Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 264 (2003) 485–490
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

A new zig-zag model for laminated composite beams: free vibration analysis

Hemendra Arya*

Aerospace Engineering Department, Center For Aerospace Systems Design and Engineering, Indian Institute of Technology, Powai, Mumbai 400 076, India

Received 25 March 2002; accepted 11 November 2002

1. Introduction

High specific strength and specific modulus makes the polymer-matrix composites a promising candidate for high performance structures. The low value of shear modulus of composites as compared to conventional materials becomes a major issue in the analysis of the structures made out of such materials. This results in high transverse shear deformation and it is to be taken care at formulation level itself. The classical laminated beam theory does not model transverse shear deformation and is only appropriate for beams of high aspect ratio where it is not prominent.

Lo et al. [1,2] proposed a higher order displacement model consisting of third order polynomial in thickness direction for analysis of composite plates. This displacement model has been widely used with modifications to satisfy transverse shear stress boundary condition at the top and bottom of the plate [3,4]. In such models the displacement and its slope with respect to the thickness direction is continuous. This property of the displacement field leads to discontinuous transverse shear stress, and is also not able to model zig-zag nature of the displacement field in thick laminated composite structures. These are also known as equivalent single layer (ESL) theories.

In the layerwise displacement models, the displacement fields are layer dependent and are able to maintain in-plane displacement and shear stress continuity at the interface. The number of variables in many of these models are independent of number of layers [5,6]. In this note, a displacement model is presented in which the number of variables are independent of layers. The displacement field consists of a trigonometric sine term to represent the non-linear variation across the thickness in addition to classical beam theory terms. In conventional theories, a cubic term is used to represent this. The sine term in in-plane displacement leads to a cosine variation in transverse shear stress as compared to parabolic variation in conventional theories. A simplified ESL model of a similar class is also presented and results are compared. Stein [7] has used a

*Corresponding author. Tel.: +91-22-2576-7840; fax: +91-22-2572-9511.
E-mail address: arya@aero.iitb.ac.in (H. Arya).

doi:10.1016/S0022-460X(02)01489-X

similar plate theory for the post-buckling analysis of plate, his theory does not satisfy the transverse shear stress condition at top and bottom of the plate. The efficacy of the present displacement model in case of static analysis is presented in Ref. [8].

2. Displacement field

A composite laminate of N lamina is shown in Fig. 1. x-axis is at the center of laminate and the z-axis is perpendicular to it. The displacement field at any point (x, z) inside the laminate is given below

$$u^k = u_0 - z\frac{\partial w_0}{\partial x} + \left(A^k + zB^k + \sin\frac{\pi z}{h}\right)\eta_x, \tag{1}$$

$$w = w_0, \tag{2}$$

where, u^k is in-plane displacement in kth layer, w_0 is transverse displacement in z direction, η_x is higher order term, h is the total thickness of the laminated beam and

$$A^k = -\frac{\pi}{h}\sum_{n=2}^{k} h_n \left[\left(\cos\frac{\pi h_n}{h}\left(\frac{G_{xz}^{n-1}}{G_{xz}^{n}} - 1\right) + \sum_{p=2}^{n-1}\cos\frac{\pi h_p}{h}\frac{G_{xz}^{p}}{G_{xz}^{n}}\left(\frac{G_{xz}^{p-1}}{G_{xz}^{p}} - 1\right)\right) - \left(\cos\frac{\pi h_{n-1}}{h}\left(\frac{G_{xz}^{n-2}}{G_{xz}^{n-1}} - 1\right) + \sum_{p=2}^{n-2}\cos\frac{\pi h_p}{h}\frac{G_{xz}^{p}}{G_{xz}^{n-1}}\left(\frac{G_{xz}^{p-1}}{G_{xz}^{p}} - 1\right)\right)\right], \tag{3}$$

$$B^k = \frac{\pi}{h}\left[\cos\frac{\pi h_k}{h}\left(\frac{G_{xz}^{k-1}}{G_{xz}^{k}} - 1\right) + \sum_{m=2}^{k-1}\cos\frac{\pi h_m}{h}\frac{G_{xz}^{m}}{G_{xz}^{k}}\left(\frac{G_{xz}^{m-1}}{G_{xz}^{m}} - 1\right)\right]. \tag{4}$$

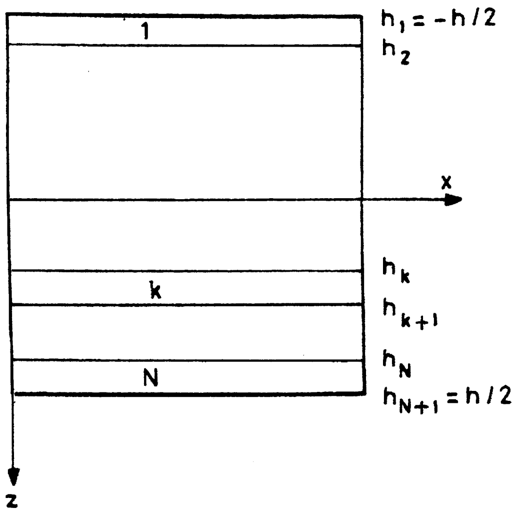

Fig. 1. Laminate geometry.

It may be noted that the displacement field is dependent on only three variables u_0, w_0 and η_x. The displacement field is different in different layers and is governed by the coefficient A^k and B^k, which are dependent on layer material and geometry. A^k and B^k are zero for $N = 1$ (single layer). Here G_{xz}^{n-1} is transverse shear modulus and superscript is layer number.

The in-plane and transverse shear strains are

$$\varepsilon_x^k = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + \left(A^k + zB^k + \sin\frac{\pi z}{h}\right)\frac{\partial \eta_x}{\partial x}, \tag{5}$$

$$\gamma_{xz}^k = \left(B^k + \frac{\pi}{h}\cos\frac{\pi z}{h}\right)\eta_x. \tag{6}$$

The governing equations associated with the above displacement functions can be derived using the virtual work principle. These variationally consistent equations are as follows:

$$\delta u_0:\quad \frac{\partial N_x}{\partial x} = I1\ddot{u}_0 + (IA + IB1 + IS)\ddot{\eta}_x,$$

$$\delta w_0:\quad \frac{\partial^2 M_x}{\partial x^2} - q = I1\ddot{w}_0 + I2\frac{\partial^2 \ddot{w}_0}{\partial x^2} - (IA1 + IB2 + IS1)\frac{\partial \ddot{\eta}_x}{\partial x}, \tag{7}$$

$$\begin{aligned}\delta \eta_x:\quad & \frac{\partial N_{Ax}}{\partial x} + \frac{\partial M_{Bx}}{\partial x} + \frac{\partial N_{Sx}}{\partial x} - T_{Bx} - \frac{\pi}{h}T_{cx} \\ &= (IA + IB1 + IS)\ddot{u}_0 - (IA1 + IB2 + IS1)\frac{\partial \ddot{w}_0}{\partial x} \\ &\quad + (IAA + 2IAB1 + 2IAS + IBB2 + 2IBS1 + ISS)\ddot{\eta}_x,\end{aligned}$$

where q is the distributed transverse load, and the resultant forces and moments are defined as follows:

$$(N_x, N_{Ax}, N_{Sx}, M_x, M_{Bx}) = \sum_{k=1}^{N}\int_{h_k}^{h_{k+1}} \sigma_x^k\left(1, A_k, \sin\frac{\pi z}{h}, z, zB_k\right)\mathrm{d}z, \tag{8}$$

$$(T_{Bx}, T_{cx}) = \sum_{k=1}^{N}\int_{h_k}^{h_{k+1}} \tau_{xz}^k\left(B_k, \cos\frac{\pi z}{h}\right)\mathrm{d}z. \tag{9}$$

The resultant inertia terms are

$$(I1, I2, IS, IS1, ISS) = \sum_{k=1}^{N}\int_{h_k}^{h_{k+1}} \rho^k\left(1, z^2, \sin\frac{\pi z}{h}, z\sin\frac{\pi z}{h}, \sin^2\frac{\pi z}{h}\right)\mathrm{d}z, \tag{10}$$

$$\begin{aligned}&(IA, IA1, IB1, IAA, IAB1, IB2, IBB2, IAS, IBS1) \\ &= \sum_{k=1}^{N}\int_{h_k}^{h_{k+1}} \rho^k\left(A^k, zA^k, zB^k, A^kA^k, zA^kB^k, z^2B^k, z^2B^kB^k, A^k\sin\frac{\pi z}{h}, zB^k\sin\frac{\pi z}{h}\right)\mathrm{d}z.\end{aligned} \tag{11}$$

The essential and natural boundary conditions obtained are listed below:

$$\begin{array}{cc} \textit{Natural boundary conditions} & \textit{Essential boundary conditions} \\ N_x & u_0 \\ M_x & \partial w_0/\partial x \\ \partial M_x/\partial x & w_0 \\ (N_{Sx}+M_{Bx}+N_{Ax}) & \eta_x \end{array} \tag{12}$$

3. Numerical example and results

The efficacy of the present zig-zag displacement model further referred as TSDT-ZZ, is shown by comparing the results with the examples available in literature. The results are compared with a higher order mixed theory [9].

The boundary conditions of the chosen beam at two ends are: $N_x = 0$, $M_x = 0$, $w_0 = 0$ and $N_{Sx} + M_{Bx} + N_{Ax} = 0$. Problem considered here is a four layer beam. The material properties of the layers used is DATA-3 and DATA-5 from Ref. [9]:

$E_1 = 1.448E8$ kN/mm^2; $E_2 = 9.65E6$ kN/mm^2; $G_{12} = 4.14E6$ kN/mm^2;
$v_{12} = 0.3$; $\rho = 1389.23$ N s^2/m^4;
Length $l = 1.5E4$ mm; Thickness $h = 1.0E3$ mm and 3.0E3 mm;
Lamination scheme [0/90/90/0] all layers of equal thickness.

The following displacement function satisfies the simply supported boundary conditions mentioned earlier:

$$u_0 = \sum_{i=1}^{\infty} U_i \cos \alpha x e^{\iota\omega t},$$

$$w_0 = \sum_{i=1}^{\infty} W_i \sin \alpha x e^{\iota\omega t},$$

$$\eta_x = \sum_{i=1}^{\infty} \eta_{xi} \cos \alpha x e^{\iota\omega t}. \tag{13}$$

In above equations $\alpha = \mathrm{i}\pi/l$, where l is length of the beam and ω is natural frequency.

The results are also compared with a ESL theory of similar class and it is further referred as TSDT-ESL, the displacement field is

$$u = u_0 - z\frac{\partial w_0}{\partial x} + \sin\frac{\pi z}{h}\eta_x, \tag{14}$$

$$w = w_0. \tag{15}$$

It may be noted that in the above displacement function the derivative with respect to thickness is continuous. This leads to discontinuous shear stress at interface in layered composites.

It can be noted that TSDT-ZZ requires one additional variable as compared to TSDT-ESL, i.e., u_0. For symmetric laminates this term is zero in the case of TSDT-ESL and it can be further

Table 1
Comparison of natural frequency $\bar{\omega}$ of composite beams for the first four modes

l/h	Theory	1	2	3	4
5	TSDT-ZZ	1.785	4.444	7.181	10.084
	TSDT-ESL	1.783	4.444	7.201	10.147
	Ref. [9]	1.814	4.530	7.234	9.931
15	TSDT-ZZ	2.505	8.569	16.0636	23.962
	TSDT-ESL	2.505	8.562	16.048	23.941
	Ref. [9]	2.513	8.660	16.330	24.436

simplified to

$$u = -z\frac{\partial w_0}{\partial x} + \sin\frac{\pi z}{h}\eta_x, \tag{16}$$

$$w = w_0, \tag{17}$$

$$\gamma_{xz} = \frac{\pi}{h}\cos\frac{\pi z}{h}\eta_x. \tag{18}$$

In above equation it can be noted that single cosine term is able to give zero shear strain at top and bottom of the beam, whereas in polynomial type displacement model at least one more term will be required to satisfy this condition. This feature of TSDT makes it elegant and results in lesser number of stress resultants in governing equations.

Numerical results are presented in Table 1 and compared with another mixed theory [9]. It can be noted that the results are matching very well. Using TSDT-ZZ transverse shear stress can be calculated using constitutive relation and it offers advantage over TSDT-ESL where one has to use equilibrium equation approach.

All the calculated frequencies are non-dimensionalised as given below:

$$\bar{\omega} = \omega l^2\left[\frac{\rho}{E_1 h^2}\right]^{1/2}.$$

4. Concluding remarks

In this note, a new type of zig-zag displacement function is presented. The displacement function uses trigonometric terms to represent the displacement across the thickness. The displacement function satisfies displacement and transverse shear stress continuity at the interface. Zero shear stress condition at top and bottom of the beam is also satisfied. The new displacement function is used for free vibration analysis of a thick simply supported layered beam and results are compared with another mixed theory. Results are also presented for a similar class of equivalent single layer beam theory.

References

[1] K.H. Lo, R.M. Christensen, E.M. Wu, A higher order theory of plate deformation, Part 1: Homogeneous plates, Journal of Applied Mechanics 44 (1977) 663–668.
[2] K.H. Lo, R.M. Christensen, E.M. Wu, A higher order theory of plate deformation, Part 2: Laminated plates, Journal of Applied Mechanics 44 (1977) 669–676.
[3] J.N. Reddy, A simple higher order theory for laminated composite plates, Journal of Applied Mechanics 51 (1984) 745–752.
[4] J.L. Doong, C. Lee, C.P. Fung, Vibration and stability of laminated plates based on a modified plate theory, Journal of Sound and Vibration 151 (2) (1991) 193–201.
[5] K.H. Lee, Z.W. Lin, S.T. Chow, Bidirectional bending of laminated composite plates using an improved zig-zag model, Composite Structures 28 (1994) 283–294.
[6] Xiaoyu Li, Dashin Liu, Zig-zag theory for composite laminates, American Institute of Aeronautics and Astronautics Journal 33 (6) (1995) 1163–1165.
[7] M. Stein, Nonlinear theory of plates and shells including the effect of transverse shearing, American Institute of Aeronautics and Astronautics Journal 24 (9) (1986) 1537–1544.
[8] Hemendra Arya, R.P. Shimpi, N.K. Naik, A zig-zag model for laminated composite beams, Composite structures 56 (2002) 21–24.
[9] M. Kameswara Rao, Y.M. Desai, M.R. Chitnis, Free vibrations of laminated beams using mixed theory, Composite Structures 52 (2001) 149–160.