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The existence of periodic motions in rub-impact rotor systems

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Abstract

The existence of rub-impact periodic motions in an eccentric rotor system is considered. A criterion for the periodicity condition of *n*-periodic impacts is derived and other conditions for real rub-impacts are also discussed. A method consisting of analytical and numerical techniques is presented to solve the existence problem of rub-impact periodic motions. Some special results are obtained by theoretical analyses for rub-impact rotor systems, including the existence of grazing circle motions and that of single-impact periodic motions.

1. Introduction

Rub-impact events often occur in high-speed rotating machinery such as generators and turbines, and cause great damage to machines. There are a number of works studying the rub-impact problem for application purposes and most of them make use of numerical and/or experimental methods. Childs [1] studied the effect of rub-impacts on the appearance of parametrically excited subharmonic vibrations of rotor systems. Choy and Padovan [2] discussed the transient rub-interactions between a rotor and a casing in a rotating structure. Li and Paidoussis [3] took account of a simplified rotor-casing system in which there is very rich dynamical behaviour including bifurcations and chaos. Ehrich [4,5] published a series of papers concerning observations of the subharmonic and chaotic regimes. The dynamics of a rotor system with bearing clearance was studied by Flowers and Wu [6] using the analytical and experimental methods, with particular interest in the influence of coupled disk/shaft vibration. Xie et al. [7] considered the steady state dynamic behaviour of a rotor system supported by auxiliary bearings

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with clearance and presented some insight into the behaviour of such systems. Recently, von Groll and Ewins [8] applied the harmonic balance method with arc-length continuation to investigate rotor/stator contact problems, including periodic responses and their stability. Other noteworthy works with regard to this topics are in Refs. [9,10]. A rub-impact rotor-dynamic system, which is operating eccentrically within a clearance and in local contact with the stator, is virtually identical in behaviour to piecewise linear oscillator systems that have been studied in a general form by a number of authors [11–17]. These systems differ greatly from ordinary linear systems. Due to the discontinuity of stiffness during rub-impacts, the flow in the phase space is not smooth, and sometimes it is even discontinuous. It is just because of the non-linearity and non-smoothness that complicated dynamical behaviour appears in rub-impact rotor systems. Usually rub-impact dynamics problems can be solved only by numerical methods.

In this paper rub-impact periodic motions in an eccentric rotor system are considered. Some criteria for rub-impact periodic motions are derived and discussed. Then the existence of rub-impact periodic responses can be treated by means of a combination of analytical and numerical methods. Under certain clearance conditions, it is verified that there exist grazing circle motions in the rotor–casing system. The existence and non-existence of single-impact periodic responses for high speed and small mass eccentricity are also shown.

2. Mathematical model

A simple rub-impact eccentric rotor system without damping is considered here. The whirling motion of the rotor between two collisions can be described as

$$\begin{cases} \ddot{x}_1 + \omega_0^2 x_1 = e\omega^2 \cos(\omega t), \\ \ddot{x}_2 + \omega_0^2 x_2 = e\omega^2 \sin(\omega t) - f, \end{cases} \quad \left(\sqrt{x_1^2 + x_2^2} < R\right), \tag{1}$$

where x_1, x_2 are the position components of the rotor centre in the Cartesian co-ordinates, ω_0 is the natural frequency, e is the mass eccentric distance of the rotor, ω is its rotation frequency, R is the clearance between the rotor and the casing (see Fig. 1), and f represents an external force, such as the gravitational force.

Assume that a rub-impact between the rotor and the casing occurs instantaneously when $t = t_0$ at (x_{10}, x_{20}) with $\sqrt{(x_1^2(t_0) + x_2^2(t_0))} = R$. From the law of impulse and from Coulomb's law of friction, it is known that the velocity components of the rotor just before and after the impact satisfy the relations

$$v_+ = -kv_-, \quad p_+ = p_- - \mu(1+k)v_-, \tag{2}$$

where v_-, v_+ denote the radial whirling velocities and p_-, p_+ the annular whirling velocities at t_{0-} and t_{0+}, which denote the time just before and after the impact, respectively; $0 < k \leqslant 1$ is the restitution coefficient and $\mu \geqslant 0$ is the annular friction coefficient. It is clear that $t_{0-} = t_{0+}$ by the assumption of instantaneous impact.

In this paper the existence of rub-impact period-n motions of system (1) is studied; that is, an impact repeats periodically with a period nT for given integer n, where $T = 2\pi/\omega$ is the rotation period of the rotor. According to the number m of impacts in the time duration nT, rub-impact

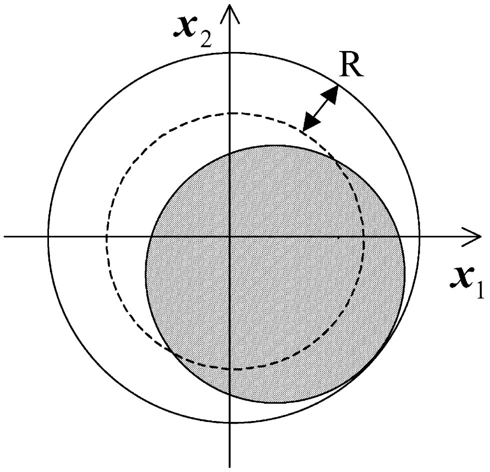

Fig. 1. Scheme of rub-impact rotor.

period-n motions can be divided into two types: single-impact ($m = 1$) motions and multiple-impact motions ($m > 1$). The periodicity conditions of the responses are given as

$$x_i(t_{0-}) = x_i(t_{0-} + 2n\pi/\omega), \quad \dot{x}_i(t_{0-}) = \dot{x}_i(t_{0-} + 2n\pi/\omega), \quad i = 1, 2. \tag{3}$$

Although the dynamics between two adjacent collisions is linear, collisions result in essential non-linearity and non-smoothness in the rotor system. Therefore, the rub-impact motions of system (1) are rather complicated because of the non-smooth features and multiple degrees of freedom. It is usually difficult to give an analytical treatment for rub-impact motions of rotor systems. Based on the piecewise-linear characteristics of system (1), however, an analytical method is presented that gives rise to some theoretical results on periodic motions of system (1) to help understand the dynamics of rub-impact systems.

Assume that the rub-impact rotor system is not in a resonant state, that is, $\omega \neq \omega_0$. The general solution of the first equation of system (1) between two adjacent collisions is given by

$$\begin{aligned} x_1(t) &= A_1 \sin \omega_0(t - t_{0+}) + B_1 \cos \omega_0(t - t_{0+}) + e\gamma\omega^2 \cos(\omega t), \\ \dot{x}_1(t) &= A_1\omega_0 \cos \omega_0(t - t_{0+}) - B_1\omega_0 \sin \omega_0(t - t_{0+}) - e\gamma\omega^3 \sin(\omega t), \end{aligned} \tag{4}$$

where A_1, B_1 are constants determined by the initial conditions $(x_{10}, \dot{x}_{10})$, and $\gamma = (\omega_0^2 - \omega^2)^{-1}$.

Now the constants A_1 and B_1 are determined. Let V_{10-} and V_{10+} represent the velocity components in the x_1 direction of the rotor just before and after the impact, respectively. Thus

$$V_{10+} = \dot{x}_1(t_{0+}) = v_+ \cos\theta - p_+ \sin\theta, \quad V_{10-} = \dot{x}_1(t_{0-}) = v_- \cos\theta - p_- \sin\theta, \tag{5}$$

where θ is the polar angle of the rub-impact position (x_{10}, x_{20}) so that

$$\cos\theta = x_{10}/R, \quad \sin\theta = x_{20}/R. \tag{6}$$

From the initial conditions $x_1(t_{0+}) = x_{10}, \dot{x}_1(t_{0+}) = V_{10+}$ it turns out that

$$A_1 = [V_{10+} + e\gamma\omega^3 \sin(\omega t_{0+})]/\omega_0, \quad B_1 = x_{10} - e\gamma\omega^2 \cos(\omega t_{0+}). \tag{7}$$

By considering the periodicity condition (3), that is, $x_1(t_1) = x_{10}$ and $\dot{x}_1(t_1) = V_{10-}$ when $t_1 = t_{0-} + 2n\pi/\omega$, it follows that

$$x_{10} = A_1 S_n + B_1 C_n + e\gamma\omega^2 C_0, \tag{8a}$$

$$V_{10-} = A_1\omega_0 C_n - B_1\omega_0 S_n - e\gamma\omega^3 S_0, \tag{8b}$$

where $S_0 = \sin(\omega t_0), C_0 = \cos(\omega t_0), S_n = \sin(2n\pi\omega_0/\omega)$ and $C_n = \cos(2n\pi\omega_0/\omega)$.

Similarly, within the time interval $(t_{0-}, t_{0-} + 2n\pi/\omega)$ the general solution of the second equation of system (1) between two adjacent collisions is given by

$$\begin{aligned} x_2(t) &= A_2 \sin\omega_0(t - t_{0+}) + B_2\cos\omega_0(t - t_{0+}) + e\gamma\omega^2\sin(\omega t) - f/\omega_0^2, \\ \dot{x}_2(t) &= A_2\omega_0\cos\omega_0(t - t_{0+}) - B_2\omega_0\sin\omega_0(t - t_{0+}) + e\gamma\omega^3\cos(\omega t), \end{aligned} \tag{9}$$

where A_2 and B_2 are constants determined by the initial conditions.

Let V_{20-} and V_{20+} stand for the velocities of the rotor just before and after the impact, respectively. Then

$$V_{20+} = \dot{x}_2(t_{0+}) = v_+\sin\theta + p_+\cos\theta, \quad V_{20-} = \dot{x}_2(t_{0-}) = v_-\sin\theta + p_-\cos\theta. \tag{10}$$

From the initial conditions $x_2(t_{0+}) = x_{20}$ and $\dot{x}_2(t_{0+}) = V_{20+}$, it turns out that

$$A_2 = [V_{20+} - e\gamma\omega^3\cos(\omega t_{0+})]/\omega_0, \quad B_2 = x_{20} - e\gamma\omega^2\sin(\omega t_{0+}) + f/\omega_0^2. \tag{11}$$

Once more by the periodicity conditions (3), that is, $x_2(t_2) = x_{20}$ and $\dot{x}_2(t_2) = V_{20-}$ when $t_2 = t_{0-} + 2n\pi/\omega$, it follows that

$$x_{20} = A_2 S_n + B_2 C_n + e\gamma\omega^2 S_0 - f/\omega_0^2, \tag{12a}$$

$$V_{20-} = A_2\omega_0 C_n - B_2\omega_0 S_n + e\gamma\omega^3 C_0, \tag{12b}$$

in which the definitions of S_n, C_n, S_0 and C_0 are the same as those in Eqs. (8a) and (8b).

Solving Eqs. (8a) and (12a) simultaneously and using Eqs. (2), (5), (7), (10) and (11), it can be seen that

$$\begin{aligned} v_- = {} & [S_n(h_1x_{10} + h_2x_{20})\omega_0]^{-1}\{-(-1 + C_n)x_{10}^2\omega_0^2 - x_{10}\omega_0(e\gamma\omega^2 S_0 S_n \\ & - (-1 + C_n)(e\gamma\omega^2 C_0 - x_{10})\omega_0) + x_{20}(f + e\gamma\omega^2 C_0 S_n\omega_0 \\ & - e\gamma\omega^2 S_0\omega_0^2 + C_n(-f + e\gamma\omega^2 S_0\omega_0^2))\}, \end{aligned} \tag{13a}$$

$$\begin{aligned} p_- = {} & [S_n(h_1x_{10} + h_2x_{20})\omega_0]^{-1}\{R[h_2\omega_0(e\gamma\omega^2 S_0 S_n \\ & - (-1 + C_n)(e\gamma\omega^2 C_0 - x_{10})\omega_0 + h_1(f + e\gamma\omega^2 C_0 S_n\omega_0 \\ & - e\gamma\omega^2 S_0\omega_0^2 + x_{20}\omega_0^2 - C_n(f - e\gamma\omega^2 S_0\omega_0^2 + x_{20}\omega_0^2))]\}, \end{aligned} \tag{13b}$$

where

$$h_1 = -(k/R)x_{10} + (\mu(1 + k)/R)x_{20}, \quad h_2 = -(k/R)x_{20} - (\mu(1 + k)/R)x_{10}. \tag{14}$$

Obviously, it is essential to ensure

$$S_n(h_1x_{10} + h_2x_{20})\omega_0 \neq 0 \tag{15}$$

so that the formula (13) holds. Furthermore, the inequality (15) implies that

$$2n\omega_0 \neq \tilde{k}\omega, \quad \tilde{k} = 0, 1, 2, \ldots. \tag{16}$$

Substituting Eqs. (13a) and (13b) into Eqs. (8b), (12b) and using Eqs. (2), (5), (7), (10) and (11), the equations for C_0 and S_0 are given as

$$\alpha_{11} + \alpha_{12}C_0 + \alpha_{13}S_0 = 0, \tag{17}$$

$$\alpha_{21} + \alpha_{22}C_0 + \alpha_{23}S_0 = 0, \tag{18}$$

where

$$\begin{aligned}\alpha_{11} = &-Rh_1(f(-1+C_n)x_{20} + (-C_n + C_n^2 + S_n^2)x_{10}^2\omega_0^2 + (-1+C_n)x_{20}^2\omega_0^2 \\ &+ x_{10}((-1+C_n)x_{10}^2\omega_0^2 + (-1+C_n)x_{20}^2\omega_0^2 + x_{20}(-f - RC_n^2h_2\omega_0^2 \\ &- Rh_2(1+S_n^2)\omega_0^2 + C_n(f + 2Rh_2\omega_0^2))),\end{aligned}$$

$$\begin{aligned}\alpha_{12} = &e\gamma\omega^2\omega_0(R\omega h_1S_nx_{20} - \omega S_nx_{10}x_{20} + Rh_1(-C_n + C_n^2 + S_n^2)x_{10}\omega_0 \\ &- (-1+C_n)x_{10}^2\omega_0 + Rh_2(1 - 2C_n + C_n^2 + S_n^2)x_{20}\omega_0),\end{aligned}$$

$$\alpha_{13} = -e\gamma\omega^2(Rh_1 - x_{10})\omega_0(\omega S_nx_{10} - (-1+C_n)x_{20}\omega_0),$$

$$\begin{aligned}\alpha_{21} = &-fRh_2S_n^2x_{20} - fx_{20}^2 + Rh_2x_{10}^2\omega_0^2 - x_{10}^2x_{20}\omega_0^2 - Rh_2S_n^2x_{20}^2\omega_0^2 - x_{20}^3\omega_0^2 \\ &- Rh_1(1 - 2C_n + C_n^2 + S_n^2)x_{10}(f + x_{20}\omega_0^2) - RC_n^2h_2x_{20}(f + x_{20}\omega_0^2) \\ &+ C_n(Rh_2(fx_{20} - x_{10}^2\omega_0^2 + x_{20}^2\omega_0^2) + x_{20}(fx_{20} + x_{10}^2\omega_0^2 + x_{20}^2\omega_0^2)),\end{aligned}$$

$$\alpha_{22} = e\gamma\omega^2(Rh_2 - x_{20})\omega_0(\omega S_nx_{20} + (-1+C_n)x_{10}\omega_0),$$

$$\begin{aligned}\alpha_{23} = &e\gamma\omega^2\omega_0(\omega S_nx_{10}x_{20} + Rh_1S_n^2x_{10}\omega_0 + (-1+C_n)(R(-1+C_n)h_1x_{10} - x_{20}^2)\omega_0 \\ &+ Rh_2(-\omega S_nx_{10} + (-1+C_n)C_nx_{20}\omega_0 + S_n^2x_{20}\omega_0))\end{aligned}$$

with the initial impact position (x_{10}, x_{20}) as parameters. Considering Eqs. (17) and (18) as a system of linear algebraic equations with respect to C_0 and S_0, for $n_d \neq 0$ this follows that

$$C_0 = n_c/n_d, \quad S_0 = n_s/n_d \quad \text{for } n_d \neq 0, \tag{19}$$

where

$$\begin{aligned}n_d = &e\gamma\omega^2\omega_0^2((-1+C_n)(x_{10}^2 + x_{20}^2)\omega_0 - Rh_1(\omega S_nx_{20} + (-1+C_n)C_nx_{10}\omega_0 \\ &+ S_n^2x_{10}\omega_0) + Rh_2(\omega S_nx_{10} - (-1+C_n)C_nx_{20}\omega_0 - S_n^2x_{20}\omega_0),\end{aligned}$$

$$\begin{aligned}n_c = &x_{10}(-Rh_2S_n^2x_{20}\omega_0^3 + (-1+C_n)(x_{10}^2 - RC_nh_2x_{20} + x_{20}^2)\omega_0^3 \\ &+ \omega S_nx_{10}(f + Rh_2\omega_0^2) - Rh_1((-1+C_n)C_nx_{10}\omega_0^3 + S_n^2x_{10}\omega_0^3 + \omega S_n(f + x_{20}\omega_0^2))),\end{aligned}$$

$$n_s = \omega S_n x_{10} x_{20}(f + Rh_2\omega_0^2) - Rh_2 S_n^2 x_{20}\omega_0(f + x_{20}\omega_0^2)$$
$$+ (-1 + C_n)(x_{10}^2 + x_{20}(-RC_n h_2 + x_{20}))\omega_0(f + x_{20}\omega_0^2)$$
$$- Rh_1(\omega S_n x_{20} + (-1 + C_n)C_n x_{10}\omega_0 + S_n^2 x_{10}\omega_0)(f + x_{20}\omega_0^2). \quad (20)$$

Using Eq. (19), the identity $C_0^2 + S_0^2 = 1$ gives

$$n_c^2 + n_s^2 = n_d^2. \quad (21)$$

Substituting Eq. (14) into Eq. (21), the following equation can be deduced:

$$n_{10} + n_{20}\sqrt{R^2 - x_{10}^2} + (n_{11} + n_{21}\sqrt{R^2 - x_{10}^2})x_{10} + n_{22}x_{10}^2 = 0, \quad (22)$$

where

$$n_{10} = R^4(-(1+k)\mu\omega S_n + (-1 + C_n)(1 + kC_n)\omega_0 + kS_n^2\omega_0)^2(f^2 + (R^2 - e^2\gamma^2\omega^4)\omega_0^4),$$

$$n_{20} = 2fR^4\omega_0^2(-(1+k)\mu\omega S_n + (-1 + C_n)(1 + kC_n)\omega_0 + kS_n^2\omega_0)^2,$$

$$n_{11} = -2f(1+k)R^4\omega S_n\omega_0^2((1+k)\mu\omega S_n - (-1 + C_n)(1 + kC_n)\omega_0 - kS_n^2\omega_0),$$

$$n_{21} = -2f^2(1+k)R^2\omega S_n((1+k)\mu\omega S_n - (-1 + C_n)(1 + kC_n)\omega_0 - kS_n^2\omega_0),$$

$$n_{22} = -f^2(1+k)R^2\omega S_n((1+k)(-1+\mu^2)\omega S_n - 2\mu(-1 + C_n)(1 + kC_n)\omega_0 - 2k\mu S_n^2\omega_0).$$

Eq. (22), which represents the periodicity condition of rub-impact, gives an important criterion for n-periodic impacts of the rub-impact rotor system (1). There are still other requirements for a real impact at (x_{10}, x_{20}), such as $\sqrt{(x_{10}^2 + x_{20}^2)} = R$ for the impact position and $v_- > 0$ for the impact radial velocity. The number of impacts in the n-period time should also be discussed further.

3. Existence of rub-impacting periodic motions

Now consider the existence of rub-impact periodic motions in the rotor system (1). Obviously, when Eq. (22) has a real solution $x_{10} \leqslant R$ for given n, there may exist an impact period-n motion with (x_{10}, x_{20}) as the original impact position, where $\sqrt{(x_{10}^2 + x_{20}^2)} = R$. Although there is no explicit expression for the solutions of the non-linear algebraic Eq. (22), this existence problem can be solved through the combination of both analytical and numerical techniques in general. Under the given parameters of the rotor system and the value of n, at first one finds the real roots, x_{10}, of Eq. (22). Then C_0, S_0 and the initial time t_0 are calculated by Eq. (20). The velocity components just before and after the impact, that is, V_{10-} and V_{10+}, follow from Eqs. (13a), (13b), (2) and (5). The other requirements for an impact, such as $v_- > 0$, should also be considered. Finally, the corresponding impact period-n solution of the rub-impact system (1) can be obtained by numerical integration under the initial conditions $(x_{10}, x_{20}, V_{10-}, V_{10+})$ at $t = t_0$. By means of the above method, one can find all impact period-n motions and then solve the existence problem.

Although the existence of impacting periodic motions of the rub-impact rotor system (1) cannot be solved in a wholly analytical way, some theoretical analysis will be discussed in what follows.

Firstly, the Eq. (22) can be transformed into the form

$$\lambda_0 + \lambda_1 x_{10} + \lambda_2 x_{10}^2 + \lambda_3 x_{10}^3 + \lambda_4 x_{10}^4 = 0, \tag{23}$$

where

$$\lambda_0 = n_{10}^2 - R^2 n_{20}^2, \quad \lambda_1 = 2n_{10}n_{11} - 2R^2 n_{20}n_{21},$$
$$\lambda_2 = 2n_{22}n_{10} + n_{11}^2 + n_{20}^2 - R^2 n_{21}^2,$$
$$\lambda_3 = 2n_{22}n_{11} + 2n_{20}n_{21}, \quad \lambda_4 = n_{22}^2 + n_{21}^2.$$

Especially, if $\lambda_4 \neq 0$ it is equivalent to

$$\beta_4 + \beta_3 x_{10} + \beta_2 x_{10}^2 + \beta_1 x_{10}^3 + x_{10}^4 = 0, \tag{24}$$

where

$$\beta_1 = \lambda_3/\lambda_4, \quad \beta_2 = \lambda_2/\lambda_4, \quad \beta_3 = \lambda_1/\lambda_4, \quad \beta_4 = \lambda_0/\lambda_4.$$

Solving Eq. (24) yields

$$x_{10}^{1,2} = \frac{\sqrt{\Omega}}{2} - \frac{\beta_1}{4} \pm \frac{1}{2}\sqrt{\Delta + \frac{-\beta_1^3 + 4\beta_1\beta_2 - 8\beta_3}{4\sqrt{\Omega}}},$$
$$x_{10}^{3,4} = -\frac{\sqrt{\Omega}}{2} - \frac{\beta_1}{4} \pm \frac{1}{2}\sqrt{\Delta - \frac{-\beta_1^3 + 4\beta_1\beta_2 - 8\beta_3}{4\sqrt{\Omega}}}, \tag{25}$$

where

$$\Phi = 2\beta_2^3 - 9\beta_1\beta_2\beta_3 + 27\beta_3^2 + 27\beta_1^2\beta_4 - 72\beta_2\beta_4,$$

$$\Gamma = -4(\beta_2^2 - 3\beta_1\beta_3 + 12\beta_4)^3 + \Phi^2,$$

$$\Omega = \frac{(\sqrt{\Gamma} + \Phi)^{1/3}}{3 \times 2^{1/3}} + \frac{\beta_1^2}{4} - \frac{2\beta_2}{3} + \frac{2^{1/3}(\beta_2^2 - 3\beta_1\beta_3 + 12\beta_4)}{3(\sqrt{\Gamma} + \Phi)^{1/3}},$$

$$\Delta = -\frac{(\sqrt{\Gamma} + \Phi)^{1/3}}{3 \times 2^{1/3}} + \frac{\beta_1^2}{2} - \frac{4\beta_2}{3} - \frac{2^{1/3}(\beta_2^2 - 3\beta_1\beta_3 + 12\beta_4)}{3(\sqrt{\Gamma} + \Phi)^{1/3}}.$$

Obviously, only the real solutions of Eq. (23) (or (24)) with $x_{10} \leqslant R$ can be considered as the initial values in this rub-impact problem.

Secondly, consider a special case, in which $f = 0$ (that is, there is no external force), and let the original impact position be $(x_{10}, x_{20}) = (R, 0)$. Thus one of the solutions of Eq. (22) is

$$x_{10} = R = e\omega^2|\gamma|. \tag{26}$$

Furthermore, it follows from Eqs. (13a) and (13b) that

$$v_- = 0, \quad p_- = R\omega. \tag{27}$$

This implies that in this case there exists a grazing rub-impact if the clearance condition (26) is satisfied. By means of Eqs. (7) and (11), it is shown that $A_1 = B_1 = A_2 = B_2 = 0$. Thus the rub-impact solution is

$$x_1(t) = e\gamma\omega^2 \cos(\omega t), \quad x_2(t) = e\gamma\omega^2 \sin(\omega t). \tag{28}$$

Thus $x_1^2(t) + x_2^2(t) = e^2\gamma^2\omega^4 = R^2$ for all time t, and then the rotor contacts the stator in a whole circle. This special motion is named as a grazing circle motion of the rotor.

Thirdly, consider other special cases in which the rotation frequency ω is sufficiently large. Suppose that there is at least a real solution of Eq. (22) in the following discussion. This hints that the periodicity condition of a rub-impact motion in system (1) is met, and it is still necessary to identify whether the condition $v_- > 0$ holds at (x_{10}, x_{20}) or not. Considering the case of $\omega \to +\infty$, Eqs. (7) and (11) become

$$A_1 = -\frac{f(-1 + C_n)x_{10}(x_{10} - \mu x_{20})}{\mu S_n(x_{10}^2 + x_{20}^2)\omega_0^2} + O(\omega^{-1}), \quad B_1 = \frac{f x_{10}(x_{10} - \mu x_{20})}{\mu(x_{10}^2 + x_{20}^2)\omega_0^2} + O(\omega^{-1}),$$
$$A_2 = -\frac{f(-1 + C_n)x_{10}(\mu x_{10} + x_{20})}{\mu S_n(x_{10}^2 + x_{20}^2)\omega_0^2} + O(\omega^{-1}), \quad B_2 = \frac{f x_{10}(\mu x_{10} + x_{20})}{\mu(x_{10}^2 + x_{20}^2)\omega_0^2} + O(\omega^{-1}).$$

Therefore, the whirling components are

$$\begin{aligned} x_1(t) &= e\gamma\omega^2 \cos(\omega t) + fA(x_{10}, \mu)u(t) + O(\omega^{-1}), \\ x_2(t) &= e\gamma\omega^2 \sin(\omega t) + fB(x_{10}, \mu)u(t) - f/\omega_0^2 + O(\omega^{-1}), \end{aligned} \tag{29}$$

where

$$A = x_{10}\left(x_{10} - \mu\sqrt{R^2 - x_{10}^2}\right)/R^2\mu\omega_0^2, \quad B = x_{10}\left(\mu x_{10} + \sqrt{R^2 - x_{10}^2}\right)/R^2\mu\omega_0^2,$$

$$u(t) = ((1 - C_n)/S_n)\sin[\omega_0(t - t_0)] + \cos[\omega_0(t - t_0)].$$

From Eq. (13a), v_- can be formulated as

$$v_- = -2f(1 - C_n)x_{10}/(1 + k)\mu S_n R\omega_0 + O(\omega^{-1}). \tag{30}$$

Without loss of generality, the case of $x_{10} > 0$ is considered. By the condition of $v_- > 0$ at the original impact position, it follows from Eq. (30) that $n \in (\omega/2\omega_0, 3\omega/4\omega_0)$ or $n \in (3\omega/4\omega_0, \omega/\omega_0)$ can assure this requirement. In fact, for instance, the former corresponds to $2n\pi\omega_0/\omega \in (\pi, 3\pi/2)$, and then $(1 - C_n)/S_n < -1$ and $v_- > 0$ as $\omega \to +\infty$. This means that if the rotation frequency ω is sufficiently large, then only the rub-impact period-n motions for $n \in (\omega/2\omega_0, 3\omega/4\omega_0)$ or $n \in (3\omega/4\omega_0, \omega/\omega_0)$ may exist in system (1).

Now make more discussion on the number of impacts in the time duration nT of a rub-impact period-n motion. Remembering that $\gamma = (\omega_0^2 - \omega^2)^{-1} \to 0$ as $\omega \to +\infty$, it is clear from (29) that for $t \in (t_{0-}, t_{0-} + 2n\pi/\omega)$

$$\begin{aligned} x_1^2(t) + x_2^2(t) = & f^2 C^2(x_{10}, \mu) u^2(t) + 2efu(t)[A(x_{10}, \mu)\cos(\omega t) \\ & + B(x_{10}, \mu)\sin(\omega t)]\omega^2/(\omega_0^2 - \omega^2) + e^2\omega^4/(\omega_0^2 - \omega^2)^2 \\ & + (f/\omega_0^4)[f - 2fB\omega_0^2 u[t] - 2e\gamma\omega^2\omega_0^2 \sin(\omega t)] + O(\omega^{-1}), \end{aligned} \tag{31}$$

where

$$C^2(x_{10}, \mu) = A^2(x_{10}, \mu) + B^2(x_{10}, \mu) = (1 + \mu^2)x_{10}^2/R^2\mu^2\omega_0^4.$$

Let φ be given by $\tan\varphi = A(x_{10}, \mu)/B(x_{10}, \mu)$. Then Eq. (31) becomes

$$\begin{aligned} x_1^2(t) + x_2^2(t) = & f^2 C^2 u^2(t) - 2efCu(t)\sin(\omega t + \varphi) + e^2 \\ & + (f/\omega^4)[f - 2fB\omega_0^2 u[t] + 2e\omega_0^2 \sin(\omega t)] + O(\omega^{-1}) \\ = & [fCu(t) - e\sin(\omega t + \varphi)]^2 + e^2\cos^2(\omega t + \varphi) \\ & + (f/\omega^4)[f - 2fB\omega_0^2 u[t] + 2e\omega_0^2 \sin(\omega t)] + O(\omega^{-1}) \\ \equiv & h(t) + O(\omega^{-1}), \end{aligned} \tag{32}$$

where

$$\begin{aligned} h(t) = & [fCu(t) - e\sin(\omega t + \varphi)]^2 + e^2\cos^2(\omega t + \varphi) \\ & + (f/\omega_0^4)[f - 2fB\omega_0^2 u[t] + 2e\omega_0^2 \sin(\omega t)]. \end{aligned}$$

As e is small enough usually, the function $h(t)$ possesses the expression

$$\begin{aligned} h(t) = & [fCu(t)]^2 + (f/\omega_0^4)[f - 2fB\omega_0^2 u(t)] + O(e) \\ = & (f/\omega_0^4)[C^2\omega_0^4 u^2(t) - 2B\omega_0^2 u(t) + 1] + O(e) \\ = & (f/\omega_0^4)[(B\omega_0^2 u(t) - 1)^2 + A^2\omega_0^4 u^2(t)] + O(e) \equiv \tilde{h}(t) + O(e), \end{aligned} \tag{33}$$

where $\tilde{h}(t) = f\omega_0^{-4}[(B\omega_0^2 u(t) - 1)^2 + A^2\omega_0^4 u^2(t)]$.

Now when the rotation frequency ω is large enough and the mass eccentricity e is small enough, the number of impacts in the time duration nT of a rub-impact period-n motion for $n \in (\omega/2\omega_0, 3\omega/4\omega_0)$ or $n \in (3\omega/4\omega_0, \omega/\omega_0)$ is determined by the property of the function $\tilde{h}(t)$.

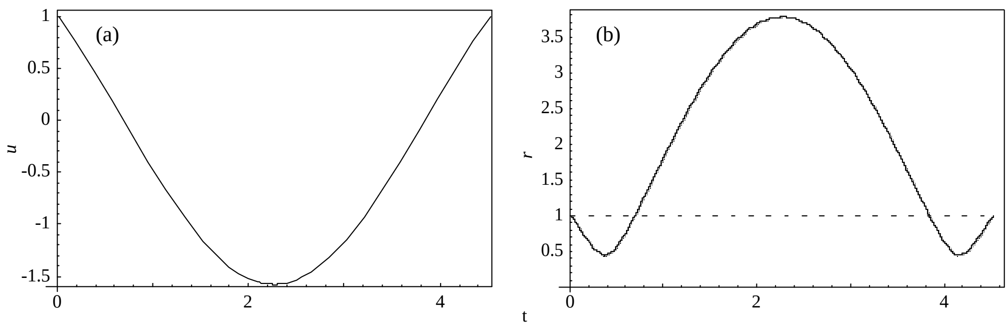

Fig. 2. Time histories for (a) $u(t)$ and (b) $r(t)$, $(x_0, x_{20}) = (0.8, 0.6)$ where $e = 0.01, \omega = 50.1, \mu = 0.5, k = 1, \omega_0 = 1, R = 1, f = 1.01, n = 36$.

For example, if $|\tilde{h}(t)| < R^2$, then $x_1^2(t) + x_2^2(t) < R^2$ for $t \in (t_{0-}, t_{0-} + 2n\pi/\omega)$. Hence there is no collision between two adjacent impacts in the time duration nT, and this corresponds to a single-impact period-n motion. On the other hand, if there is a $t^* \in (t_{0-}, t_{0-} + 2n\pi/\omega)$ such that $|\tilde{h}(t^*)| > R^2$, then an impact may occur just before the time t^* because $x_1^2(t^*) + x_2^2(t^*) > R^2$ and the number of rub-impacts is larger than 1 in the time duration nT. Thus the rub-impact period-n motion is not a single-impact motion, but a multiple-impact one.

Fig. 2 presents the numerical results for the case of $k = 1, \omega_0 = 1, R = 1, e = 0.01, \omega = 50.1, \mu = 0.5, f = 1 - e\gamma\omega_1^2 = 1.01, n = 36 \in (\omega/2\omega_0, 3\omega/4\omega_0)$. Consider the rub-impact period-36 motions in the time interval [0, nT) = [0, 4.51). Fig. 2(a) demonstrates the variation of $u(t)$, and Fig. 2(b) shows the time history of the rotor motion originated from the initial impact position $(x_{10}, x_{20}) = (0.8, 0.6)$, where $r(t) = \sqrt{(x_1^2(t) + x_2^2(t))}$. It is clear from Fig. 2(b) that $r(t)$ would exceed the boundary $R = 1$ if there were no collision occurring, and this actually means that more rub-impacts may happen in this time interval [0, 4.51), except the original impact at $t = 0$. Thus there is no single-impact period-36 motion in the system (1) under the given values of parameters, and the rub-impact period-36 motion should be a multiple-impact one. More numerical simulations show that the existence of rub-impact periodic motions is very sensitive to the parameters of the rotor system. Moreover, when ω is large enough and e is small enough, single-impact periodic motions appear rarely in rotor systems and most of rub-impact periodic motions are multiple-impact periodic, quasi-periodic or chaotic ones.

4. Conclusions

The existence of rub-impact periodic motions in an eccentric rotor system was discussed by means of an analytical method. A criterion for the periodicity condition of periodic rub-impacts was derived and other conditions for real rub-impacts were discussed. A method consisting of analytical and numerical techniques was presented to study the existence of rub-impact periodic motions, and the number of rub-impacts was also discussed.

Two special cases were discussed in detail by theoretical analyses for rub-impact periodic motions:

1. Under the clearance condition $R = e\omega^2|\gamma|$, there may exist a grazing circle motion for the case of $f = 0$, that is, there is no external force. In this case, the rotor moves along the whole casing surface and leads to serious effects on rotor systems.
2. Rub-impact period-n motions may occur in the rotor system (1) only for $n \in (\omega/2\omega_0, 3\omega/4\omega_0) \cup (3\omega/4\omega_0, \omega/\omega_0)$ when the rotation frequency ω is large enough. Moreover, rub-impacting usually results in multiple-impact periodic, quasi-periodic or chaotic motions in rotor systems when the rotation speed is high and the mass eccentricity is small simultaneously.

The effects of damping were not considered in this paper. There are various damping effects, which play an important role in rub-impacts. In general, the rub-impact behaviour is likely to be sensitive to damping and more investigation should be made. Only the simplest effect of damping included in the rotor model can be treated theoretically in a way similar to that in this paper, but more complicated. Other effects of damping, such as the internal damping, the visco-elastic damping and the damping of oil-film bearings, should be studied mainly by numerical simulation. These will be considered in further work.

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References

[1] D.W. Childs, Rub induced parametric excitation in rotors, American Society of Mechanical Engineers Journal of Mechanical Design 10 (1979) 640–644.
[2] F.K. Choy, J. Padovan, Nonlinear transient analysis of rotor–casing rub events, Journal of Sound and Vibration 113 (1987) 529–545.
[3] G.-X. Li, M.P. Paidoussis, Impact phenomena of rotor-casing dynamical systems, Nonlinear Dynamics 5 (1994) 53–70.
[4] F. Ehrich, The dynamic stability of rotor/stator radial rubs in rotating machinery, American Society of Mechanical Engineers Journal of Engineering for Industry 91 (1969) 1025–1028.
[5] F. Ehrich, Observation of subcritical superharmonic and chaotic response in rotor dynamics, American Society of Mechanical Engineers Journal of Vibration and Acoustics 114 (1992) 93–100.
[6] G.T. Flowers, Fangsheng Wu, Disk/shaft vibration induced by bearing clearance effects: analysis and experiment, Transactions of the American Society of Mechanical Engineers, Journal of Vibration and Acoustics. 118 (1996) 204–208.
[7] H. Xie, G.T. Flowers, L. Feng, C. Lawrence, Steady-state dynamic behavior of a flexible rotor with auxiliary support from a clearance bearing, Transactions of the American Society of Mechanical Engineers, Journal of Vibration and Acoustics. 121 (1999) 78–83.
[8] G. von Groll, D.J. Ewins, The harmonic balance method with arc-length continuation in rotor/stator contact problems, Journal of Sound and Vibration 241 (2001) 223–233.
[9] R.G. Kirk, Evaluation of AMB turbomachinery auxiliary bearings, Transactions of the American Society of Mechanical Engineers, Journal of Vibration and Acoustics. 121 (1999) 156–161.
[10] J. Zapomel, C.H.J. Fox, E. Malenovsky, Numerical investigation of a rotor system with disc-housing impact, Journal of Sound and Vibration 243 (2001) 215–240.
[11] S.W. Shaw, P.J. Holmes, A periodically forced impact oscillator with large dissipation, American Society of Mechanical Engineers Journal of Applied Mechanics 50 (1983) 849–857.
[12] S.W. Shaw, The dynamics of a harmonically excited system having rigid amplitude constraints (parts I and II), American Society of Mechanical Engineers Journal of Applied Mechanics 52 (1985) 453–464.
[13] J.M.T. Thompson, R. Ghaffari, Chaos after period doubling bifurcations in the resonance of an impact oscillator, Physics Letters A 91 (1982) 5–8.
[14] G.S. Whiston, The vibro-impact response of a harmonically excited and preloaded one-dimensional linear oscillator, Journal of Sound and Vibration 115 (1987) 303–319.
[15] G.S. Whiston, Global dynamics of a vibro-impacting linear oscillator, Journal of Sound and Vibration 118 (1987) 395–424.
[16] A.V. Ivanov, Stabilization of an impact oscillator near grazing incidence owing to resonance, Journal of Sound and Vibration 118 (1993) 395–424.
[17] J.O. Aidanpaa, R.B. Gupta, Periodic and chaotic behaviour of a threshold-limited two-degree-of-freedom system, Journal of Sound and Vibration 165 (1993) 305–327.