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Letter to the Editor

A novel formulation of the receptance matrix of non-proportionally damped dynamic systems

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1. Introduction

The calculation of the inverse of the summation of matrices is an operation that one comes across frequently in different scientific disciplines. One of the most known examples is the receptance matrix which plays a very important role in the mechanical vibration area.

As is known the receptance matrix (also called the frequency response matrix) is an important matrix which interrelates the input and output of a damped linear discrete mechanical system which is subject to harmonical forcing as input. There are many publications in the literature on this subject. Some of the recent publications are Refs. [1–3]. Yang presented in Ref. [1] an exact method for evaluating the receptances of non-proportionally damped dynamic systems. Based on a decomposition of the damping matrix, an iteration procedure is developed which does not require matrix inversion. In Ref. [2], Lin and Lim developed a new and effective method to derive structural design sensitivities which include both frequency response function sensitivities and eigenvalue and eigenvector sensitivities from limited vibration test data. The study of Mottershead [3] was concerned with the zeros of structural frequency response functions and their sensitivities.

The recent study in Ref. [4] is concerned with a viscously damped linear mechanical system, the co-ordinates of which are assumed to be subject to several constraint equations. The frequency response matrix of the constrained system described above is established in terms of the frequency response matrix of the unconstrained system and the coefficient vectors of the constraint equations.

This study has been motivated by Yang's article [1]. In his paper, an iterative method was developed for the calculation of the receptance matrix when the damping matrix was decomposed into the sum of dyadic products. It was specifically pointed out that there was no need to applying an inverse operation of matrices during the iteration procedure. In Yang's method, first the iteration process starts with the receptance matrix of the undamped system. Then the application of iteration, using as many iterations as the number of dyadic products in the damping matrix,

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gives the receptance matrix of the damped system. In getting to the final form of the iteration equation, a formula which is known as the Sherman–Morrison formula [5] in the matrix literature, used for obtaining the inverse of the summation of a regular matrix, and a dyadic product was also utilized.

The aim of this study is to show that it is possible to obtain the receptance matrix directly without using the iterations which were used in Ref. [1]. In this frame, a more general procedure of obtaining the inverse of the sum of a regular matrix and any symmetric, positive semi-definite matrix, is taken into account. Subsequently, a new formula is given to obtain the inverse of the general problem mentioned before. The present formulation does not require any iterations but it needs one more inverse operation in addition to Yang's procedure. Actually, both Ref. [1] and the present procedure require the inverse operation for obtaining the receptance matrix of the undamped system. The new formulation is based on the fact that a symmetric and positive semi-definite matrix can be expressed as the sum of dyadic products [6].

As it is known, the Sherman–Morrison formula is useful in obtaining the inverse of a regular matrix and only one dyadic product (i.e., rank 1). On the other hand, the Woodbury formula gives the inverse of the sum of a regular matrix and a matrix product whose rank can be grater than 1 (rank $r \geqslant 1$) [5].

The new methodology formulated in this study makes it possible to put any number of dyadic products by expressing it in terms of a matrix product ($r \geqslant 1$) into a form where the Woodbury formula can be used.

In the following section, after a brief introduction, the procedure followed in Ref. [1] will be summarized, both, from the point of completeness and that of clarity for the readers' understanding. In Section 3, the derivation of the new formula which was mentioned above will be given. In the last section, the calculation of receptance matrix by using the new formula without iteration will be given.

2. Theory

The motion of a viscously damped linear mechanical system with n-degrees-of-freedom which is harmonically excited, is governed in the physical space by the matrix differential equation of order two

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \bar{\mathbf{F}}\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the $(n \times n)$ mass, damping and stiffness matrices, respectively. $\mathbf{q}$ is the $(n \times 1)$ vector of generalized co-ordinates. $\bar{\mathbf{F}}$ is the forcing vector and ω denotes the forcing frequency.

Substitution of

$$\mathbf{q}(t) = \bar{\mathbf{q}}\mathrm{e}^{\mathrm{i}\omega t} \tag{2}$$

into Eq. (1) yields the relation

$$\bar{\mathbf{q}} = \mathbf{H}(\omega)\bar{\mathbf{F}} \tag{3}$$

between the constant part of the input and response vectors. The complex matrix

$$\mathbf{H}(\omega) = (-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{D} + \mathbf{K})^{-1} \tag{4}$$

is referred to as the (complex) frequency response matrix or the receptance matrix. It is also referred to as the admittance matrix or dynamic influence coefficient matrix [7]. It is assumed that the mechanical system given in Eq. (1) is non-proportionally damped.

2.1. The iteration method in Ref. [1]

A brief summary of the iteration method in Ref. [1] will be given to familiarize the reader with the procedure.

The starting point of the method is the relation

$$\mathbf{H}(\omega) = [\mathbf{I} + \mathrm{i}\omega\mathbf{H}_0(\omega)\mathbf{D}]^{-1}\mathbf{H}_0(\omega) \tag{5}$$

used also in Ref. [8]. The derivation of Eq. (5) is given in the appendix. $\mathbf{H}_0(\omega)$, represents the receptance matrix of the undamped system where $\mathbf{D} = \mathbf{0}$, and has the following form:

$$\mathbf{H}_0(\omega) = (-\omega^2\mathbf{M} + \mathbf{K})^{-1}. \tag{6}$$

Let the rank of damping matrix $\mathbf{D}$ be $r \leqslant n$. It is possible to express $\mathbf{D}$ as

$$\mathbf{D} = \boldsymbol{\psi}\boldsymbol{\Lambda}_D\boldsymbol{\psi}^T = \sum_{k=1}^{r} d_k\boldsymbol{\psi}_k\boldsymbol{\psi}_k^{\mathrm{T}} \tag{7}$$

since $\mathbf{D}$ is a symmetric, positive semi-definite matrix.

In Eq. (7), $\boldsymbol{\Lambda}_D$ is

$$\boldsymbol{\Lambda}_D = \mathbf{diag}(d_1 \ldots d_r \quad 0 \ldots 0) \in R^{n\times n}, \quad \boldsymbol{\psi} = \left[\boldsymbol{\psi}_1 \ldots \boldsymbol{\psi}_n\right] \in R^{n\times n}. \tag{8}$$

Also in Eq. (7) d_k's represent real and positive scalars and $\boldsymbol{\psi}_k$'s represent real and linear independent vectors. Even though in the expansion of $\mathbf{D}$ in the form of (7), d_k and $\boldsymbol{\psi}_k$ can be used as the eigensolutions or eigenvalues and eigenvectors of $\mathbf{D}$, but it is not necessary.

If relation (7) is used for the damping matrix, then Eq. (5) will be as follows:

$$\mathbf{H}^{-1}(\omega) = \mathbf{H}_0^{-1}(\omega) + \mathrm{i}\omega\sum_{k=1}^{r} d_k\boldsymbol{\psi}_k\boldsymbol{\psi}_k^{\mathrm{T}}, \tag{9}$$

where $\mathbf{H}^{-1}(\omega)$ and $\mathbf{H}_0^{-1}(\omega)$ are inverses of the matrices $\mathbf{H}(\omega)$ and $\mathbf{H}_0(\omega)$, respectively.

Let $\mathbf{H}_\ell(\omega)$ be a sequence of intermediate receptance matrices and be defined as

$$\mathbf{H}_\ell^{-1}(\omega) = \mathbf{H}_0^{-1}(\omega) + \mathrm{i}\omega\sum_{k=1}^{\ell} d_k\boldsymbol{\psi}_k\boldsymbol{\psi}_k^{\mathrm{T}} \quad (\ell = 1, \ldots, r). \tag{10}$$

It can easily be seen that $\mathbf{H}_\ell(\omega)$ are symmetric and because of Eqs. (9) and (10),

$$\mathbf{H}(\omega) = \mathbf{H}_r(\omega). \tag{11}$$

On the other hand, based on Eq. (10), it is obvious that one can write the following expression:

$$\mathbf{H}_{\ell+1}^{-1}(\omega) = \mathbf{H}_\ell^{-1}(\omega) + \mathrm{i}\omega d_{\ell+1}\boldsymbol{\psi}_{\ell+1}\boldsymbol{\psi}_{\ell+1}^{\mathrm{T}}. \tag{12}$$

The last relation can be put into the following form:

$$\mathbf{H}_{\ell+1}(\omega) = \left[\mathbf{I} + \mathrm{i}\omega d_{\ell+1}\mathbf{H}_{\ell}(\omega)\boldsymbol{\psi}_{\ell+1}\boldsymbol{\psi}_{\ell+1}^{\mathrm{T}}\right]^{-1}\mathbf{H}_{\ell}(\omega), \tag{13}$$

when the formula

$$(\mathbf{K} + \mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1} = \mathbf{K}^{-1} - \mathbf{K}^{-1}\mathbf{u}(1 + \mathbf{v}^{\mathrm{T}}\mathbf{K}\mathbf{u})^{-1}\mathbf{v}^{\mathrm{T}}\mathbf{K}^{-1} \tag{14}$$

is used for obtaining the inverse of the sum of a square matrix and a dyadic product and known as the Sherman–Morrison formula in the literature, expression (13) will take the following form:

$$\mathbf{H}_{\ell+1}(\omega) = \left[\mathbf{I} - \frac{\mathrm{i}\omega d_{\ell+1}\mathbf{H}_{\ell}(\omega)\boldsymbol{\psi}_{\ell+1}\boldsymbol{\psi}_{\ell+1}^{\mathrm{T}}}{1 + \mathrm{i}\omega d_{\ell+1}\boldsymbol{\psi}_{\ell+1}^{\mathrm{T}}\mathbf{H}_{\ell}(\omega)\boldsymbol{\psi}_{\ell+1}}\right]\mathbf{H}_{\ell}(\omega). \tag{15}$$

Thus, after calculating the receptance matrix $\mathbf{H}_0(\omega)$ of the undamped system, the receptance matrix of the damped system can be obtained by using Eq. (15) at the end of $\ell = 0, 1, \ldots, r-1$ successive iterations.

2.2. *The derivation of the new formula*

In this section, it will be shown that it is possible to calculate the receptance matrix of a non-proportionally damped system directly without using any iteration. In this context, a new formula for calculating the inverse of the sum of a square matrix and another matrix which is composed of any number of (r) dyadic products, will be derived.

Let $\mathbf{A}$ represent the regular matrix and $\mathbf{B}$ represent the sum of the dyadic products. The question is to find the inverse of $\mathbf{A}+\mathbf{B}$.

Let the dimension of $\mathbf{A}$ be ($n \times n$). The dimensions of column vectors $\mathbf{x}_i$ and $\mathbf{y}_i$ are ($n \times 1$). Let $\mathbf{B}$ represent the sum of r dyadic products of vectors $\mathbf{x}_i$ and $\mathbf{y}_i$:

$$\mathbf{B} = \sum_{i=1}^{r} \mathbf{x}_i\mathbf{y}_i^{\mathrm{T}}. \tag{16}$$

It is obvious that the sum of r dyadic products can be written as

$$\mathbf{B} = [\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n]\begin{bmatrix}\mathbf{y}_1^{\mathrm{T}}\\ \mathbf{y}_2^{\mathrm{T}}\\ \vdots\\ \mathbf{y}_n^{\mathrm{T}}\end{bmatrix}. \tag{17}$$

At this point the following definitions are made:

$$\mathbf{X} \equiv [\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n]; \quad \mathbf{Y} \equiv \left[\mathbf{y}_1\mathbf{y}_2\ldots\mathbf{y}_n\right]. \tag{18}$$

With the aid of these matrices, the matrix $\mathbf{B}$ in (16) can be expressed as follows:

$$\mathbf{B} = \mathbf{X}\mathbf{Y}^{\mathrm{T}}. \tag{19}$$

At this step, the formula

$$(\mathbf{K} + \mathbf{U}\mathbf{V}^{\mathrm{T}})^{-1} = \mathbf{K}^{-1} - \mathbf{K}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{V}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{U})^{-1}\mathbf{V}^{\mathrm{T}}\mathbf{K}^{-1} \tag{20}$$

will be used. Eq. (20) is also known as the Woodbury formula in the literature. As it can be seen from Eq. (20), the Woodbury formula gives the inverse of the sum of a regular square matrix ($n \times n$), and a product of two matrices whose rank is r. It is obvious that when the rank $r = 1$, then Woodbury formula reduces to that of Sherman–Morrison [5].

In the present case, if the following notations

$$\mathbf{U} = \mathbf{X}, \quad \mathbf{V} = \mathbf{Y} \tag{21}$$

are used in (20), then

$$(\mathbf{A} + \mathbf{X}\mathbf{Y}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{X}(\mathbf{I} + \mathbf{Y}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{X})^{-1}\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{-1} \tag{22}$$

is obtained. This is no more than the relation that is required.

2.3. The expression of the receptance matrix with the aid of the new formula

After obtaining Eq (22) in the preceeding section, it is now quite easy to calculate the receptance matrix $\mathbf{H}(\omega)$ of the dynamical system defined by Eq. (1).

If the following correspondence:

$$\begin{aligned} &\mathbf{A} \hat{=} \mathbf{K} - \omega^2\mathbf{M}, \quad \mathbf{X} = \left[d_1\boldsymbol{\psi}_1 d_2\boldsymbol{\psi}_2 \ldots d_n\boldsymbol{\psi}_n\right], \\ &\mathbf{Y} = \left[\boldsymbol{\psi}_1\boldsymbol{\psi}_2 \ldots \boldsymbol{\psi}_n\right], \end{aligned} \tag{23}$$

between the general formula and the dynamical system and also Eq. (6)–(8) are used, then the receptance matrix $\mathbf{H}(\omega)$ can be obtained as

$$\begin{aligned} \mathbf{H}(\omega) &= (\mathbf{K} - \omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{D})^{-1} \\ &= \mathbf{H}_0(\omega)[\mathbf{I} - (\mathrm{i}\omega\mathbf{X})(\mathbf{I} + \mathbf{Y}^{\mathrm{T}}\mathbf{H}_0(\omega)(\mathrm{i}\omega\mathbf{X}))^{-1}\mathbf{Y}^{\mathrm{T}}\mathbf{H}_0(\omega)]. \end{aligned} \tag{24}$$

As it is seen from Eq. (24), the receptance matrix $\mathbf{H}(\omega)$ of the non-proportionally damped system can be calculated with the aid of the receptance matrix, $\mathbf{H}_0(\omega)$, of the undamped system and the eigencharacteristics of the damping matrix $\mathbf{D}$ without using any iteration.

3. Numerical evaluations

This section is devoted to the numerical evaluation of the formulae obtained. The simple system in Fig. 1 is taken as an illustrative example. Assume that the following numerical values are chosen for the physical parameters of the system:

$$\begin{aligned} &k_1 = 2\,\mathrm{N/m}, \quad k_2 = 1\,\mathrm{N/m}, \quad m_1 = 2\,\mathrm{kg}, \quad m_2 = 1\,\mathrm{kg}, \\ &c_1 = 0.35\,\mathrm{N/m/s}, \quad c_2 = 0.15\,\mathrm{N/m/s}, \quad c_3 = 0.05\,\mathrm{N/m/s}, \\ &\omega = 1\,\mathrm{rad/s}. \end{aligned}$$

The numerical values above yield the following system matrices:

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.40 & -0.05 \\ -0.05 & 0.20 \end{bmatrix}.$$

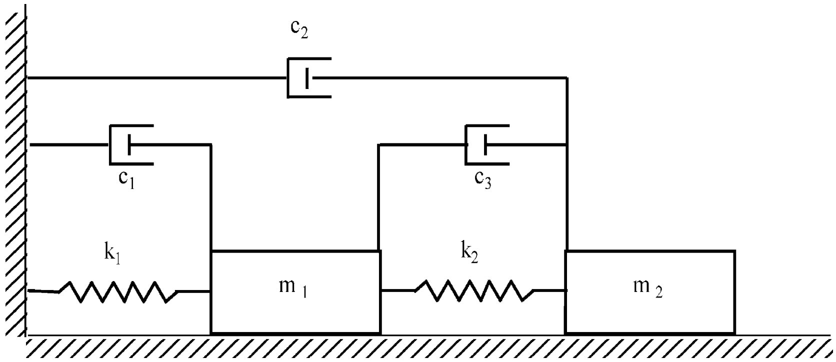

Fig. 1. The sample system.

The receptance matrix given in Eq. (4) is obtained as

$$\mathbf{H}(\omega) = \begin{bmatrix} 0.01707933 - 0.18402976\mathrm{i} & -0.91587897 - 0.13140408\mathrm{i} \\ -0.91587897 - 0.13140408\mathrm{i} & -0.88599015 - 0.45345616\mathrm{i} \end{bmatrix}.$$

The eigencharacteristics of the damping matrix $\mathbf{D}$ are

$$d_1 = 0.41180340, \quad d_2 = 0.18819660$$

$$\boldsymbol{\psi}_1 = \begin{bmatrix} -0.97324899 \\ 0.22975292 \end{bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{bmatrix} -0.22975292 \\ -0.97324899 \end{bmatrix}.$$

By using d_1, d_2, $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$, the matrices $\mathbf{X}$ and $\mathbf{Y}$ can be obtained in the following form:

$$\mathbf{X} = \begin{bmatrix} -0.40078724 & -0.04323872 \\ 0.09461303 & -0.18316215 \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} -0.97324899 & -0.22975292 \\ 0.22975292 & -0.97324899 \end{bmatrix}.$$

The correctness of the formula can be checked by forming the following matrix product:

$$\mathbf{X}\mathbf{Y}^{\mathrm{T}} = \begin{bmatrix} 0.40 & -0.05 \\ -0.05 & 0.20 \end{bmatrix} = \mathbf{D}.$$

As it is seen from the above equation, the matrix product $\mathbf{X}\mathbf{Y}^{\mathrm{T}}$ equals the damping matrix $\mathbf{D}$.

On the other hand, the matrix $\mathbf{H}_0(\omega)$ given by Eq. (6) can easily be obtained as follows:

$$\mathbf{H}_0(\omega) = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}.$$

If the matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{H}_0(\omega)$ given above are taken into Eq. (24), then exact correspondence between the new and old $\mathbf{H}(\omega)$, is obtained:

$$\mathbf{H}(\omega) = \begin{bmatrix} 0.01707933 - 0.18402976\mathrm{i} & -0.91587897 - 0.13140408\mathrm{i} \\ -0.91587897 - 0.13140408\mathrm{i} & -0.88599015 - 0.45345616\mathrm{i} \end{bmatrix}.$$

This is the same as the one obtained directly from expression (4).

4. Conclusions

This study is concerned with a novel representation of the receptance matrix, which plays a very important role in the investigation of the linear vibrational systems excited harmonically. In this context, first the damping matrix is written as the sum of dyadic products then the sum is put into the form of the product of two matrices.

Consequently, it is possible to express the receptance matrix of the damped system in terms of the receptance matrix of the undamped system and the product of matrices, which represent the damping by using the Woodbury formula.

Appendix A

Let us assume that Eq. (5) is correct:

$$\mathbf{H}(\omega) = [\mathbf{I} + \mathrm{i}\omega\mathbf{H}_0(\omega)\mathbf{D}]^{-1}\mathbf{H}_0(\omega).$$

First, the explicit form of $\mathbf{H}_0(\omega)$ can be put into the above equation:

$$\mathbf{H}(\omega) = [\mathbf{I} + \mathrm{i}\omega(-\omega^2\mathbf{M} + \mathbf{K})^{-1}\mathbf{D}]^{-1}(-\omega^2\mathbf{M} + \mathbf{K})^{-1}.$$

If the inverse of the last relation is taken, then the relation

$$\mathbf{H}^{-1}(\omega) = (-\omega^2\mathbf{M} + \mathbf{K})[\mathbf{I} + \mathrm{i}\omega(-\omega^2\mathbf{M} + \mathbf{K})^{-1}\mathbf{D}]$$

will be obtained. After some manipulations, the following expression:

$$\mathbf{H}^{-1}(\omega) = (-\omega^2\mathbf{M} + \mathbf{K}) + \mathrm{i}\omega\mathbf{D}$$

can be obtained. If the inverse of the last equation is taken, then it is arrived at the

$$\mathbf{H}(\omega) = (-\omega^2\mathbf{M} + \mathbf{K} + \mathrm{i}\omega\mathbf{D})^{-1}.$$

This is the definition of the receptance matrix given in Eq. (4).

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