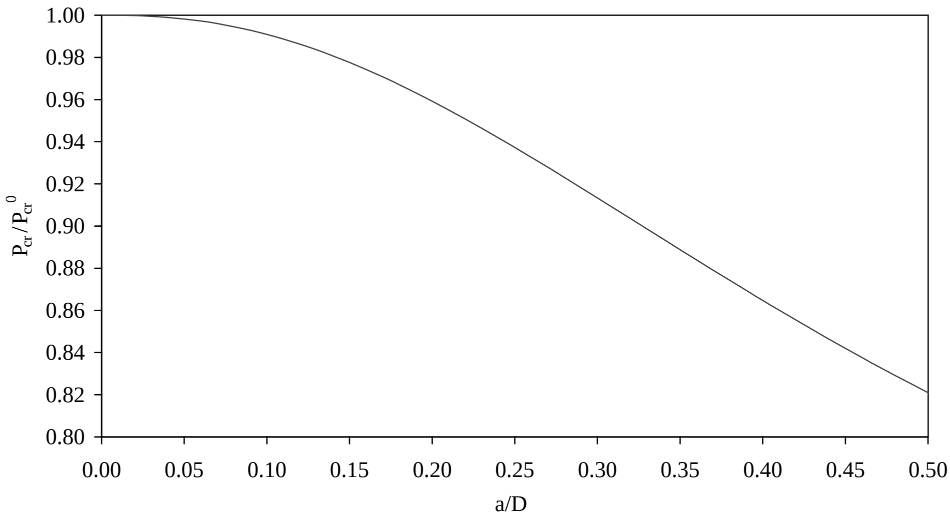

Fig. 6. Buckling load reduction versus relative crack depth of a column with one open crack at its bottom.

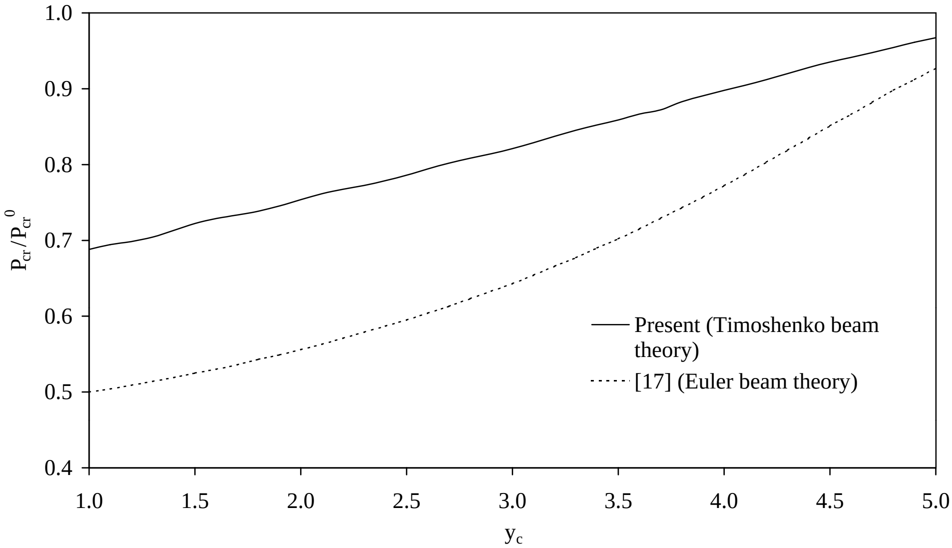

Fig. 7. Buckling load reduction versus crack location of a column.

3. Numerical examples

3.1. Example 1. Effect of slenderness on the buckling load for a pin–pin column without cracks

For verification of the proposed theory and the relevant program coding, we first computed the buckling loads of a pin–pin column without any cracks (Fig. 3). The beam column has rectangular cross-section with width b and height h $(b = h)$. The effect of slenderness ratio (l/h) on the buckling loads of the beam column was studied by the present method. The results are shown in Fig. 4. In Fig. 4, the vertical axis stands for the ratio of the buckling loads of Timoshenko column and Euler column over the value $P^*_{cr} = \pi^2 EI/l^2$; the horizontal axis stands for the slenderness ratio (l/h). From Fig. 4 we can see that, in the case of slender column, when l/h is greater than 10, the difference between the Timoshenko column and the Euler column is insignificant. But, in the case of short columns, i.e., when the slender ratio l/h decreases, the shear deformation in the column becomes more significant and must be taken into account.

3.2. Example 2. A column with a crack at the bottom

Fig. 5 shows a column ($l = 6\,\text{m}$, $D = 1\,\text{m}$; $E = 28\,\text{GPa}$, $\mu = 0.2$) with an open crack at its bottom. The depth of the crack is variable. The computed results are shown in Fig. 6. In Fig. 6, the vertical axis stands for the buckling load reduction while the horizontal axis stands for the relative crack depth.

3.3. Example 3. A column with a crack with variable location

The same column as in Example 2 is considered. The column has an open crack with depth $a = 0.5\,\text{m}$ but variable location. The computed results are shown in Fig. 7. The same problem can also be analyzed by using Euler beam theory [17] and the corresponding results are also shown in Fig. 7.

4. Conclusions

A new Modified Fourier Series (MFS) was presented. It was developed to tackle the problem in beam columns with an arbitrary number of open cracks. The Modified Fourier Series can approach a function with internal geometrical discontinuities effectively. Via the Euler–Lagrangian approach, we can treat the stability analysis of a cracked beam column in the most usual way. It thus renders the problem-solving procedures rather simple. In the formulation, an open crack assumes having stiffness, which is simply added to the stiffness matrix of the beam column. The beam column can be of non-uniform cross-section and the number of cracks can be arbitrary. In solving the buckling load of a cracked beam column, only a standard linear eigenvalue equation needs to be solved. All the formulae are expressed in matrix form and therefore computer coding is straightforward. Numerical examples showed that the present method is versatile and effective.

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