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Letter to the Editor

Evaluation of motion of the Duffing equation from its general properties

Y.Z. Chen

Division of Engineering Mechanics, Jiangsu University, Zhenjiang, Jiangsu 212013, People's Republic of China

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1. Introduction

Solution of the Duffing equation in a non-linear vibration problem is studied in the paper. The governing equation for the problem was formulated in Refs. [1,2]. In the case of ε being a small parameter, the equation is solved by using the Lindstedt–Poincare technique, the method of multiple scales, and the method of averaging [1,2]. Almost all perturbation methods are based on small parameters so that the approximate solutions can be expressed in a series of small parameters. The limitation of the perturbation method was pointed out in Refs. [3,4]. Clearly, in the case of ε being a larger value, the perturbation method is no longer valid.

Recently, the target function method is used to evaluate the motion of the Duffing equation [5]. The method is an accurate one without the limitation of ε being a small value. The method is effective and it mainly depends on the computer computation.

In this paper, one more method for the solution of the Duffing equation is suggested. In the method, the motion of the Duffing equation ($\mathrm{d}^2u/\mathrm{d}t^2 + \omega_0^2 u(1+\varepsilon u^2) = 0$) under some initial boundary conditions is evaluated by using its general properties. The approximate motion is assumed in the form of a Fourier series. The main idea developed is that let the quantitative properties in the approximate motion be close to those in real motion as much as possible. For example, let the trajectory of the velocity–displacement (v versus u) in the approximate motion approach the counterpart in the real motion. The used quantitative properties include: (a) the trajectory of the velocity–displacement (v versus u) on the phase plane, (b) the imposed initial boundary conditions, (c) the acceleration at the starting point and (d) the maximum velocity achieved in the real motion. The qualitative property used is that the actual circular frequency ω_p must be higher than ω_0, which is present at pure harmonic motion ($\varepsilon = 0$ case). A three-parameter (ω_p, c_1, c_3) formulation is suggested, where the frequency ω_p is the actual circular frequency, and c_1, c_3 are the coefficients in the Fourier series. Four-parameter (ω_p, c_1, c_3, c_5) formulation is also suggested. All these parameters can be achieved from a solution by using the mentioned properties.

E-mail address: chens@ujs.edu.cn (Y.Z. Chen).

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In order to verify the particular advantage of the suggested method, several numerical examples are given. It is proved that a higher accuracy of solution is achieved in the four-parameter formulation. Note that, in the present study, the parameter ε need not be a small value.

2. Analysis

In following analysis, the Duffing equation is defined by [1,2]

$$N(u) = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega_0^2 u(1 + \varepsilon u^2) = 0, \tag{1}$$

where ω_0 is the circular frequency, ε is a constant which may not be a small value. In Eq. (1), $N(u)$ is a non-linear operator as indicated. The imposed initial boundary conditions take the form

$$u|_{t=0} = A, \quad \frac{\mathrm{d}u}{\mathrm{d}t}|_{t=0} = 0, \tag{2a, b}$$

where A is a positive value. In the formulation, the values of ω_0, ε and A are given beforehand. In solution, the velocity of motion is defined as

$$v = \frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3}$$

The general properties of the Duffing equation are defined such that they are derived from Eqs. (1) and (2). On the contrary, they are not independent in general. The first property is about the trajectory of motion on the phase plane. Clearly, after multiplying both sides of Eq. (1) by $2\mathrm{d}u$ and making integration, we have

$$v^2 + F(u) = E, \tag{4}$$

where

$$F(u) = 2\omega_0^2 \int_0^u u(1 + \varepsilon u^2)\, \mathrm{d}u = \omega_0^2 \left(u^2 + \frac{\varepsilon u^4}{2} \right), \tag{5}$$

$$E = F(u)|_{u=A} = \omega_0^2 \left(A^2 + \frac{\varepsilon A^4}{2} \right). \tag{6}$$

The second property of motion is about the acceleration $(\mathrm{d}^2u/\mathrm{d}t^2)$. Clearly, the acceleration at the starting time is as follows:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}|_{t=0} = [-\omega_0^2 u(1 + \varepsilon u^2)]\big|_{t=0} = -\omega_0^2 A(1 + \varepsilon A^2). \tag{7}$$

The third property is qualitative, and it is about the circular frequency of the motion. Obviously, since $(1 + \varepsilon u^2) > 1$ in general, the actual circular frequency ω_p must be larger than ω_0, or $\omega_p > \omega_0$.

From the above analysis, we see that none of the mentioned properties of motion is independent, and they are derived form Eq. (1) and initial condition (2a,b). The aim of this study is to derive an approximate motion such that:

(a) The trajectory in approximate motion on the phase plane is very near the one defined by Eq. (4).

(b) The approximate motion satisfies conditions (2a,b) and (7) exactly.
(c) The approximate motion may not satisfy governing equation (1) exactly.
(d) We assume a circular frequency ω_p in the approximate motion, which is equal to the actual circular frequency in motion of the Duffing equation. From analysis cited below, we see that the assumed frequency ω_p can be obtained in the process of solution.

The motion of Duffing equation can be assumed in the form

$$u(t) = \sum_{j=1,3,5} c_j \cos(j\omega_p t), \quad 0 \leqslant \omega_p t \leqslant 2\pi. \tag{8}$$

This form is obtained on the fact that if at time $\omega_p t$ $(0 < \omega_p t < \pi/2)$, the displacement and the velocity (u, v) are $(u_0, -v_0)$, then at times $\pi - \omega_p t$, $\pi + \omega_p t$, and $2\pi - \omega_p t$, the relevant values should be $(-u_0, -v_0)$, $(-u_0, v_0)$, and (u_0, v_0), respectively.

On the basis of the above-mentioned assumption, the three-parameter formulation (ω_p, c_1, c_3) is introduced first. In this case, the approximate motion is assumed in the form [5]

$$u_a(t) = c_1 \cos(\omega_p t) + c_3 \cos(3\omega_p t), \quad 0 \leqslant \omega_p t \leqslant 2\pi, \tag{9}$$

where the subscript "a" denotes the approximate solution.

In order to evaluate the three parameters (ω_p, c_1, c_3), the following conditions are imposed:

$$u|_{t=0} = u_a|_{t=0}, \tag{10}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}|_{\max} = \frac{\mathrm{d}u_a}{\mathrm{d}t}|_{\max} \text{ or } v|_{\max} = v_a|_{\max} \text{ or } \frac{\mathrm{d}u}{\mathrm{d}t}\Big|_{\omega_p t = 3\pi/2} = \frac{\mathrm{d}u_a}{\mathrm{d}t}\Big|_{\omega_p t = 3\pi/2}, \tag{11}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}|_{t=0} = \frac{\mathrm{d}^2 u_a}{\mathrm{d}t^2}|_{t=0}. \tag{12}$$

The formulation gives accurate results in some ranges of "A" and "ε". Since a very accurate solution for the problem has been obtained previously, the relative deviation for the circular frequency can be estimated immediately. For examples, in the case of $A = 1$ and $\varepsilon = 10$, the maximum deviation of $\alpha(\omega_p/\omega_0)$ is 1.102%, and in the case of $A = 2$ and $\varepsilon = 10$, the maximum deviation of $\alpha(\omega_p/\omega_0)$ is 1.333%.

3. Four-parameter formulation

Clearly, the four-parameter formulation $(\omega_p, c_1, c_3, c_5)$ is more accurate. Four-parameter formulation for the solution of the Duffing equation is studied below. In the present case, the displacement, velocity and acceleration are assumed in the form

$$u_a(t) = c_1 \cos(\omega_p t) + c_3 \cos(3\omega_p t) + c_5 \cos(5\omega_p t), \quad 0 \leqslant \omega_p t \leqslant 2\pi, \tag{13}$$

$$v_a(t) = \frac{\mathrm{d}u_a(t)}{\mathrm{d}t} = -\omega_p[c_1 \sin(\omega_p t) + 3c_3 \sin(3\omega_p t) + 5c_5 \sin(5\omega_p t)], \quad 0 \leqslant \omega_p t \leqslant 2\pi, \tag{14}$$

$$\frac{\mathrm{d}^2 u_a(t)}{\mathrm{d}t^2} = -\omega_p^2[c_1 \cos(\omega_p t) + 9c_3 \cos(3\omega_p t) + 25c_5 \cos(5\omega_p t)], \quad 0 \leqslant \omega_p t \leqslant 2\pi. \tag{15}$$

In the formulation, the mentioned four parameters are $\omega_p,\ c_1,\ c_3,\ c_5$. In the following analysis, we denote

$$h = \varepsilon A^2, \quad \alpha = \frac{\omega_p}{\omega_0}, \quad g_1 = \frac{c_1}{A}, \quad g_3 = \frac{c_3}{A}, \quad g_5 = \frac{c_5}{A}. \tag{16}$$

Clearly, in order to find the four parameters $\omega_p,\ c_1,\ c_3,\ c_5$, it is necessary to propose four conditions. Three conditions shown by Eqs. (10)–(12) are still used in the present case. As before, after using these conditions, we have

$$c_1 = A - c_3 - c_5 \quad \text{or} \quad g_1 = 1 - g_3 - g_5, \tag{17}$$

$$\omega_0^2\left(1 + \frac{h}{2}\right) = \omega_p^2(1 - 4g_3 + 4g_5)^2, \tag{18}$$

$$\omega_0^2(1 + h) = \omega_p^2(1 + 8g_3 + 24g_5). \tag{19}$$

One more condition is derived as follows. Letting $\omega_p t = 7\pi/4$, from Eqs. (13) and (14), the relevant displacement and velocity are obtained as follows:

$$u_{a(d)} = \frac{1}{\sqrt{2}}[c_1 - c_3 - c_5], \tag{20}$$

$$v_{a(d)} = \frac{\omega_p}{\sqrt{2}}[c_1 + 3c_3 - 5c_5]. \tag{21}$$

In fact, the $(u_{a(d)},\ v_{a(d)})$ pair corresponds the point $p_{(d)}$ on the trajectory on the phase plane (Fig. 1).

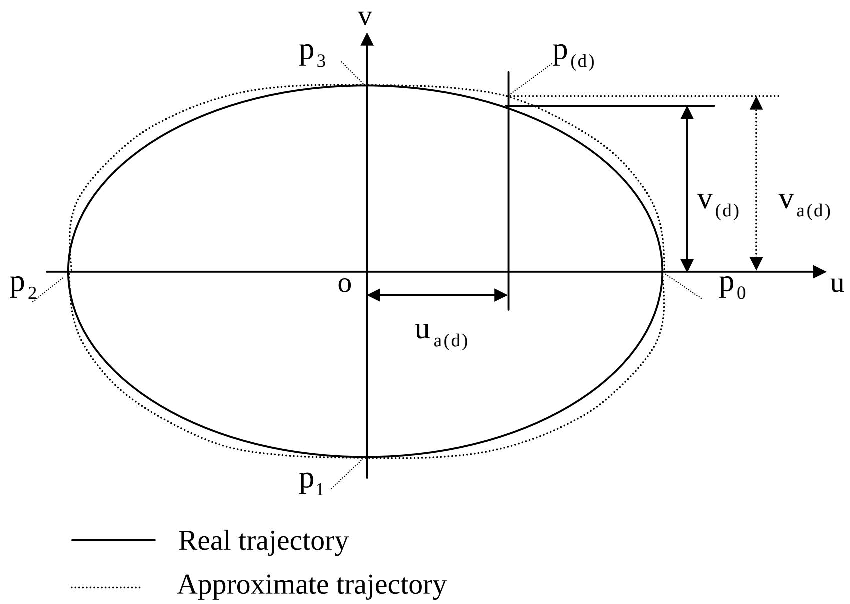

Fig. 1. The "v" versus "u" trajectory for solution of the Duffing equation on the phase plane.

Meantime, we can get the real velocity $v_{(d)}$ at the assumed displacement $u_{a(d)}$ from Eq. (4), and it will result in

$$v_{(d)}^2 = \omega_0^2\left(A^2 + \frac{\varepsilon A^4}{2}\right) - \omega_0^2\left(u_{a(d)}^2 + \frac{\varepsilon u_{a(d)}^4}{2}\right). \tag{22}$$

Finally, the fourth condition is obtained as follows:

$$v_{(d)} = v_{a(d)} \quad \text{or} \quad v_{(d)}^2 = v_{a(d)}^2. \tag{23}$$

Note that the value $v_{(d)}(v_{a(d)})$ is the velocity in the real motion (the approximate motion) respectively, for the same assumed displacement $u_{a(d)}$ shown by Eq. (20). Generally, the $v_{(d)}$ value may not be equal to $v_{a(d)}$, as indicated in Fig. 1. In fact, the imposed condition (23) will make the approximate motion to be closer to the real motion. From Eqs. (21)–(23) we get the fourth condition

$$\omega_0^2\left(2 + h - q_1^2 - \frac{h}{4}q_1^4\right) = \omega_p^2(1 + 2g_3 - 6g_5)^2, \tag{24}$$

Table 1
α value and the calculated Fourier coefficients for the solution of the Duffing equation by using the four-parameter formulation (see Eqs. (13), (16))

A=1 case										
ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α	1.0367	1.0720	1.1060	1.1389	1.1708	1.2017	1.2318	1.2612	1.2898	1.3178
c_1	0.9971	0.9945	0.9923	0.9903	0.9885	0.9869	0.9854	0.9840	0.9828	0.9817
c_3	0.0029	0.0054	0.0077	0.0096	0.0114	0.0130	0.0144	0.0157	0.0169	0.0180
c_5	0.0000	0.0000	0.0001	0.0001	0.0001	0.0002	0.0002	0.0003	0.0003	0.0003
ε	1	2	3	4	5	6	7	8	9	10
α	1.3178	1.5691	1.7844	1.9760	2.1503	2.3115	2.4620	2.6039	2.7383	2.8665
c_1	0.9817	0.9740	0.9697	0.9670	0.9652	0.9638	0.9627	0.9619	0.9612	0.9606
c_3	0.0180	0.0253	0.0294	0.0319	0.0337	0.0350	0.0359	0.0367	0.0373	0.0379
c_5	0.0003	0.0007	0.0009	0.0011	0.0012	0.0013	0.0013	0.0014	0.0015	0.0015
A=2 case										
$\varepsilon =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α	1.1389	1.2612	1.3719	1.4739	1.5691	1.6586	1.7435	1.8243	1.9017	1.9760
c_1	1.9805	1.9681	1.9594	1.9530	1.9480	1.9441	1.9409	1.9382	1.9360	1.9341
c_3	0.0193	0.0314	0.0398	0.0460	0.0507	0.0544	0.0574	0.0599	0.0620	0.0638
c_5	0.0002	0.0005	0.0008	0.0011	0.0013	0.0015	0.0017	0.0019	0.0020	0.0021
ε	1	2	3	4	5	6	7	8	9	10
α	1.9760	2.6039	3.1069	3.5390	3.9237	4.2739	4.5975	4.8998	5.1845	5.4543
c_1	1.9341	1.9238	1.9195	1.9173	1.9158	1.9148	1.9141	1.9136	1.9131	1.9128
c_3	0.0638	0.0734	0.0773	0.0794	0.0808	0.0817	0.0823	0.0829	0.0832	0.0836
c_5	0.0021	0.0028	0.0031	0.0033	0.0034	0.0035	0.0035	0.0036	0.0036	0.0037

where

$$q_1 = 1 - 2q_2 = 1 - 2g_3 - 2g_5, \quad q_2 = g_3 + g_5. \tag{25}$$

Finally, from Eqs. (17)–(19) and (24), the solution of four parameters $(\omega_p, c_1, c_3, c_5)$ (or ω_p, g_1, g_3, g_5) can be obtained. For compactness of the present note, the detail solution technique for obtaining the parameters will not be mentioned.

By using the four-parameter formulation for the cases of $A = 1$ and 2, $\varepsilon = 0.1, 0.2, \ldots, 1.0, 2.0, \ldots, 10.0$, the obtained results are listed in Table 1. As before, the obtained results are compared with the previously obtained accurate results in Ref. [5]. The following relative deviations are found: (a) 0.00% for the case of $A = 1$ and $\varepsilon = 1$, (b) –0.0035% for the case of $A = 1$ and $\varepsilon = 10$, (c) 0.00% for the case of $A = 2$ and $\varepsilon = 1$ and (d) –0.0092% for the case of $A = 2$ and $\varepsilon = 10$. From the mentioned results, we see that the solution from the four-parameter formulation is a very accurate one within the range of assumed values for A and ε.

After the solution is obtained, the deviation to satisfy the ODE can be evaluated by the value $N(u_a)$, which is obtained by substituting the obtained solution into the left-hand term of Eq. (1). The obtained $N(u_a)$ may expressed in turn as

$$N(u_a) = G(\omega_p t), \quad 0 \leqslant \omega_p t \leqslant \pi. \tag{26}$$

From Eqs. (1) and (13), it is easy to find the following property:

$$\int_0^{\pi} G(T)\,\mathrm{d}T = 0 \quad (\text{letting } T = \omega_p t). \tag{27}$$

This result means that the average of $N(u_a)$ on the interval $0 \leqslant \omega_p t \leqslant \pi$ always vanishes.

From the above mentioned analysis, a rule for the motion of the Duffing equation is found. The rule can be summarized as a theorem as follows:

Theorem. *Assume that there are many pairs of the values (ε_i, A_i), if $\varepsilon_i A_i^2$ preserves a constant $(\varepsilon_i A_i^2 = c)$, the magnitude factors "α" of the circular frequency for all pairs must be the same, and the magnitude of motion is directly proportional to "A_i".*

Proof. It is assumed that the equation

$$N_1(u_1) = \frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \omega_0^2 u_1(1 + \varepsilon_1 u_1^2) = 0, \quad u_1|_{t=0} = A_1, \ \frac{\mathrm{d}u_1}{\mathrm{d}t}|_{t=0} = 0 \tag{28}$$

has a solution $u_1(\omega_p t)$. Clearly, the statement in the theorem is equivalent to prove the following alternative statement. In the condition of satisfying $\varepsilon_1 A_1^2 = \varepsilon_2 A_2^2$, the equation

$$N_2(u_2) = \frac{\mathrm{d}^2 u_2}{\mathrm{d}t^2} + \omega_0^2 u_2(1 + \varepsilon_2 u_2^2) = 0, \quad u_2|_{t=0} = A_2, \quad \frac{\mathrm{d}u_2}{\mathrm{d}t}|_{t=0} = 0 \tag{29}$$

must have a solution as follows:

$$u_2(\omega_p t) = \frac{A_2}{A_1} u_1(\omega_p t). \tag{30}$$

In fact, substituting Eq. (30) into Eq. (29) yields

$$N_2(u_2) = \frac{A_2}{A_1}\left(\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \omega_0^2 u_1\left(1 + \frac{\varepsilon_2 A_2^2}{A_1^2} u_1^2\right)\right) = \frac{A_2}{A_1}\left(\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \omega_0^2 u_1(1 + \varepsilon_1 u_1^2)\right) = 0. \tag{31}$$

Also, it is easy to see that the function $u_2(\omega_p t)$ satisfies the initial conditions in Eq. (29). Thus, the proof of theorem is completed. $\square$

The introduced theorem can also be verified from the calculated results in Table 1.

4. Conclusions

Previously, when the straightforward expansion method or the Lindstedt–Poincare technique were used, it was invariable to meet the secular term. In this case, one has to use a lot of effort to explain why this term is not reasonable [2, Chapter 4]. In the present study, instead of using the ODE and initial boundary value conditions, the properties of the Duffing equation were used to get the final solution of the equation. It is found in the present study that there is no step, which is relating to the direct solution of ODE. This is a particular advantage of the present study. Secondly, since the basic properties of the real motion were modelled in the approximate motion, the obtained solution must possess a higher accuracy, particularly, in the case of using the four-parameter formulation. This situation can be seen from Table 1 and the previously obtained results [5]. If more terms are assumed in the Fourier series of motion, it is expected to use the suggested method in the case of more wide range of the parameters of "ε" and "A".

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