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Dynamics of linear discrete systems connected to local, essentially non-linear attachments

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Abstract

The dynamics of a linear periodic substructure, weakly coupled to an essentially non-linear attachment are studied. The essential (non-linearizable) non-linearity of the attachment enables it to resonate with any of the linearized modes of the subtructure leading to energy pumping phenomena, e.g., passive, one-way, irreversible transfer of energy from the substructure to the attachment. As a specific application the dynamics of a finite linear chain of coupled oscillators with a non-linear end attachment is examined. In the absence of damping, it is found that the dynamical effect of the non-linear attachment is predominant in neighborhoods of internal resonances between the attachment and the chain. When damping exists energy pumping phenomena are realized in the system. It is shown that energy pumping strongly depends on the topological structure of the non-linear normal modes (NNMs) of the underlying undamped system. This is due to the fact that energy pumping is caused by the excitation of certain damped invariant NNM manifolds that are analytic continuations for weak damping of NNMs of the underlying undamped system. The bifurcations of the NNMs of the undamped system help explain resonance capture cascades in the damped system. This is a series of energy pumping phenomena occurring at different frequencies, with sudden lower frequency transitions between sequential events. The observed multi-frequency energy pumping cascades are particularly interesting from a practical point of view, since they indicate that non-linear attachments can be designed to resonate and extract energy from an a priori specified set of modes of a linear structure, in compatibility with the design objectives.

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1. Introduction

In this work the dynamics of a linear structure weakly coupled to a local non-linear attachment possessing essential stiffness non-linearity is studied. It was shown recently [1–3] that under certain conditions this type of essentially non-linear attachment can passively absorb energy from a linear non-conservative (damped) structure, in essence, acting as non-linear energy sink. Then, *energy pumping* from the linear structure to the attachment occurs, namely, a one-way, irreversible transfer of energy.

As discussed in Ref. [2], energy pumping from the linear non-conservative structure to the attachment is due to *resonance capture*. This is a transient dynamical phenomenon that has been theoretically studied in previous works [4–7]. It occurs (among other types of dynamical systems) in coupled non-conservative oscillators and leads to transient capture of the dynamical flow on a resonance manifold of the system. The main aim of this work is to show that the physics of the energy pumping/resonance capture phenomenon in the non-conservative system under consideration can be understood and explained by studying the energy dependence of the non-linear free periodic solutions (non-linear normal modes—NNMs [8]) of the corresponding conservative system that is obtained when all damping forces are eliminated. Hence, a dynamical phenomenon will be presented that, although it takes place only when damping (non-conservativeness) exists, it is mainly influenced by the dynamics of the underlying undamped (conservative) system.

After providing a general mathematical formulation of the problem of structure–attachment interaction the work will focus on the specific application of a finite linear chain of coupled oscillators weakly coupled to an essentially non-linear end attachment. The dynamics of linear/non-linear periodic chains with local attachments ('defects') is a research area with many interesting applications, such as, optical and magneto-optical waveguide periodic arrays, semiconductor superlattices, layered composite media, micro- or nano-lattices as thermal barriers, photonic band-gap materials (photonic crystals) and bio-molecular engines [9,10]. Analytical works on solitons and solitary waves in linear or non-linear periodic systems with local non-linear defects have also been reported [11–13]. In these works solitary waves and their bifurcations were studied, and techniques were provided for analyzing the stability of the derived solutions under certain classes of perturbations. In this work the response of the finite linear chain will be expressed in terms of its modes and then the dynamic interaction of the modal oscillators with the weakly coupled non-linear attachment will be studied. The existence of localized and non-localized periodic oscillations in the free problem will be proved, and the occurrence of energy pumping from a modal oscillator to the attachment in the forced case. In the later case, it will be shown that there exists even the possibility of *resonance capture cascades* whereby, the attachment resonates sequentially with a set modal oscillators, extracting energy from each at a different frequency range. The important role of damping for the realization of these phenomena will be discussed.

2. Preliminaries: modal formulation

Consider the system of Fig. 1, composed of a linear substructure with $(N+1)$ degrees of freedom (d.o.f.) that is weakly coupled to a local essentially non-linear attachment at point O. The

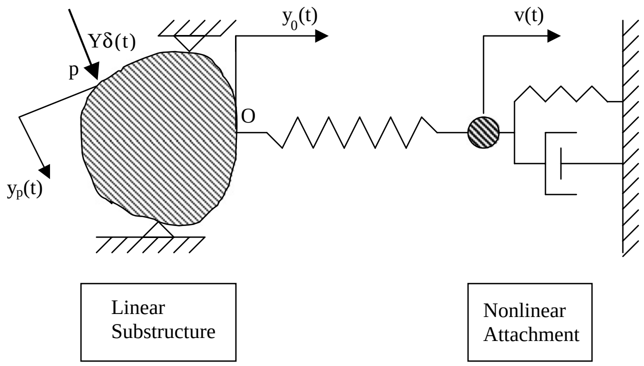


Fig. 1. Linear (main) substructure with a weakly connected local non-linear attachment.

attachment consists of a non-linearizable stiffness non-linearity of the third order in parallel with a viscous dashpot that models energy dissipation; throughout this work the mass of the NES will be taken $m = 1$. The coupling stiffness between the linear and non-linear parts is assumed to be linear and weak, of order $\varepsilon, 0 < \varepsilon \ll 1$. In addition, the connection between the two systems is assumed to be one dimensional.

Introducing modal co-ordinates $a_m(t)$, $m = 0, \ldots, N$ for the linear substructure, its response $y_0(t)$ at the point of attachment O is expressed in the following modal form:

$$y_0(t) = \sum_{k=0}^{N} \phi_0^{(k)} a_k(t), \tag{1}$$

where $\phi_0^{(k)}$ denotes the element at position O of the kth mass normalized eigenvector; in general, $\phi_j^{(i)}$ denotes the element at position j of the mass-normalized eigenvector $\underline{\phi}^{(i)}$ of the uncoupled linear substructure with $\varepsilon = 0$. In Eq. (1) it is assumed that the uncoupled linear substructure possesses $(N+1)$ mass-normalized eigenvectors $\underline{\phi}^{(i)}$ corresponding to $(N+1)$ distinct eigenfrequencies ω_i, $i = 0, 1, \ldots, N$. Taking into account Eq. (1), the $(N+1)$ equations of motion of the combined system are expressed in the following form:

$$\ddot{v}(t) + Cv^3(t) + \varepsilon\lambda\dot{v}(t) + \varepsilon\left(v - \sum_{k=0}^{N} \phi_0^{(k)} a_k(t)\right) = 0,$$

$$\ddot{a}_m(t) + \omega_m^2 a_m(t) + \varepsilon\lambda\dot{a}_m(t) + \varepsilon\left(\sum_{k=0}^{N} \phi_0^{(k)}\phi_0^{(m)} a_k(t) - \phi_0^{(m)} v(t)\right) = 0, \quad m = 0, 1, \ldots, N, \tag{2}$$

where the viscous damping forces are also weak and ordered by the small parameter ε. In Eq. (2) the equations of motion are expressed in terms of the $(N+1)$ modal oscillators of the uncoupled linear substructure.

Of interest is to study resonance interactions between the non-linear attachment and individual modes of the linear substructure. As shown in Refs. [1–3,14] such resonance interactions can lead to interesting *resonance capture* [5] and *energy pumping* phenomena, whereby externally imparted energy in the linear system gets transferred (pumped) to the non-linear attachment in a one-way irreversible fashion. In that case the non-linear attachment acts, in essence, as a non-linear energy sink.

Assuming that the linear substructure is excited by the impulsive excitation $F(t) = Y\delta(t)\delta(y - y_p)$ at position p (where y and y_p are body co-ordinates), the response of the system at $t = 0+$ is a free oscillation with initial conditions, $a_i(0+) = 0, \quad \dot{a}_i(0+) = Y\phi_p^{(i)}, \quad v(0+) = \dot{v}(0+) = 0, \; i = 0, 1, \ldots, N$. For the sake of simplicity, we assume that the non-linear attachment is in resonance interaction with only the leading modal oscillator 0 of the linear substructure. Hence, the study will focus on oscillations of the non-linear attachment at frequencies close to the lowest eigenfrequency ω_0 of the substructure. The same methodology, however, can be followed to study resonance interactions of the attachment with any of the other modal oscillators in Eq. (2). Then, correct to $O(1)$ the responses of the remaining N modal oscillators are not affected by the non-linear attachment, and thus decay exponentially in time,

$$a_i(t) = \frac{Y\phi_p^{(i)}\mathrm{e}^{-\varepsilon\lambda t/2}}{\omega_i\sqrt{1-\zeta_i^2}}\sin\omega_i\sqrt{1-\zeta_i^2}\,t + O(\varepsilon), \quad i = 1, \ldots, N, \tag{3}$$

where ζ_i denotes the viscous critical damping ratio for the ith linearized mode. Taking into account Eq. (3), the non-linear resonance interaction is governed by the following reduced system:

$$\begin{aligned}
&\ddot{v}(t) + Cv^3(t) + \varepsilon\lambda\dot{v}(t) + \varepsilon(v - \phi_0^{(0)}a_0(t)) \\
&\quad = \varepsilon\sum_{k=1}^{N}\frac{\phi_0^{(k)}\phi_p^{(k)}Y}{\omega_k\sqrt{1-\zeta_k^2}}\mathrm{e}^{-\varepsilon\lambda t/2}\sin\omega_k\sqrt{1-\zeta_k^2}\,t + O(\varepsilon^2), \\
&\ddot{a}_0(t) + \omega_0^2 a_0(t) + \varepsilon\lambda\dot{a}_0(t) + \varepsilon(\phi_0^{(0)2}a_0(t) - \phi_0^{(0)}v(t)) \\
&\quad = -\varepsilon\sum_{k=1}^{N}\frac{\phi_0^{(k)}\phi_p^{(0)}\phi_p^{(k)}Y}{\omega_k\sqrt{1-\zeta_k^2}}\mathrm{e}^{-\varepsilon\lambda t/2}\sin\omega_k\sqrt{1-\zeta_k^2}\,t + O(\varepsilon^2)
\end{aligned} \tag{4}$$

ntioned initial conditions. From Eq. (4) note that the non-resonant modal oscillators introduce $O(\varepsilon)$ non-homogeneous, high-frequency terms on the right-hand sides of the above equations. Moreover, it is implicitly assumed in Eq. (4) that there are no resonance interactions of the form $(\omega_p : \omega_m : \ldots : \omega_k)$ involving purely modal oscillators, e.g., that there are no additional internal resonances in (2) other than the resonances $(0 : \omega_m), m = 0, \ldots, N$ between the attachment and each of the modal oscillators.

Eq. (4) is similar in form to the two-degree-of-freedom system studied in Refs. [1,2,14]. As shown in these works, under certain conditions *energy pumping* occurs in these systems, whereby energy from a linearized mode gets transferred into the non-linear attachment in a one-way irreversible fashion. As discussed in Ref. [2] the mechanism that governs energy pumping is resonance capture on a 1–1 resonance manifold of the combined system. It is well established that resonance capture is only possible in the non-conservative system (e.g., damping is required for its realization), and that for fixed structural parameters energy pumping can occur only above a certain threshold of the externally induced energy.

A different perspective regarding the dynamics governing energy pumping will be presented in this work. In particular, it will be shown that the topological structure in parameter space of the non-linear normal modes (NNMs) of the undamped, unforced system (2) is mainly responsible for energy pumping in the corresponding damped and forced system. By NNMs are denoted the free periodic and synchronous oscillations of the undamped, unforced system, that are, in essence, the non-linear analogs of the linear modes of classical vibration theory [8]. Hence, it will be shown that even though damping and forcing are prerequisites for energy pumping, this phenomenon is governed in essence by the dynamics of the corresponding undamped and unforced system.

3. Mathematical analysis of the undamped and unforced system

Setting $\lambda = 0$ in Eq. (2) and omitting the initial conditions (e.g., forcing) one introduces the following partition of Eqs. (2):

$$\ddot{v}(t) + Cv^3(t) + \varepsilon\left(v(t) - \phi_0^{(0)}a_0(t) - \sum_{k=1}^{N}\phi_0^{(k)}a_k(t)\right) = 0,$$
$$\ddot{a}_0(t) + \omega_0^2 a_0(t) + \varepsilon\left(\phi_0^{(0)2}a_0(t) - \phi_0^{(0)}v(t) + \sum_{k=1}^{N}\phi_0^{(k)}\phi_0^{(0)}a_k(t)\right) = 0,$$
$$\ddot{a}_m(t) + \omega_m^2 a_m(t) + \varepsilon\left(\sum_{k=0}^{N}\phi_0^{(k)}\phi_0^{(m)}a_k(t) - \phi_0^{(m)}v(t)\right) = 0, \quad m = 1,\dots,N. \tag{5}$$

The study of the NNMs of Eq. (5) will be performed at frequencies close to the linearized eigenfrequency ω_0; that is, the free, synchronous periodic oscillations of the system will be analyzed when it vibrates in-unison with frequency close to the eignfrequency of the zeroth modal oscillator. In this way the resonance interaction of the non-linear attachment with linear mode 0 of the undamped substructure will be investigated. As mentioned previously, the primary focus will be in single mode resonance interaction with the attachment and possible complications resulting from additional mutual interactions between modes of the linear substructure due to internal resonances will be omitted.

The analysis is complexified by introducing the following new complex variables:

$$\psi_v = \dot{v} + \mathrm{j}\omega_0 v,$$
$$\psi_0 = \dot{a}_0 + \mathrm{j}\omega_0 a_0,$$
$$\psi_k = \dot{a}_k + \mathrm{j}\omega_k a_k, \quad k = 1,\dots,N, \tag{6}$$

where $\mathrm{j} = (-1)^{1/2}$. Expressing the real variables in Eqs. (5) and (6) (where superscript star denotes complex conjugate),

$$v = \frac{\psi_v - \psi_v^*}{2\mathrm{j}\omega_0}, \quad \ddot{v} = \dot{\psi}_v - \frac{\mathrm{j}\omega_0}{2}(\psi_v + \psi_v^*), \quad v^3 = \frac{\mathrm{j}}{8\omega_0^3}(\psi_v + \psi_v^*)^3, \dots . \tag{7}$$

Eqs. (5) are expressed in the following complex form:

$$\dot{\psi}_v - \frac{j\omega_0}{2}(\psi_v + \psi_v^*) + \frac{jC}{8\omega_0^3}(\psi_v - \psi_v^*)^3$$
$$- \frac{j\varepsilon}{2\omega_0}\left[(\psi_v - \psi_v^*) - \phi_0^{(0)}(\psi_0 - \psi_0^*) - \sum_{k=1}^{N}\phi_0^{(k)}\left(\frac{\omega_0}{\omega_k}\right)(\psi_k - \psi_k^*)\right] = 0,$$
$$\dot{\psi}_0 - j\omega_0\psi_0 - \frac{j\varepsilon}{2\omega_0}\left[\phi_0^{(0)2}(\psi_0 - \psi_0^*) - \phi_0^{(0)}(\psi_v - \psi_v^*) + \sum_{k=1}^{N}\phi_0^{(k)}\phi_0^{(0)}\left(\frac{\omega_0}{\omega_k}\right)(\psi_k - \psi_k^*)\right] = 0,$$
$$\dot{\psi}_m - j\omega_m\psi_m - \frac{j\varepsilon}{2\omega_0}\left[\sum_{k=0}^{N}\phi_0^{(k)}\phi_0^{(m)}\left(\frac{\omega_0}{\omega_k}\right)(\psi_k - \psi_k^*) - \phi_0^{(m)}(\psi_v - \psi_v^*)\right] = 0, \quad m = 1, \ldots, N. \tag{8}$$

Expressions (8) are exact up to this point. The complex variables (6) are expressed in the following form:

$$\psi_v = \varphi_v e^{j\omega_0 t}, \quad \psi_0 = \varphi_0 e^{j\omega_0 t}, \quad \psi_m = \varphi_m e^{j\omega_m t}, \quad m = 1, \ldots, N, \tag{9}$$

where the complex time-varying amplitudes φ_i represent '*slowly*' *varying* quantities compared to the *fast* multiplying exponentials; hence, with Eq. (9) a partition between slow and fast dynamics in the problem is introduced. The specific forms of the multiplying exponentials in (9) (that govern the fast dynamics) are dictated by the physics of the resonance interaction under consideration herein; namely, the amplitudes of the zeroth modal oscillator and of the non-linear attachment are assumed to possess a fast frequency equal to the zeroth linearized eigenvalue ω_0, whereas the fast frequencies of each of the higher (non-resonant) modal oscillators are identical to their eigenfrequencies. These fast frequencies are perturbed by slow modulations introduced by the (slow) variation of the complex amplitudes φ_i.

Substituting Eq. (9) into Eq. (8), the first two equations are averaged over their (common) fast frequency ω_0, and each of the remaining equations over its corresponding fast frequency ω_m, $m = 1, \ldots, N$. Then, the following approximate averaged equations are obtained governing the evolutions of the slow modulations φ_i:

$$\dot{\varphi}_v + \frac{j\omega_0}{2}\varphi_v - \frac{3jC}{8\omega_0^3}|\varphi_v|^2\varphi_v^* - \frac{j\varepsilon}{2\omega_0}(\varphi_v - \phi_0^{(0)}\varphi_0) = 0,$$
$$\dot{\varphi}_0 - \frac{j\varepsilon}{2\omega_0}(\phi_0^{(0)2}\varphi_0 - \phi_0^{(0)}\varphi_v) = 0,$$
$$\dot{\varphi}_m - \frac{j\varepsilon}{2\omega_0}\phi_0^{(m)2}\left(\frac{\omega_0}{\omega_m}\right)\varphi_m = 0, \quad m = 1, \ldots, N. \tag{10}$$

The slowly varying complex amplitudes are expressed in polar form

$$\varphi_v = c_v e^{j\gamma_v}, \quad \varphi_0 = c_0 e^{j\gamma_0}, \quad \varphi_m = c_m e^{j\gamma_m}, \quad m = 1, \ldots, N, \tag{11}$$

where c_i represent real, slowly varying amplitudes and γ_i slowly varying (e.g., at most of $O(\varepsilon)$) real phases that govern the slow frequency modulations. In essence, the approximate set (10) governs the slow flow (with time-scale εt) of the dynamical system (5). Substituting Eq. (11) into Eq. (10) and equating the real parts to zero one obtains the following modulation equations for the real amplitudes c_i:

$$\left.\begin{aligned}\dot{c}_v - \frac{\varepsilon}{2\omega_0}\phi_0^{(0)} c_0 \sin(\gamma_0 - \gamma_v) = 0\\ \dot{c}_0 + \frac{\varepsilon}{2\omega_0}\phi_0^{(0)} c_v \sin(\gamma_0 - \gamma_v) = 0\end{aligned}\right\} \Rightarrow c_v^2 + c_0^2 = \rho^2,$$
$$\dot{c}_m = 0 \quad \Rightarrow \quad c_m = A_m, \quad m = 1, \ldots, N, \tag{12a}$$

where A_m are real constants. The integral relation derived by combining the first two equations is a consequence of the conservation of energy during the resonance interaction between the non-linear attachment and the zeroth linearized mode of the main substructure; hence, ρ^2 is an energy-like quantity. The remaining N equations in (12) indicate that the amplitudes of the non-resonant N modal oscillators in (5) are approximately [correct to $O(\varepsilon)$] constant.

Equating the imaginary parts of Eq. (10) to zero one obtains the following real phase modulation equations:

$$c_v\dot{\gamma}_v + \frac{\omega_0}{2}c_v - \frac{3C}{8\omega_0^3}c_v^3 - \frac{\varepsilon}{2\omega_0}\left[c_v - \phi_0^{(0)}c_0\cos(\gamma_0 - \gamma_v)\right] = 0,$$
$$c_0\dot{\gamma}_0 - \frac{\varepsilon}{2\omega_0}\left[\phi_0^{(0)2}c_0 - \phi_0^{(0)}c_v\cos(\gamma_0 - \gamma_v)\right] = 0,$$
$$\dot{\gamma}_m = \frac{\varepsilon}{2\omega_0}\phi_0^{(m)2}\left(\frac{\omega_0}{\omega_m}\right), \quad m = 1, \ldots, N. \tag{12b}$$

Note that the last N equations in Eq. (12b) indicate that the (slow) phase modulations of the non-resonant modes are of $O(\varepsilon)$, hence small. This is in agreement with the earlier assertion that during free oscillations the fast frequencies of the modal oscillators are perturbed by slowly varying frequency modulations. From the results above one concludes that during free oscillation the mth non-resonant modal oscillator possesses the approximate frequency,

$$\Omega_m \approx \omega_m + \frac{\varepsilon}{2\omega_0}\phi_0^{(m)2}\left(\frac{\omega_0}{\omega_m}\right).$$

For the sake of simplicity, it will be assumed that the initial conditions of the system are such that the (constant) amplitudes of the non-resonant modes in Eq. (12a) are equal to zero, e.g., that $A_m = 0, \quad m = 1, \ldots, N$. Further, assuming that $c_v c_0 \neq 0$ one combines the first two of Eq. (12b) by introducing the slow phase difference variable $\theta = \gamma_0 - \gamma_v$:

$$\dot{\theta} - \frac{\varepsilon}{2\omega_0}\left[\phi_0^{(0)2} - \phi_0^{(0)}\frac{c_v}{c_0}\cos(\gamma_0 - \gamma_v)\right] - \frac{\omega_0}{2} + \frac{3C}{8\omega_0^3}c_v^2 + \frac{\varepsilon}{2\omega_0}\left[1 - \phi_0^{(0)}\frac{c_0}{c_v}\cos(\gamma_0 - \gamma_v)\right] = 0. \tag{12c}$$

Eq. (12) govern the slow flow evolution of the real amplitude and phase modulations of the modal oscillators and the non-linear attachment during unforced and undamped free oscillation.

Of interest is to study the free periodic solutions of the slow flow, e.g., the NNMs of the combined system for frequencies close to ω_0. This is performed by requiring that the derivatives in

Eqs. (12a) and (12c) are equal to zero, obtaining the following set of equations:

$$
\begin{aligned}
&c_v = A_v, \qquad c_0^2 = \rho^2 - A_v^2,\\
&\sin(\gamma_0 - \gamma_v) = 0 \quad \Rightarrow \quad \gamma_0 - \gamma_v = 0,\\
&\frac{\varepsilon}{2\omega_0}\left[\phi_0^{(0)2} - \phi_0^{(0)}\frac{A_v}{A_0}\right] + \frac{\omega_0}{2} - \frac{3CA_v^2}{8\omega_0^3} - \frac{\varepsilon}{2\omega_0}\left[1 - \phi_0^{(0)}\frac{A_0}{A_v}\right] = 0.
\end{aligned}
\tag{13}
$$

For fixed structural parameters and specified energy-like quantity ρ^2, this represents a system of two non-linear coupled algebraic equations with unknowns the amplitudes A_v of the attachment and A_0 of the zeroth modal oscillator of the linear substructure. Provided that the frequency modulations (12b) are kept small (much less than the fast frequency ω_0) Eq. (13) solve approximately [correct to $O(\varepsilon)$] the problem of resonance interaction between the essentially non-linear attachment and the linear substructure for frequencies close to the lowest eigenfrequency. Recalling the co-ordinate transformations introduced previously, one derives the following approximate solutions for the motions of the attachment and the connecting point of the main substructure:

$$
\begin{aligned}
&v(t) = \frac{A_v}{\omega_0}\sin(\omega_0 t + \gamma_v(t) + O(\varepsilon^2)) + O(\varepsilon), \quad \dot{v}(t) = A_v\cos(\omega_0 t + \gamma_v(t) + O(\varepsilon^2)) + O(\varepsilon),\\
&y_0(t) = \phi_0^{(0)}\frac{A_0}{\omega_0}\sin(\omega_0 t + \gamma_v(t) + O(\varepsilon^2)) + O(\varepsilon),\\
&\dot{y}_0(t) = \phi_0^{(0)}A_0\cos(\omega_0 t + \gamma_v(t) + O(\varepsilon^2)) + O(\varepsilon).
\end{aligned}
\tag{14}
$$

The corresponding frequency of the synchronous oscillation of the combined system when it vibrates on the NNM is approximated as

$$
\Omega_0 \approx \omega_0 + \dot{\gamma}_v = \omega_0 + \dot{\gamma}_0 = \omega_0 + \frac{\varepsilon}{2\omega_0}\left[\phi_0^{(0)2} - \phi_0^{(0)}\frac{A_v}{A_0}\right]. \tag{15}
$$

Consistent with the earlier assertions, Ω_0 is in the vicinity of the eigenfrequency ω_0.

As an application, the system depicted in Fig. 2 is considered, consisting of a linear repetitive chain of $(N+1)$ coupled oscillators weakly connected to a non-linear attachment; setting $N = 1$, attention is focused in a chain of two coupled oscillators, with $\phi_0^{(0)} = 1/\sqrt{2}$. In Fig. 3 the amplitudes A_v and A_0 are depicted as functions of the energy-like parameter ρ for $\omega_0 = 0.9487$, $\omega_1 = 1.3784$, $d = 0.5$, $C = 5.0$, and $\varepsilon = 0.1$; all particles of the system are of unit mass

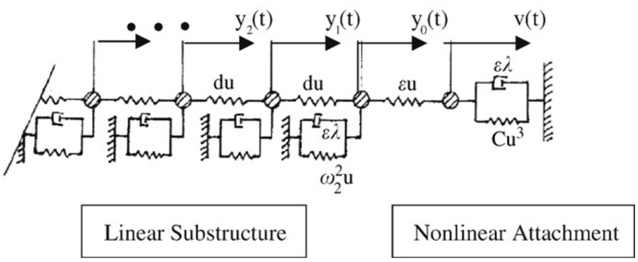


Fig. 2. The $(N+1)$ d.o.f. linear periodic chain with a weakly coupled non-linear end attachment.

$m = 1$, i.e., equal to the mass of the NES. Note that there exist four branches of synchronous free periodic oscillations (NNMs), one of which is unstable. These numbers are labelled from 1 to 4. For sufficiently small energies there exist two solutions; namely NNM branch 1 corresponding to oscillations localized at the attachment, and NNM branch 2 with oscillations predominantly confined to the zeroth modal oscillator with small movement for the attachment. For sufficiently

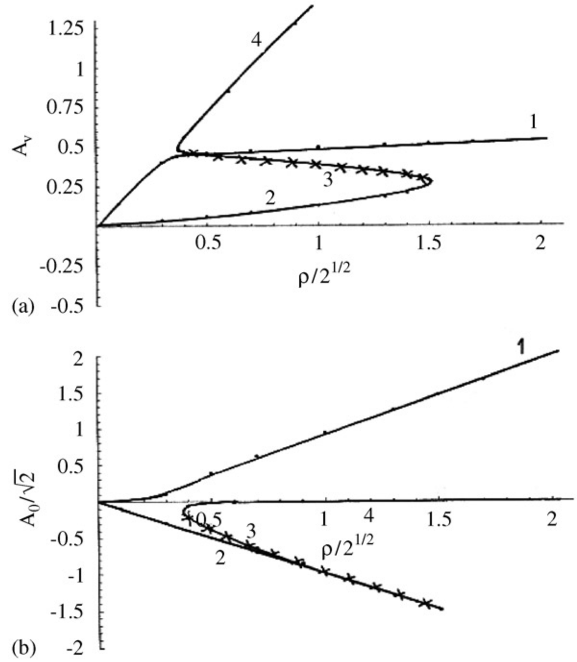


Fig. 3. Amplitudes A_v and A_0 as functions of the energy-like parameter ρ for NNMs in the vicinity of the lowest linearized mode: ——— Stable, × × × unstable NNMs.

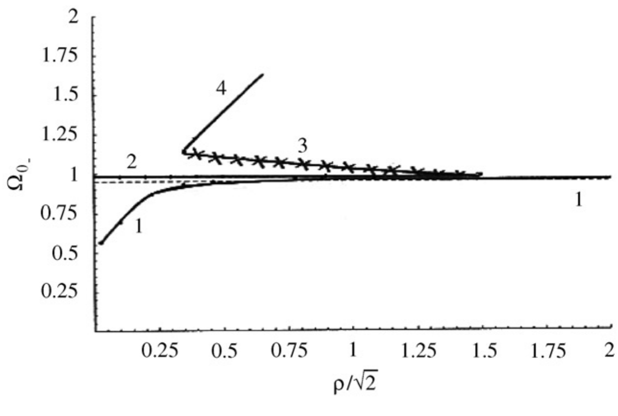


Fig. 4. Frequencies $\Omega_0(t)$ as functions of the energy-like parameter ρ for NNMs in the vicinity of the lowest linearized mode: ——— Stable, × × × unstable NNMs. Dashed line indicates the lowest eigenfrequency ω_0.

large energies, again there exist two solution branches: Branch 1 now corresponds to relatively large motion of the linear substructure in its zeroth mode, and moderate oscillation of the attachment; and branch 4 corresponding to strongly localized motion at the attachment. In Fig. 4 the corresponding frequencies of the four branches of NNMs are depicted as functions of the energy-like parameter ρ.

For the 3-d.o.f. under consideration the analysis is repeated for free periodic oscillations close to the highest eigenfrequency $\omega_1 = 1.3784$. In doing one studies the effect of the non-linear attachment on the highest linearized mode of the combined system. The analysis was performed by assuming fast oscillation with frequency ω_1, and it is similar to that performed previously for the lower mode. The results are depicted in the frequency–energy plot of Fig. 5, where there are also shown the previous results derived near the frequency ω_0. In Fig. 5 the modulated frequencies Ω_0 and Ω_1 of the combined system are depicted for free periodic oscillations close to the lower and higher linearized modes, respectively. Since these results are valid only in the vicinity of the corresponding eigenfrequencies, e.g., $\Omega_0 \approx \omega_0$ and $\Omega_1 \approx \omega_1$, there is the need to 'calibrate' them by appropriately defining a common energy measure that is valid close to each linearized mode. This was achieved by considering in each case the corresponding (conserved) physical energy of the combined system. Hence, for NNMs in the vicinity of the lowest linearized mode this energy measure was selected as

$$R^{(0)} = \frac{CA_v^4}{4\omega_0^4} + \phi_0^{(0)} A_0^2 + O(\varepsilon),$$

whereas, for NNMs close to the higher linearized mode the physical energy was

$$R^{(1)} = \frac{CB_v^4}{4\omega_2^4} + \phi_1^{(1)} B_1^2 + O(\varepsilon).$$

B_v and B_1 denote the approximately constant amplitudes of the non-linear attachment and highest linearized mode, respectively, that correspond to the NNM under consideration. Also depicted in the plot of Fig. 5 are the 'backbone curves' of the uncoupled system with $\varepsilon = 0$. These curves show

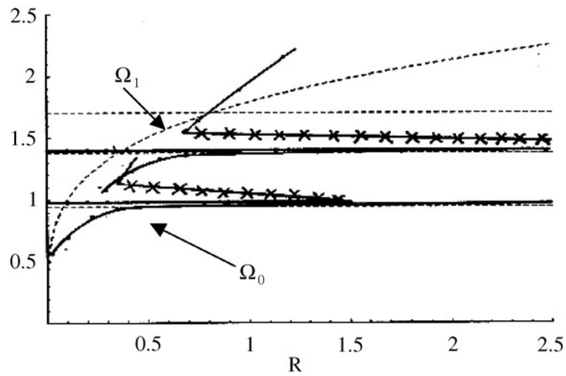


Fig. 5. NNM Frequencies Ω_0 and Ω_1 as functions of the (conserved) energy R of the combined system: ——— Stable, × × × unstable NNMs. Dashed lines indicate the 'backbone curves' of the uncoupled system with $\varepsilon = 0$.

the energy dependence of the frequency of free oscillation of the system. Clearly, the disjoint linear substructure possesses two (energy-independent) modes at frequencies ω_0 and ω_1, whereas the non-linear attachment possesses the frequency $\omega = [\pi C^{1/2}/2K(1/2)](4R/C)^{1/4}$, where $K(1/2)$ is the complete elliptic integral of the first kind, and R the level of physical energy.

From the results of Fig. 5 one notes that the effect of the non-linear attachment on the free dynamics of the system is more profound at points of internal resonance, e.g., at neighborhoods of the points of crossing of the 'backbone curves.' In addition, one notes that there are similar topological structures for the NNMs in each of the neighborhoods of ω_0 and ω_1, with two similar saddle-node bifurcations taking place near each of the linearized eigenfrequencies. In these bifurcations a stable/unstable pair of NNMs is generated or eliminated with increasing energy. For the NNMs in the vicinity of ω_1, the bifurcation where the pair of modes is eliminated occurs at the energy value $R_{bif}^{(1)} \approx 6.9182$ (the plot of Fig. 5 is not extended up to that energy level for reasons of clarity), which is much higher than the corresponding value for the lower frequency modes, $R_{bif}^{(0)} \approx 1.50046$. As discussed in the next section, this observation has important implications on the occurrence of *resonance capture cascades* in the strongly forced system.

Taking into account that at relatively high energies the motion of the combined system tends to nearly localize to the non-linear attachment, and 'bridging' the two solutions $\Omega_0(t) \approx \omega_0$ and $\Omega_1(t) \approx \omega_1$, one can construct a sketch of the synthesized frequency–energy plot for the system by synthesizing the disjoint solutions that appear in Fig. 5. This sketch is depicted in Fig. 6. Extending the labelling system introduced in Fig. 4 one numbers the NNM branches in the synthesized sketch from 1 to 7, with NNMs 3 and 6 being unstable. NNMs 1 and 7 enter the lower and upper *attenuation zones* of the infinite periodic chain, respectively. Both these NNMs are localized at the non-linear attachment, with branch 1 corresponding to in-phase oscillations of the particles of the chain, and branch 7 to anti-phase ones. There are two attenuation zones of the infinite uncoupled chain of Fig. 2, namely, a low-frequency (lower), $[0, \omega_0)$, and a high-frequency (upper) one,$(\sqrt{\omega_0^2 + 4d}, \infty)$. Inside these zones the chain can only support standing waves with exponentially decaying envelopes, e.g., nearfield solutions. The other NNMs of the system are located always in the propagation zone of the infinite chain, $(\omega_0, \sqrt{\omega_0^2 + 4d})$. Inside this zone the

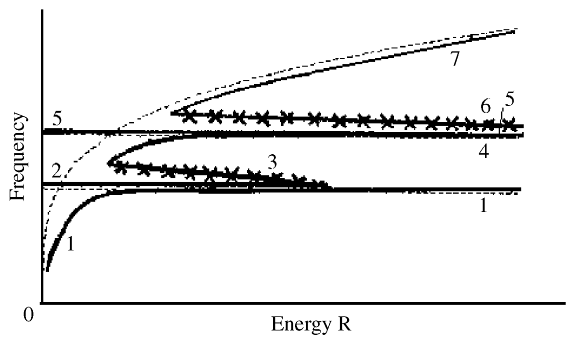


Fig. 6. Sketch of the synthesized NNM Frequencies as functions of the (conserved) energy R of the combined system: ——— Stable, × × × unstable NNMs. With dashed lines we depict the 'backbone curves' of the uncoupled system with $\varepsilon = 0$.

chain supports travelling wave solutions. At the bounding frequencies between attenuation and propagation zones the chain performs in-phase and out-of-phase normal oscillations.

4. Energy pumping and resonance capture cascades in the excited, damped system

In this section the excited and damped system is examined. The aim is to show that, provided that no conditions for internal resonances between the modes of the linear substructure exist (leading to non-linear modal interactions [15]), the energy variation of the NNMs of the undamped, unforced system helps to explain the energy pumping phenomena in the damped and forced one.

One starts by considering energy pumping in the system involving non-linear interaction of the attachment with only a *single mode* of the linear substructure. As mentioned previously, energy pumping is the transient one-way irreversible transfer of energy from a modal oscillator of the linear substructure to the non-linear attachment. In essence then the attachment acts as a *non-linear energy sink*. Considering again Eqs. (3) and (4), one recalls that they were derived under the assumption of impulsive excitation of the linear substructure at position p, and non-linear resonance interaction of the zeroth (lowest) mode with the attachment. Applying the complexification analysis of the previous section in Eq. (4) and averaging over the 'fast' eigenfrequency ω_0, the higher frequency terms on the right-hand side of Eq. (4) are averaged out and the equations reduce to the form of a 2-d.o.f. damped, forced system similar to the ones discussed in detail in Refs. [2,3]. Referring to the averaged analogs of (4) it can be analytically proven that provided that the external impulse is above a certain threshold, there occurs energy pumping from the modal oscillator to the non-linear attachment. Since this analysis was performed in the aforementioned works it will not be repeated here, but instead numerical simulations will be presented that substantiate our assertions.

For the simulations the system depicted in Fig. 2 was considered with $N = 9$ (that is, a 10-d.o.f. linear chain with end non-linear attachment), with each oscillator (including the non-linear attachment) possessing a grounded weak viscous damper with constant equal to $\varepsilon\lambda$. The parameters of the system are chosen as, $\omega_0^2 = 0.4$, $d = 3.5$, $C = 5.0$, $\lambda = 0.5$, and $\varepsilon = 0.1$, with initial conditions, $v(0) = \dot{v}(0) = 0$, $y_m(0) = 0$, $m = 0, 1, \ldots, N$, and $\dot{y}_m(0) = 0$, $m = 0, 1, 3, \ldots, N$, $\dot{y}_2(0) = 4$; these correspond to impulsive excitation of magnitude Y applied to the third from the right end particle of the chain. In Fig. 7 the responses $v(t), y_0(t)$ and $y_9(t)$ are depicted, together with the variation of the instantaneous frequency of oscillation of the non-linear attachment, $\Omega(t)$, versus time. This instantaneous frequency was computed elsewhere [2] as

$$\Omega(t) = \Xi I_1^{1/3}(t),$$

where

$$\Lambda = \left(\frac{1}{4C}\right)^{1/6}\left(\frac{3\pi}{K(1/2)}\right)^{1/3}, \quad \Xi = \left(\frac{3\pi^4 C}{8K^4(1/2)}\right)^{1/3}, \quad I_1(t) = \left(\frac{\pi^2\dot{v}^2(t)}{2\Lambda^2\Xi^2K^2(1/2)} + \frac{v^4(t)}{\Lambda^4}\right)^{3/4}.$$

Note, that this frequency differs from the frequencies of oscillation of the particles of the chain during the damped motion of the system, since the *transient dynamics* is considered. This is in contrast to the results of the previous section (cf. Figs. 3–6) where in the absence of damping and

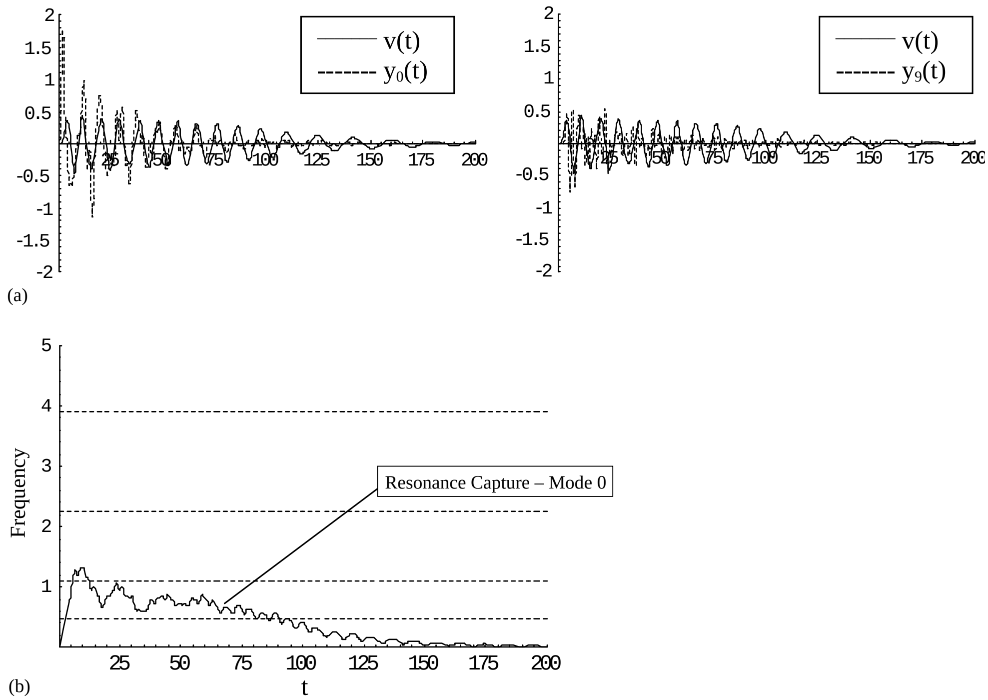


Fig. 7. Resonance interaction of the attachment with the lowest linearized mode leading to energy pumping for a moderately forced system: (a) transient responses $v(t)$, $y_0(t)$ and $y_9(t)$ (solid line corresponds to the attachment); (b) instantaneous frequency of the attachment versus time (dashed lines indicate the linearized eigenfrequencies).

transient forcing the coupled system performs synchronous vibrations (e.g., NNMs) with a single common frequency for the linear substructure and the attachment.

From the simulations of Figs. 7a and b it is clearly evidenced that there occurs energy pumping from the linear chain to the non-linear attachment; indeed, for $t > 70$ the motion is nearly confined to the non-linear attachment. Moreover, as indicated by the frequency plot of Fig. 7c, after some early time multi-frequency transients (that involve very short resonance interaction with modal oscillator 1 at frequency near ω_1), the energy pumping is mainly due to resonant interactions of the non-linear attachment and the lowest modal oscillator 0. Note that even though the impulsive force excites additional NNMs in the vicinities of higher linearized modes of the chain, the imparted initial energy is not sufficiently large to cause resonance interactions of the attachment with these higher frequency NNMs.

It is of interest to interpret the transient energy pumping phenomenon of Fig. 7 in terms of the dynamics (e.g., the NNMs) of the corresponding undamped, unforced system, namely the dynamics described in the plots of Figs. 3 and 4. First, one recognizes that since the excitation used in the simulations is the impulse $Y\delta(t)$, starting at $t = 0+$ the system performs free oscillations with initial energy equal to $(Y^2/2)$. For sufficiently strong excitation (initial energy) there exist two NNMs in the undamped system, namely, branches 1 and 4 (cf., Figs. 3 and 4). For sufficiently small damping (as is the case herein) it can be shown that the NNMs are preserved in the form of damped NNMs, e.g., decaying periodic (synchronous or asynchronous) oscillations of the system that take place on two-dimensional invariant manifolds in the phase space of the motion [16,17]. These damped modes can be considered as analytical continuations for weak damping of the NNMs of the undamped system. In what follows the damped analog of NNM p are denoted as *NNM damped invariant manifold p* or simply, *damped NNM p*.

The external impulsive excitation excites precisely these damped NNM invariant manifolds (or damped NNMs). Since the excitation is applied directly to the chain and the non-linear attachment is initially at rest, the initial state of the system at $t = 0+$favors excitation of only damped NNM 1 and not of damped NNM 4 (this mode cannot be excited at $t = 0+$ since it corresponds to strongly localized motions at the attachment and negligible motion of the chain, e.g., has incompatible initial conditions). Indeed, it can be shown that the maximum amplitude of oscillation reached by the attachment at the early stage of the motion is nearly identical to that predicted by NNM 1. As time increases and energy decreases (due to damping dissipation) the motion of the system mainly takes place on the damped invariant NNM 1 (once the motion reaches this non-linear manifold it can never leave it due to invariance), which at low energies corresponds to motions localized in the non-linear attachment. This is precisely what is observed in the numerical simulations of Fig. 7.

The topological structure of the NNM branches of Figs. 3 and 4 also helps explain the existence of a minimum threshold for the impulse magnitude, above which energy pumping can only occur. Indeed, if the impulse (or equivalently, the energy at $t = 0+$) is of small magnitude, either the damped NNMs 1 and 2, or all four damped NNMs 1–4 (with mode 3 being unstable) are excited. In this case, the initial state of the system at $t = 0+$ favors excitation of NNM invariant manifold 2, with nearly all energy being confined to the zeroth modal oscillator. In that case the damped motion takes place predominantly on the two-dimensional invariant manifold of damped NNM 2, and, as a result most of the externally induced energy remains confined in the chain, with only a small portion of this energy being ‘spreading’ to the attachment; hence, no energy pumping occurs. In this case the maximum amplitude of vibration reached by the attachment at the early stage of the motion is the one predicted by NNM 2. Similar conclusions can be drawn for the case when the initial energy is such that four damped NNMs exist in the system; no energy pumping can occur in that case either. From the discussion above one concludes, that in order for energy pumping to occur it is necessary to excite the damped NNM 1 at sufficiently high energies, namely, above the energy level corresponding to the bifurcation where the stable/unstable pair of NNMs 2 and 3 is eliminated with increasing energy. Hence, it appears that the energy level where this NNM bifurcation occurs can be regarded as a first estimate for the minimum energy threshold required for energy pumping in the system.

It is now shown that by increasing the magnitude of the impulse, it is possible to get resonance capture cascades in the system. By this, one denotes a sequence of multiple resonance interactions of the non-linear attachment with more than one modal oscillators of the linear substructure. In Fig. 8 the responses of the 11-d.o.f. system considered earlier are depicted, with the same parameters and initial conditions, except for $\dot{y}_2(0) = 10$; this amounts to increasing the magnitude of the impulse from 4 to 10. Judging from the instantaneous frequency of the non-linear attachment, one notes that there occurs a cascade of resonance captures involving the lowest three linearized modes of the chain. Referring to Fig. 8b, at the initial stage of the motion when the energy is relatively high, the attachment resonates with linearized mode 2 and energy pumping from the chain to the attachment takes place at frequencies near ω_2. As energy decreases due to

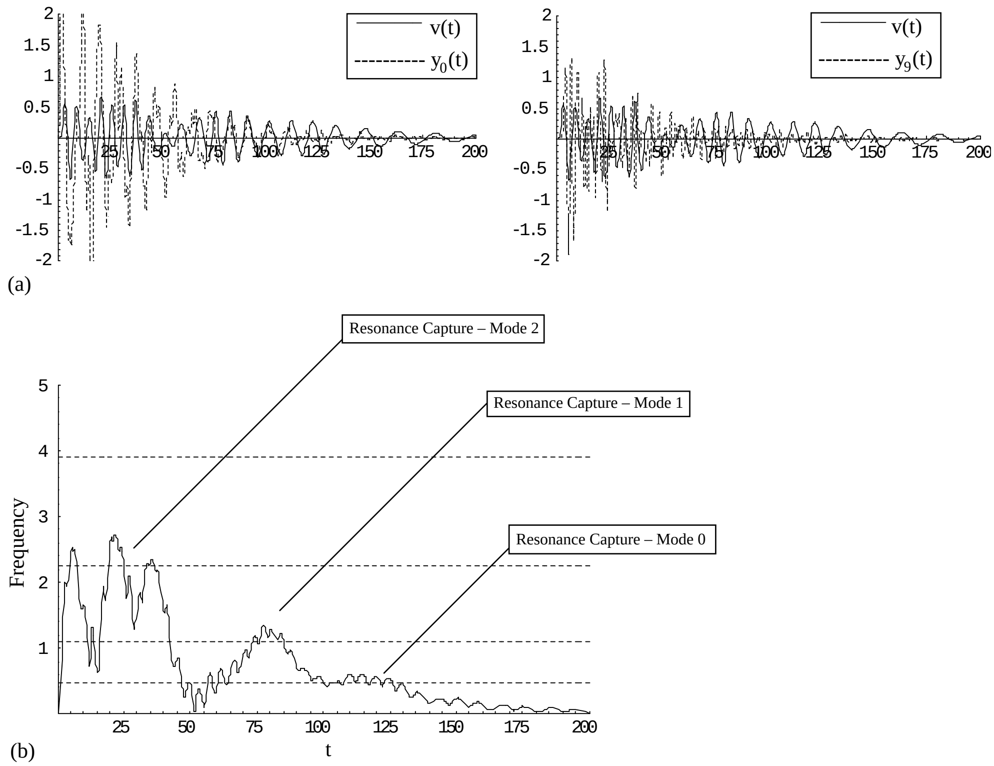


Fig. 8. Resonance capture cascades leading to energy pumping for the strongly forced system: (a) transient responses $v(t)$, $y_0(t)$ and $y_9(t)$ (solid line corresponds to the attachment); (b) instantaneous frequency of the attachment versus time (dashed lines indicate the linearized eigenfrequencies).

damping there occurs a sudden lower frequency transition (jump) into a regime where the attachment resonates with linearized mode 1; this results in energy pumping at frequencies near ω_1. With further decrease of the energy, there occurs a second lower frequency transition in the neighborhood of ω_0; at this third stage pumping is realized at frequencies near the lowest eigenfrequency of the linear chain. One concludes that for sufficiently strong excitations energy pumping from the chain to the attachment occurs at different frequency regimes that are reached through sudden transitions (jumps) from higher to lower frequencies; these regimes of energy pumping and resonance in the system represent resonance capture cascades.

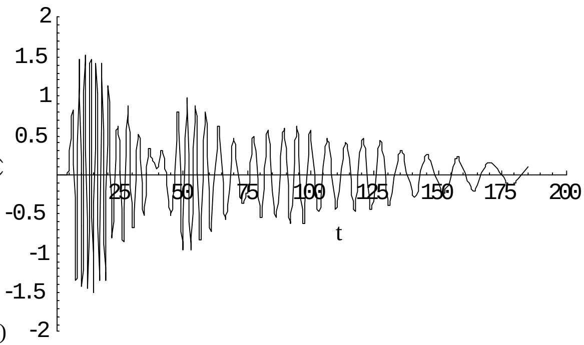
As a final comment, note that the analytical results of Section 3 lead to an interesting conjecture regarding the relation of resonance capture cascades to the topology of the NNMs of the underlying undamped, unforced system. To show this, one considers for simplicity the synthesized sketch of the NNM frequency branches of the 3-d.o.f. system depicted in Fig. 6. For sufficiently strong impulses the initial state of the system at $t = 0+$ favors motion of the system on the two-dimensional NNM invariant manifolds that are the damped analogs of NNMs 4 and 1; indeed both these modes correspond to moderate-amplitude motions of the attachment and large-amplitude responses of the first (anti-phase) and zeroth (in-phase) linearized modes of the chain. Moreover, it can be proven that 'competitor' damped invariant NNM manifolds that co-exist at high energies with the damped NNM manifolds 1 and 4 are either unstable (and thus not physically realizable), or cannot be significantly excited by the impulse due to incompatibility of the required initial conditions and the actual initial state of the system at $t = 0+$. As energy decreases due to damping the motion on the invariant manifold 1 leads to energy pumping at frequencies near ω_0 and eventual localization of the motion in the attachment, as discussed previously. The motion on the invariant manifold 4 gives rise to energy pumping at higher frequencies near ω_1 and gradual localization of the motion to the attachment. However, near the energy level where the NNMs 4 and 3 are eliminated through a saddle node bifurcation, there occurs a sudden transition (jump) of the damped motion from the damped NNM manifold 4 to the damped NNM manifold 1, driving the energy pumping process to lower frequencies. Hence the conjecture is formulated, that the cascades of resonance captures are caused by bifurcations of stable/unstable pairs of damped NNMs that abruptly eliminate as energy decreases the NNM manifolds responsible for high-frequency energy pumping. Note that excitation of the damped NNM 4 requires significantly large impulsive excitation, capable of inducing energy at $t = 0+$ above the level where the bifurcation of NNMs 6 and 4 occurs (cf., Fig. 6). This is compatible with the previous numerical simulations, indicating that resonance capture cascading requires sufficiently strong excitation of the system.

In an additional illustrative example of energy pumping, consider the 10-d.o.f. damped linear chain with the non-linear end attachment considered previously, with the same parameters, and initial conditions, $v(0) = \dot{v}(0) = 0$, $y_m(0) = 0$, $m = 0, 1, \ldots, N$, and $\dot{y}_m(0) = 0$, $m = 0, 1, 2, \ldots, 8$, $\dot{y}_9(0) = 70$. Hence, an impulsive excitation is applied to the most distant from the attachment particle of the chain. In Fig. 9 the response of the attachment $v(t)$ is depicted, together with the variation of the instantaneous frequency of oscillation of the non-linear attachment versus time. Note the vigorous resonance capture cascading involving as many as six of the linearized modes of the chain (including both the highest and lowest linearized modes in the frequency domain).

5. Discussion

The dynamics of a linear periodic substructure, weakly coupled to an essentially non-linear attachment was studied. The requirement of essential (non-linearizable) non-linearity is an important one, since this introduces a series of $(0 : \omega_m)$ internal resonances between the attachment and each of the modal oscillators of the linear substructure. In other words, there is no 'preferential' resonance frequency for the attachment, and, depending on the energy and the initial conditions, it can resonate with any of the modal oscillators. This feature increases the versatility of the non-linear attachment to act as energy sink. Moreover, since no specific restriction was posed on the configuration of the linear structure (other than it is discrete), the analysis is quite general and applies to a wide class of spatially extended, discrete linear structures with local essentially non-linear attachments. Placing more than one local non-linear attachments can also be considered with no particular difficulty by extending the methodology presented here.

For the free and undamped system it was found that there exist saddle-node NNM bifurcations where stable/unstable pairs of NNMs are generated or annihilated with increasing energy. There exist two such bifurcations in the neighborhood of each of the linearized eigenfrequencies of the



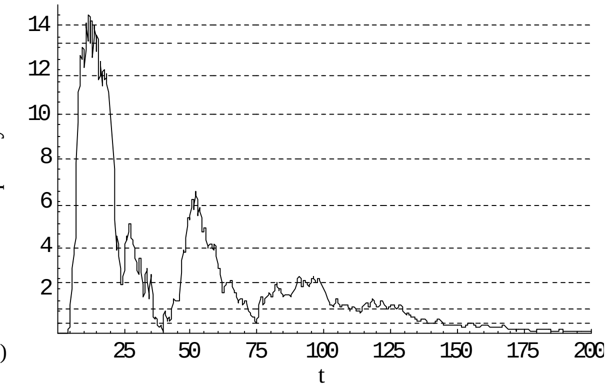


Fig. 9. Resonance capture cascades leading to energy pumping for the strongly forced system: (a) transient response of the attachment; (b) instantaneous frequency of the attachment versus time (dashed lines indicate the linearized eigenfrequencies).

uncoupled linear substructure. In addition, two of the NNMs enter the attenuation zones of the uncoupled infinite periodic structure and are localized to the non-linear attachment. These two modes are essentially non-linear (e.g., are not analytic continuations of linear modes), having no linear counterparts; the mode localizing at lower frequencies corresponds to in-phase oscillations of the particles of the chain, whereas the one localizing at higher frequencies to anti-phase motions. In general, the study of free dynamics indicates that the non-linear attachment has a profound effect on the dynamics mainly in neighborhoods of internal resonances between the attachment and the chain, e.g., at points of crossing of the 'backbone curves' of the two uncoupled systems.

The impulsively forced and damped system was then considered. It was shown that this system has energy pumping properties. Indeed, with proper choice of the system parameters and sufficiently strong external forcing, energy gets passively transferred from the chain to the attachment in a one way, irreversible action. Damping in the system is a prerequisite for occurrence of energy pumping, since in the undamped system, at most, one can induce non-linear beat phenomena between the chain and the attachment (energy exchanges), but not energy pumping. An interesting finding of this work is that although energy pumping is realized only in the presence of damping, the energy pumping phenomenon strongly depends on the topological structure of the NNMs of the underlying undamped system. Indeed, energy pumping occurs due to the excitation of certain damped invariant NNM manifolds that are analytic continuations of NNMs for small damping of the underlying undamped system. Which of these invariant manifolds are eventually excited depends on the specific forcing distribution used and the initial conditions of the system. Having this knowledge one may perform an optimization study, designing the NNMs of the undamped system with the aim of achieving optimal energy pumping in the damped case.

The bifurcations of the NNMs in the undamped system help explain another interesting phenomenon observed in the dynamics of the damped system, namely, resonance capture cascades. Provided that the forcing is sufficiently strong there occur multiple resonance captures, e.g., a series of resonance interactions between the attachment and certain of the modal oscillators of the linear substructure. This results in a series of energy pumping events occurring at different frequencies, with sudden transitions to lower frequencies between sequential events. It is conjectured that this cascading is caused by the sudden elimination through bifurcation of certain damped NNM invariant manifolds with decreasing energy, forcing the motion to lower frequency NNM invariant manifolds. The observed multi-frequency energy pumping cascades are particularly interesting from a practical point of view, since they prove that non-linear attachments can be designed to resonate and extract energy from an a priori specified set of modes of a linear structure. Indeed, the set of modes that participates in the resonance capture cascades can be selected to be compatible to the design objectives of the problem.

Acknowledgements

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