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Journal of Sound and Vibration 264 (2003) 973–980

JOURNAL OF SOUND AND VIBRATION

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Letter to the Editor

Harmonic oscillations of non-conservative, asymmetric, two-degree-of-freedom systems

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Received 16 July 2002; accepted 20 November 2002

1. Introduction

In this note, stability of the equilibrium of a linear two-degree-of-freedom system described by

$$\mathbf{M}\ddot{\mathbf{z}} = -\mathbf{R}\dot{\mathbf{z}} - \mathbf{B}\mathbf{z} \tag{1}$$

is studied. Here, $\mathbf{M}$, $\mathbf{R}$, and $\mathbf{B}$ are (2×2) real, constant matrices, and $\mathbf{z}$ is a (2×1) vector of generalized co-ordinates. The superposed dots denote differentiation with respect to time. The matrices $\mathbf{R} = \mathbf{D} + \mathbf{G}$ and $\mathbf{B} = \mathbf{K} + \mathbf{H}$ are asymmetric with $\mathbf{D} = \mathbf{D}^{\mathrm{T}}$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, $\mathbf{G} = -\mathbf{G}^{\mathrm{T}}$, and $\mathbf{H} = -\mathbf{H}^{\mathrm{T}}$.

Several papers have appeared that deal with stability of equilibrium of the general asymmetric systems, as given by Eq. (1). Apart from the application of Routh's criterion and Liapunov's method, other methods for analyzing the stability of a general asymmetric system and its interesting sub-class have been presented. A good account of these up to 1986 is given in Ref. [1]. Subsequently, study of such systems essentially has taken place extending the existing methods [2–4] with an emphasis on the stability of the gyroscopic systems [5–9]. A recent work by Kounadis and Simitses [10] deals with the non-linear analysis of non-conservative, dissipative systems under partial follower load.

Adams and Padovan [11] applied the method of energy balance over the critical orbit while studying the oil whirl instability of a rigid rotor supported by hydrodynamic journal bearings. In this note, their method is extended to determine the stability of the linear systems shown in Eq. (1). The same idea is also applied to study the stability of a non-linear system without having to recourse for Liapunov's method.

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doi:10.1016/S0022-460X(02)01540-7

2. Analysis

The general asymmetric two-degree-of-freedom system described by Eq. (1) are characterized by positive definite diagonal matrix **M**, and asymmetric **R** and **B** matrices in which **D** is positive semi-definite, and **K** is positive definite. Without any loss of generality, Eq. (1) is written explicitly as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = - \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} - \gamma \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}, \quad \gamma > 0 \quad (2)$$

for further analysis. Herein, γ may be termed as the circulatory parameter.

It is to be mentioned that in the case when the two masses (diagonal elements of mass matrix) are different, equations similar to Eq. (2) with identity mass matrix can be obtained by pre-multiplying both sides of Eq. (1) with $\mathbf{M}^{-1}$, decomposing the resulting $\mathbf{M}^{-1}\mathbf{B}$ into its symmetric and antisymmetric parts followed by diagonalization of the symmetric part and transformation of all other resulting matrices accordingly. The resulting matrices must, of course, satisfy the definiteness properties mentioned earlier. It is to be emphasized that due to this transformation, the antisymmetric part preserves its form. Thus for such cases, the results obtained in this paper can still be used by setting $m = 1$.

Let it be assumed that the solutions of Eq. (2) with certain initial conditions at time $t = 0$ are such that a closed orbit in the x, y-plane exists and it is traced out in the counter-clockwise sense. From Eq. (2), the work done by the circulatory force

$$\mathbf{F}_H = \gamma(-y\hat{i} + x\hat{j}) \quad (3)$$

for each cycle of motion is given by

$$W_H = \oint_C \mathbf{F}_H \, \mathrm{d}\mathbf{l} = \int\int_S (\boldsymbol{\nabla} \times \mathbf{F}_H) \, \mathrm{d}\mathbf{s}. \quad (4)$$

Herein, C is a simple closed path in the x, y-plane representing the orbit with enclosed area S, $\mathrm{d}\mathbf{l}$ is a line element on C, $\mathrm{d}\mathbf{s}$ is an elemental area vector perpendicular to S, and $\boldsymbol{\nabla}$ is the usual vector differential operator. The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are associated with x, y, and z directions, respectively (Fig. 1). Evaluation of the surface integral in Eq. (4) with Eq. (3) and $\mathrm{d}\mathbf{s} = \mathrm{d}s\hat{k}$ (consistent with the counter-clockwise sense of circulation assumed for the orbit) yields

$$W_H = 2\gamma S. \quad (5)$$

From Eq. (5) it is clear that the work done W_H is directly proportional to the area S. It, however, does not depend on the shape of the orbit. Hence for the assumed orbit and the direction of traverse of a point, W_H turns out to be positive; as a result it supplies energy to the system in each cycle of motion. This positive work is termed as the regenerative work for future reference. Owing to the presence of positive semi-definite damping matrix **D**, however, the loss of energy from the system through dissipative work given by (refer to Eq. (2))

$$W_D = -\oint_C [(R_{xx}\dot{x} + R_{xy}\dot{y})\hat{i} + (R_{yx}\dot{x} + R_{yy}\dot{y})\hat{j}] \, \mathrm{d}\mathbf{l} \quad (6)$$

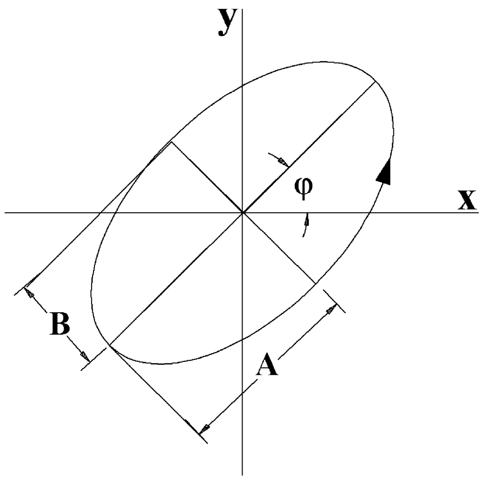

Fig. 1. Generalized elliptical orbit in the x, y-plane.

may balance the energy gained by the regenerative work. Under this condition, harmonic solution for both x and y leading to orbital motion in the counter-clockwise direction becomes possible. Notice that when $\gamma = 0$, from Eq. (5) $W_H = 0$, the sum $(W_H + W_D)$ is negative since from Eq. (6) $W_D < 0$ for positive semi-definite **D**. Thus for $\gamma > 0$, $(W_H + W_D)$ is expected to be a well-behaved increasing function of γ such that at $\gamma = \gamma_{th}$

$$(W_H + W_D)|_{\gamma=\gamma_{th}} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\gamma}(W_H + W_D)|_{\gamma=\gamma_{th}} > 0. \tag{7}$$

This implies that asymptotically stable equilibrium exists for all $\gamma < \gamma_{th}$ and instability will prevail if $\gamma > \gamma_{th}$. Finally at $\gamma = \gamma_{th}$, an orbital motion in the anticlockwise direction exists on which $W_H = -W_D$.

In the case where the antisymmetric elements of the matrix **H** in Eq. (2) exchange sign, similar arguments show that motion in a closed orbit is possible only in the clockwise direction.

Since the system is linear and asymmetric, the corresponding orbit is likely to assume a generalized elliptical shape about the origin of the x, y-plane, as shown in Fig. 1. For such an orbit

$$x = A\cos\omega_o t\cos\varphi - B\sin\omega_o t\sin\varphi, \tag{8}$$

$$y = A\cos\omega_o t\sin\varphi + B\sin\omega_o t\cos\varphi, \tag{9}$$

where the axis with length A is oriented at an angle φ with respect to the x-axis, as shown in the same figure. The orbital frequency is denoted by ω_o.

The regenerative work in this case follows directly from Eq. (5) as

$$W_H = 2\pi\gamma_{th}AB \tag{10}$$

and Eq. (6) with the help of Eqs. (8) and (9) yields, upon integration, the dissipative work as

$$W_D = -\frac{\pi\omega_o}{2}[(A^2 + B^2)(R_{xx} + R_{yy}) + (A^2 - B^2)\{(R_{xx} - R_{yy})\cos 2\varphi + (R_{xy} + R_{yx})\sin 2\varphi\}]. \tag{11}$$

Following relationships may be derived from Eq. (2) by substitution of Eqs. (8), (9) and subsequent algebraic manipulations with $\gamma = \gamma_{th}$. Thus

$$\begin{aligned}\tan\varphi &= \frac{(m\omega_o^2 - K_{xx}) - \alpha R_{xy}\omega_o}{\gamma_{th} - \alpha R_{xx}\omega_o} = -\left[\frac{\alpha\gamma_{th} - R_{xx}\omega_o}{\alpha(m\omega_o^2 - K_{xx}) - R_{xy}\omega_o}\right] \\ &= \frac{\alpha(m\omega_o^2 - K_{yy}) + R_{yx}\omega_o}{\alpha\gamma_{th} - R_{yy}\omega_o} = -\left[\frac{\gamma_{th} - \alpha R_{yy}\omega_o}{(m\omega_o^2 - K_{yy}) - \alpha R_{yx}\omega_o}\right]\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{1+\alpha^2}{\alpha} &= \frac{m\omega_o^2(m\omega_o^2 - 2K_{xx}) + K_{xx}^2 + \gamma_{th}^2 + (R_{xx}^2 + R_{xy}^2)\omega_o^2}{\omega_o[(m\omega_o^2 - K_{xx})R_{xy} + \gamma_{th}R_{xx}]} \\ &= \frac{m\omega_o^2(m\omega_o^2 - 2K_{yy}) + K_{yy}^2 + \gamma_{th}^2 + (R_{yy}^2 + R_{yx}^2)\omega_o^2}{\omega_o[-(m\omega_o^2 - K_{yy})R_{yx} + \gamma_{th}R_{yy}]}\end{aligned} \tag{13}$$

with $\alpha = B/A$.

Rearrangement of latter two fractions in Eq. (13) results in a polynomial in ω_o, which is independent of the aspect ratio α, as presented below:

$$\begin{aligned}& m^3(R_{xy} + R_{yx})\omega_o^6 + [m^2\{\gamma_{th}(R_{xx} - R_{yy}) - K_{xx}R_{xy} - K_{yy}R_{yx}\} + mR_{xy}(R_{yy}^2 + R_{yx}^2 - 2mK_{yy}) \\ &\qquad + mR_{yx}(R_{xx}^2 + R_{xy}^2 - 2mK_{xx})]\omega_o^4 \\ &\quad + \{(\gamma_{th}R_{xx} - K_{xx}R_{xy})(R_{yy}^2 + R_{yx}^2 - 2mK_{yy}) - (\gamma_{th}R_{yy} + K_{yy}R_{yx})(R_{xx}^2 + R_{xy}^2 - 2mK_{xx}) \\ &\qquad + (K_{yy}^2 + \gamma_{th}^2)mR_{xy} + (K_{xx}^2 + \gamma_{th}^2)mR_{yx}\}\omega_o^2 \\ &\quad + \{(K_{yy}^2 + \gamma_{th}^2)(\gamma_{th}R_{xx} - K_{xx}R_{xy}) - (K_{yy}^2 + \gamma_{th}^2)(\gamma_{th}R_{yy} + K_{yy}R_{yx})\} = 0.\end{aligned} \tag{14}$$

The square root of the smallest positive real root of Eq. (14) provides the orbital frequency ω_o. Subsequently, φ and α can be determined from Eqs. (12) and (13), respectively. The condition for orbital motion for $\gamma = \gamma_{th}$ may now be obtained from Eq. $(7)_1$ by the use of Eqs. (10)–(12) as follows:

$$\frac{\omega_o\left[\begin{array}{c}(\gamma_{th} - \alpha R_{xx}\omega_o)^2(R_{xx} + \alpha^2 R_{yy}) + (m\omega_o^2 - K_{xx} - \alpha R_{xy}\omega_o)^2(\alpha^2 R_{xx} + R_{yy}) \\ +(1-\alpha^2)(R_{xy} + R_{yx})(\gamma_{th} - \alpha R_{xx}\omega_o)(m\omega_o^2 - K_{xx} - \alpha R_{xy}\omega_o)\end{array}\right]}{2\alpha\gamma_{th}[(\gamma_{th} - \alpha R_{xx}\omega_o)^2 + (m\omega_o^2 - K_{xx} - \alpha R_{xy}\omega_o)^2]} = 1. \tag{15}$$

Eq. (15) may be used to find the critical value of γ_{th}; above this value the equilibrium (trivial solution of Eq. (2)) is unstable. The effect of variation of any particular parameter on the stability may be easily investigated with the aid of Eqs. (13)–(15) in an iterative manner.

3. Applications and discussion

Following examples will now be presented which help to verify the foregoing results.

Example 1 (Symmetric rotor with internal damping). The simple case of instability due to internal and external dampings of a symmetric rotor of mass, m mounted symmetrically on bearings by a shaft with negligible mass is studied here (see Fig. 2). The bearings are supported

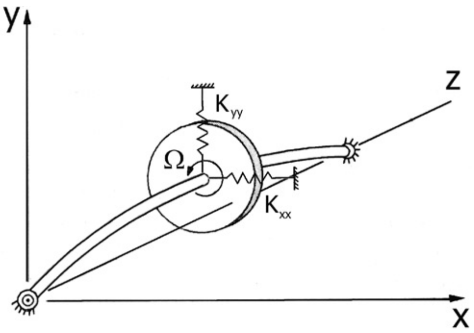

Fig. 2. Symmetric rotor–shaft system having internal and external dampings with angular speed Ω. K_{xx} and K_{yy} are equivalent principal stiffnesses of the bearings, shaft, and the flexible foundations referred to the rotor disk.

with flexible foundations providing unequal stiffness in two principal directions, x and y. Note that the K_{xx} and K_{yy} are equivalent principal stiffnesses of the bearings, shaft, and the flexible foundations referred to the rotor disk. Let R_e and R_i be the external and internal damping coefficients of the shaft, respectively. It is known in such cases that $\gamma = \Omega R_i$, where Ω is the speed of rotation of the shaft, $R_{xx} = R_{yy} = R_i + R_e$, and $K_{xx} \neq K_{yy}$. Let $R_{xy} = R_{yx} = 0$, such that there is no gyroscopic coupling.

Use of the matrix element values mentioned above in Eq. (14) results in

$$\omega_o = \sqrt{\frac{K_{xx} + K_{yy}}{2m}}. \tag{16}$$

Accordingly condition (15) for sinusoidal response gives

$$\Omega_{th} = \omega_o \left(\frac{\alpha^2 + 1}{2\alpha}\right)\left(1 + \frac{R_e}{R_i}\right). \tag{17}$$

Numerical simulations with relevant set of data have been performed at the corresponding Ω_{th} to obtain elliptical orbit. Also for $\Omega < \Omega_{th}$ and $\Omega > \Omega_{th}$, as expected, asymptotic stability and instability, respectively, have been observed.

In the case when $K_{xx} = K_{yy} = K$, i.e., when the foundation springs are equal, it can be shown that $\Omega_{th} = \omega_o(1 + R_e/R_i)$, and $\omega_o = (K/m)^{1/2}$. This result is identical with that derived earlier by other more laborious theoretical means. (See any standard book on rotor dynamics.)

Example 2 ([4, Example 1]). Consider the following two-degree-of-freedom system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} 10 & -3.25 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{18}$$

Diagonalization of the symmetric part of the **B** matrix in this case and transformation of all other matrices in Eq. (18) accordingly result in the following form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} 4.118 & 0 \\ 0 & 1.882 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = -\begin{bmatrix} 10.1864 & -1.7896 \\ 2.4604 & 3.8136 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} - 1.5\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}. \tag{19}$$

The **D** and **K** matrices which appear in Eq. (19) are positive definite. Clearly, comparison of Eq. (19) with Eq. (2) shows $m = 1$ and γ=1.5 for the associated skew symmetric matrix of the form $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Following the iteration method described earlier with the **R** and **K** matrices identified from Eq. (19), the critical value is found to be $\gamma_{th} = 17.9159$. The orbit exists at $\gamma = \gamma_{th}$, and in this case, the direction of circulation is clockwise. Since in this example γ (= 1.5) is less than γ_{th}, the equilibrium is expected to be asymptotically stable. This conclusion is in accordance with the earlier result [5]. Thus, the present method also provides the limit value of γ, below which the equilibrium will be asymptotically stable.

As an additional remark it would be appropriate to mention that for positive definite **D** and **K** matrices, if **H** is non-zero and **K** + **H** is not symmetrizable, the method proposed here may be applied successfully to determine the stability of the equilibrium.

Example 3 (Effect of gyroscopic coupling on the stability). The influence of the gyroscopic parameter $\beta(> 0)$, which may appear as either

$$\mathbf{G} = \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{G} = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

as the antisymmetric part of **R**, on the value of γ_{th} has been studied for $m = 1$, $K_{xx} = K_{yy} = 100$, $R_{xx} = 4$, $R_{yy} = 7$.

Fig. 3 shows the variation of γ_{th} with β for both the forms of the gyroscopic matrix **G** shown above. It may be observed that as β increases from zero, γ_{th} decreases in the first case and increases for the other. The foregoing method of stability analysis predicts that if the forms of antisymmetry of circulatory and gyroscopic matrices are similar, then increase in the value of β offers a larger range of the values of γ for which the static equilibrium is asymptotically stable. On the other hand, if these two matrices have dissimilar forms, larger values of the gyroscopic parameter β would have a destabilizing effect. Note that in the case of similar forms the sense of circulation

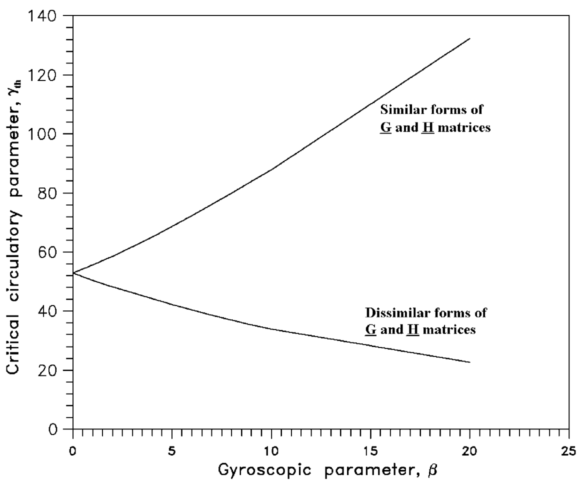

Fig. 3. Effect of gyroscopic parameter on the threshold value of the circulatory parameter for $m = 1$, $K_{xx} = K_{yy} = 100$, $R_{xx} = 7$.

produced by the gyroscopic terms is opposite to that of the circulatory terms. Thus the stability margin is wider. Reverse is the case with dissimilar forms.

Example 4 (Non-linear damping and circulatory forces). The method of balance of regenerative and dissipative work has been applied to the following non-linear equations:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = - \begin{bmatrix} R + \eta f(x,y) & 0 \\ 0 & R + \eta f(x,y) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} - \eta\xi \begin{bmatrix} 0 & f(x,y) \\ -f(x,y) & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}. \tag{20}$$

Here η, ξ are two distinct parameters and $f(x,y) = x^2 + y^2$ is an isotropic function (i.e., a function whose form is preserved upon co-ordinate rotation). It turns out that the matrices on the right side of Eq. (20) are not constant. However, this imposes no restriction in evaluating the integrals in Eqs. (4) and (6). Notice that equation of type Eq. (20) may result from non-linear modeling of internal damping in a shaft–rotor system. Assuming harmonic oscillation given by $x = A\cos\omega_o t$, $y = A\sin\omega_o t$, where $\omega_o = (K/m)^{1/2}$ (see last paragraph of Example 1) is the orbital frequency, and equating W_H with W_D, according to Eqs. (4) and (6) one obtains

$$A = \left(\frac{R\omega_o}{\eta(\xi - \omega_o)} \right)^{1/2}. \tag{21}$$

Note from Eq. (21) that a circular orbit of radius A in the x, y-plane will exist for all $\xi > \omega_o$, and $\eta > 0$. It can be proved [12] that this circular orbit is, in fact, an unstable limit orbit in the x, y-plane for $R > 0$.

For $m = 1$, $R = 1$, $K = 100$, $\eta = 10$, $\xi = 110$, and $\omega_o = (K/m)^{1/2} = 10$, from Eq. (21) $A = 0.1$. With the initial conditions $x(0) = A = 0.1$, $\dot{y}(0) = A\omega_o = 1.0$, $\dot{x}(0) = 0$, $y(0) = 0$, numerical solution of Eq. (20) yields the isolated, circular, limit orbit of radius 0.1. It also turns out from simulation that this orbit is unstable. It would be informative to point out that, in this case, reverse time solution may fail to capture the unstable limit orbit.

4. Concluding remarks

An alternative method to determine the stability of equilibrium of a class of linear, asymmetric, two-degree-of-freedom system has been suggested. Overall, it is found that for given matrices satisfying the definiteness conditions stated earlier, if the regenerative work exceeds the dissipative work, then the equilibrium is unstable. More specifically, there exists a threshold value of the circulatory parameter, γ above which instability is expected to occur. Three examples are considered where the present method is applied to prove the correctness. Note that this method fails if dissipative forces are not present in the system.

It is also concluded that depending on the relative positions of $+1$ and -1 as the antisymmetric elements in the circulatory and gyroscopic matrices the gyroscopic parameter has either stabilizing or destabilizing effect on the system stability.

Finally, the method is also applied to a two-degree-of-freedom system having non-linearities in both damping and circulatory matrices. The analysis shows that there exists an unstable circular limit orbit around the origin of the configuration plane.

Acknowledgements

Partial support from CSIR (India) under Grant No. 22(0259)/96/EMR-II is gratefully acknowledged. The authors would also like to thank Mr. H. Das and Mr. S. Sarangi for their help with the text and graphics.

References

[1] D.J. Inman, Vibration with Control, Measurement, and Stability, Prentice-Hall, Englewood Cliffs, NJ, 1986.
[2] D.W. Nicholson, Asymptotic stability conditions for asymmetric linear mechanical systems, American Society of Mechanical Engineers, Journal of Pressure Vessel Technology 111 (1989) 332–336.
[3] S.M. Shahruz, F. Ma, On symmetrizability of asymmetric non-conservative systems, American Society of Mechanical Engineers, Journal of Applied Mechanics 56 (1989) 474–476.
[4] M. Ahmedian, D.J. Inman, Some stability results for general linear lumped-parameter dynamic systems, American Society of Mechanical Engineers, Journal of Applied Mechanics 53 (1986) 10–14.
[5] S.M. Yang, C.D. Mote Jr., Stability of non-conservative linear discrete gyroscopic systems, Journal of Sound and Vibration 147 (3) (1991) 453–464.
[6] K. Huseyin, On the stability criteria for conservative gyroscopic systems, American Society of Mechanical Engineers, Journal of Vibration and Acoustics 113 (1991) 58–61.
[7] A. Seyranian, J. Stoustrup, W. Kleim, On gyroscopic stabilization, Zeitung für Angewandte Mathematik und Physik 46 (1995) 255–267.
[8] D.J. Inman, A sufficient condition for the stability of conservative gyroscopic systems, American Society of Mechanical Engineers, Journal of Applied Mechanics 55 (1988) 895–898.
[9] J.A. Walker, Stability of linear conservative gyroscopic systems, American Society of Mechanical Engineers, Journal of Applied Mechanics 58 (1991) 229–232.
[10] A.N. Kounadis, G.J. Simitses, Non-conservative systems with symmetrizable stiffness matrices exhibiting limit cycles, International Journal of Non-Linear Mechanics 32 (1997) 515–529.
[11] M.L. Adams, J. Padovan, Insights into linearized rotor dynamics, Journal of Sound and Vibration 76 (1981) 129–142.
[12] D.W. Jordan, P. Smith, Nonlinear Ordinary Differential Equations, Oxford University Press, Oxford, 1987, pp. 99–101.