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Letter to the Editor

On the necessary and sufficient conditions for the existence of classical normal modes in damped linear dynamic systems

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1. Introduction

The existence of classical real normal modes in damped linear vibrating systems was investigated by Caughey and O'Kelly [1]. According to them the necessary and sufficient condition for a damped system governed by the equations of motion:

$$[\mathbf{I}]\{\ddot{\mathbf{x}}(t)\} + [\mathbf{A}]\{\ddot{\mathbf{x}}(t)\} + [\mathbf{B}]\{\mathbf{x}(t)\} = \{\mathbf{f}(t)\} \tag{1}$$

to possess classical real normal modes is given by the theorem:

Theorem 1. *System* (1) *possesses classical normal modes if, and only if, the matrices* $[\mathbf{A}]$ *and* $[\mathbf{B}]$ *commute,* i.e., $[\mathbf{A}][\mathbf{B}] = [\mathbf{B}][\mathbf{A}]$.

Liang et al. [2] have questioned, on the basis of two examples, the classification of damped systems into proportional and non-proportional systems based on the Caughey and O'Kelly's criterion. According to them "few of us realized that using the formula $[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}]$ (in the present notation) to classify the vibration systems into two categories is misleading." If this is true then the classification of non-conservative systems into proportionally and non-proportionally damped systems in accordance with the Caughey and O'Kelly's criterion needs to be modified.

The purpose of this article is two-fold. Firstly, to address the issues raised in Ref. [2] on the classification of damped systems and secondly to place Caughey and O'Kelly's theorem in context with other well-known theorems in linear algebra on the simultaneous diagonalization of two real symmetric (complex Hermitian) matrices using a real, non-singular orthogonal (complex unitary) transformation matrix. Before addressing the issues raised in Liang et al., it is worthwhile to recall relevant theorems from linear algebra.

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2. General theorems

Theorem 2. *The conditions which must be satisfied in order that two given quadratic expressions*:

$$a_{11}x_1^2 + a_{22}x_2^2 + \cdots + 2a_{12}x_1x_2 + \cdots,$$
$$b_{11}x_1^2 + b_{22}x_2^2 + \cdots + 2b_{12}x_1x_2 + \cdots$$

may be simultaneously reducible to the form

$$a_{11}\zeta_1^2 + a_{22}\zeta_2^2 + \cdots + a_nn\zeta_n^2,$$
$$b_{11}\zeta_1^2 + b_{22}\zeta_2^2 + \cdots + b_nn\zeta_n^2$$

are that the elementary divisors of the determinant $|a_{rs}\lambda - b_{rs}|$ *are linear, where* a_{rs} *and* b_{rs} *are the elements of the matrices* [**A**] *and* [**B**], *respectively*.

The above theorem is the well-known theorem of Weierstrass developed in 1858 [3]. This theorem is extended to Hermitian complex matrices by Kronecker [4]. Rao and Mitra [5, Chapter 6] have discussed several equivalent theorems for the simultaneous diagonalization of a pair of Hermitian matrices. The theorem concerning the simultaneous diagonalization of two hermitian matrices is:

Theorem 3. *Let* [**A**] *and* [**B**] *be hermitian matrices of order n. Then there exists a unitary matrix* [**U**] *such that* $[\mathbf{U}]^*[\mathbf{A}]\,[\mathbf{U}]$ *and* $[\mathbf{U}]^*[\mathbf{B}]\,[\mathbf{U}]$ *are diagonal if and only if* [**A**] *commutes with* [**B**]. *In the later case* [**A**] [**B**] *is Hermitian and* $[\mathbf{U}]^*[\mathbf{A}][\mathbf{B}][\mathbf{U}]$ *is also diagonal. Where* $(\ )^*$ *is a complex conjugate transpose operator*.

Another important theorem concerning the simultaneous diagonalization of any number of Hermitian matrices is due to Rao and Mitra [5, Theorem 6.2.8, p. 124] which states:

Theorem 4. *Let* $[\mathbf{A}_i]$ $(i = 1, 2, \ldots, k)$ *be hermitian matrices of the same order. There exists a unitary matrix* [**U**] *such that* $[\mathbf{U}]^*\,[\mathbf{A}_i]\,[\mathbf{U}]$ *is diagonal for each* $(i = 1, 2, \ldots, k)$, *if and only if* $[\mathbf{A}_i]$ *commutes with* $[\mathbf{A}_j]$ *for each i and j*.

However the above theorem does not guarantee, in general, a real orthogonal transformation matrix [**U**]. For real symmetric matrices the analogue to Theorem 4 is Theorem 1.3.19 in Ref. [6] which, for real symmetric matrices, states:

Theorem 5. *Let* $[\mathbf{A}_i]$ $(i = 1, 2, \ldots, k)$ *be real, symmetric matrices of the same order. There exists a real orthogonal matrix* [**U**] *such that* $[\mathbf{U}]^{\mathrm{T}}\,[\mathbf{A}_i]\,[\mathbf{U}]$ *is diagonal for each* $(i = 1, 2, \ldots, k)$, *if and only if* $[\mathbf{A}_i]$ *commutes with* $[\mathbf{A}_j]$ *for each i and j*.

According to the above theorem the necessary and sufficient conditions for a viscously damped system governed by the matrices [**M**] (mass), [**K**] (stiffness) and [**C**] (damping) to be simultaneously diagonalized by a real orthogonal matrix [**U**] are:

$$
\begin{aligned}
&[\mathbf{M}][\mathbf{K}] = [\mathbf{K}][\mathbf{M}],\\
&[\mathbf{C}][\mathbf{K}] = [\mathbf{K}][\mathbf{C}],\\
&[\mathbf{M}][\mathbf{C}] = [\mathbf{C}][\mathbf{M}].
\end{aligned}
\tag{2}
$$

The proofs for the above theorems are not furnished here. For the detailed proofs the reader is encouraged to consult Refs. [3–5]. With a bit of algebraic manipulation Caughey and O'Kelly's condition $[\mathbf{C}]\,[\mathbf{M}]^{-1}\,[\mathbf{K}] = [\mathbf{K}]\,[\mathbf{M}]^{-1}\,[\mathbf{C}]$ can be derived from the commutativity relations in Eq. (2). Two more equivalent conditions: $[\mathbf{K}]\,[\mathbf{C}]^{-1}\,[\mathbf{M}] = [\mathbf{M}][\mathbf{C}]^{-1}\,[\mathbf{K}]$ and $[\mathbf{C}]\,[\mathbf{K}]^{-1}\,[\mathbf{M}] = [\mathbf{M}][\mathbf{K}]^{-1}\,[\mathbf{C}]$ can also be derived, provided of course the inverses $[\mathbf{C}]^{-1}$ and $[\mathbf{K}]^{-1}$ exist. However, the conditions given in Eq. (2) are only *sufficient* for real diagonalizing matrices, i.e., violation of conditions in Eq. (2) does not preclude the existence of real modes. In other words, systems that satisfy conditions in Eq. (2) automatically satisfy the Caughey's condition but not vice versa. All the above theorems confirm the necessity and sufficiency of Caughey and O'Kelly's conditions for simultaneous diagonalization of two real matrices by a real orthogonal matrix. It is also worth noting that these theorems are known to mathematicians since the original work of Weirestrass in 1858! However, the successful application of these results to vibrating systems was first made by Caughey and O'Kelly [1].

With this background, return now to the examples cited by Liang et al. in [2].

3. Examples

The first example given by Liang et al. is the case when the matrices in Eq. (1) are taken to be

$$
[\mathbf{A}] = \begin{bmatrix} 2.2493 & 0.8883 & -0.5744 \\ 0.8883 & 1.9462 & 0.6535 \\ -0.5744 & 0.6535 & 2.3045 \end{bmatrix},
$$
$$
[\mathbf{B}] = \begin{bmatrix} 40.0440 & 6.6221 & 28.3472 \\ 6.6221 & 31.6503 & -29.3577 \\ 28.3472 & -29.3577 & 105.7054 \end{bmatrix}.
$$

The above matrices do not commute as

$$
[\mathbf{A}][\mathbf{B}] = \begin{bmatrix} 79.6707 & 59.8731 & -23.0343 \\ 66.9839 & 48.2950 & 37.1233 \\ 46.6524 & -50.7751 & 208.1302 \end{bmatrix} \neq [\mathbf{B}][\mathbf{A}].
$$

Since the two matrices do not commute Caughey and O'Kelly's theorem says that we cannot find a real orthogonal matrix that can diagonalize both [**A**] and [**B**] simultaneously. Since the matrices do not commute it is not proportional either. However, the eigenvectors of the system are

$$[\mathbf{U}] = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 \\ -1.3361 + 0.0059\mathrm{i} & 0.7928 - 0.0000\mathrm{i} & -0.9599 + 0.0683\mathrm{i} \\ -0.7312 + 0.0576\mathrm{i} & -0.0809 + 0.0000\mathrm{i} & 2.9538 + 0.6692\mathrm{i} \end{bmatrix}.$$

It was pointed by Liang et al. that the existence of a real mode (in the second column of [**U**]) even when the damping is non-proportional as per Caughey and O'Kelly's classification, i.e., the two matrices do not commute, begs the question: how is an individual mode influenced by damping? One also wonders what is meant by "proportional" damping and whether such a concept has any physical meaning. The above example shows that even though the damping is non-proportional one or more modes can be real. If it is understood that *all* the modes of a proportionally damped system are real, not just one or a few modes, then Caughey and O'Kelly's classification is *not* violated in the above example since the complex modes are found in the damped system. In fact Caughey and O'Kelly's criterion is more concerned with the entire modal matrix [**U**] rather than the individual modes/column(s). Proportionality is an assumption made for mathematical convenience. Looked from this view point, a damped system is proportional only when there is one real *matrix* that can simultaneously diagonalize all the three system matrices in Eq. (1).

The second example given by Liang et al. is

$$[\mathbf{A}] = \begin{bmatrix} 3.0237 & 0.1470 & -0.3429 \\ 0.1470 & 3.9130 & -2.1304 \\ -0.3429 & -2.1304 & 7.9708 \end{bmatrix},$$

$$[\mathbf{B}] = \begin{bmatrix} 43.3203 & 1.9902 & -4.6437 \\ 1.9902 & 55.3646 & -28.8501 \\ -4.6437 & -28.8501 & 110.3150 \end{bmatrix}.$$

In the above case $[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}]$. However Liang et al. neglected the roundoff errors. Furthermore, this system has two identical natural frequencies (6.5574332 and 6.5574355). According to perturbation theory [7], the complex modes of a viscously damped system are related to its undamped modes by the relation

$$\{\bar{\mathbf{u}}\}^{(n)} = \{\mathbf{u}\}^{(n)} + \mathrm{i} \sum_{k \neq n} \frac{\omega_n C'_{kn} \{\mathbf{u}\}^{(k)}}{(\omega_n^2 - \omega_k^2)}, \tag{3}$$

where $\{\bar{\mathbf{u}}\}^{(n)}$ is the nth complex mode, $\{\mathbf{u}\}$'s are the undamped modes, ω's are undamped natural frequencies and $[\mathbf{C}']$ is the damping matrix in modal co-ordinates, i.e., $C'_{kn} = \{\mathbf{u}\}^{(k)^{\mathrm{T}}}[\mathbf{C}]\{\mathbf{u}\}^{(n)}$. In the current example $[\mathbf{C}'] = [\mathbf{U}]^{\mathrm{T}}\,[\mathbf{B}]\,[\mathbf{U}]$ is *almost* diagonal:

$$[\mathbf{C}'] = \begin{bmatrix} 3.0000 & 0.0001 & 0.0000 \\ 0.0001 & 3.0000 & -0.0000 \\ 0.0000 & -0.0000 & 8.9075 \end{bmatrix}.$$

However, since two natural frequencies are very close the factor $1/(\omega_n^2 - \omega_k^2)$ becomes extremely large and consequently the imaginary part in Eq. (3) becomes quite sensitive to roundoff errors. Thus it is not surprising to see complex modes in the damped system.

In the above discussion, the important issue of how an individual mode influenced by damping is ignored. It is a common experience that added damping in a structure does not have the same effect on all the modes. Some modes get damped more than the others. The clues to this mysterious behavior lie in the spatial distribution of damping sources in a structure. This issue is discussed in detail in Ref. [8]. Another important and practically relevant issue of the measurability of complex modes, almost taken for granted in analytical studies on damping, was also addressed.

4. Conclusion

General theorems concerning the simultaneous diagonalization of two real symmetric matrices using a real orthogonal transformation matrix are presented to place the Caughey and O'Kelly's theorem on classical normal modes in damped linear dynamic systems in context. It was also brought to attention that equivalent theorems similar to Caughey and O'Kelly's existed in linear algebra since the works of Weirestrass. The examples given by Liang et al. are shown to be inadequate to modify Caughey and O'Kelly's classification of damped systems. The classification of damped systems into proportional/non-proportional systems based on Caughey and O'Kelly's criterion still remains valid *if* proportional damping is meant to convey that all the modes of the damped system are real, not just one or a few and that all the three system matrices are simultaneously diagonalised by the same transformation *matrix*. In this context, it can be noted that the criterion proposed in Ref. [2] is difficult to verify in practice as damping has very small influence on the magnitude of the eigenvalues.

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References

[1] T.K. Caughey, M.E.J. O'Kely, Classical normal modes in damped linear dynamic systems, Journal of Applied Mechanics 32 (1965) 583–588.
[2] Z. Liang, M. Tong, G.C. Lee, Complex modes in damped linear dynamic systems, The International Journal of Analytical and Experimental Modal Analysis 7 (1) (1992) 1–20.
[3] E.T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies, 4th Edition, Cambridge University Press, Cambridge, 1985.
[4] F.R. Gantmacher, Applications of the Theory of Matrices, 1st Edition, Wiley Interscience, New York, 1959.
[5] C. Radhakrishna Rao, S.J. Mitra, Generalized Inverse and its Applications, 1st Edition, Wiley, New York, 1971.
[6] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[7] J. Woodhouse, Linear damping models for structural vibration, Journal of Sound and Vibration 215 (3) (1998) 547–569.
[8] A. Srikantha Phani, Damping identification in linear structural vibration, a report submitted to the Department of Engineering, University of Cambridge, Cambridge, UK, 2002.