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Letter to the Editor

On the average continuous representation of an elastic discrete medium

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1. Introduction

It is well known that difference and difference-differential equations are often used for the numerical solution of partial differential equations. A natural problem is to obtain a difference-differential equation, whose solution approximates a solution of a given partial differential equation. But an analytical study of the difference and difference-differential equations is often more difficult than a study of the corresponding partial differential equation. Therefore the following important problem appears: how can one construct a partial differential equation approximating a given difference-differential equation?

Consider an N-mass oscillator (Fig. 1). The equations of motion and boundary conditions may be written as follows:

$$M\ddot{y}_j(t) = \delta_{j+1}(t) - \delta_j(t), \tag{1}$$

$$\delta_0(t) = -f(t), \quad \delta_n(t) = 0, \quad j = 0, 1, ..., n-1, \tag{2}$$

where $y_j(t)$ is the displacement of the j-point, $\delta_j(t) = c(y_j(t) - y_{j-1}(t))$, $\delta_j(t)$ is force of interaction between the $(j-1)$- and j-point.

System (1) may be reduced to the form

$$M\delta_j(t) = c(\delta_{j+1} - 2\delta_j + \delta_{j-1}), \quad j = 0, 1, \ldots, n-1. \tag{3}$$

Consider the homogeneous initial conditions:

$$\delta_j(t) = \delta_{jt}(t) = 0 \text{ for } t = 0, \ j = 0, 1, \ldots, n-1. \tag{4}$$

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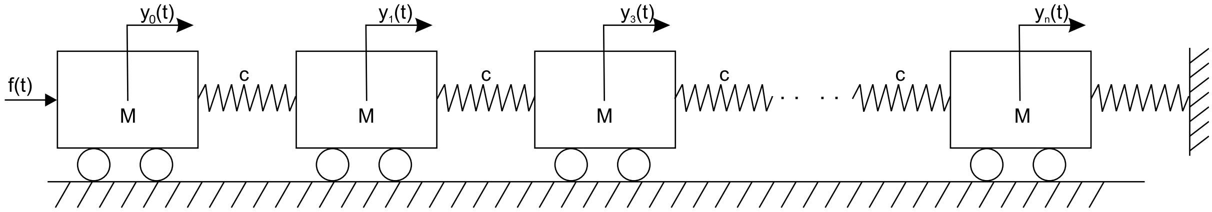

Fig. 1. The mass chain governed by Eqs. (1) and (2).

For large values of n, usually a continuous approximation to the lumped problem (2)–(4) is applied:

$$M\delta_{tt}(x,t) = cH^2\delta_{xx}(x,t), \tag{5}$$

$$\delta(0,t) = -f(t), \quad \delta((n+1)H,t) = 0, \tag{6}$$

$$\delta(x,0) = \delta_t(x,0) = 0, \tag{7}$$

where H corresponds to the distance between masses;

$$(\ldots)_x = \frac{\partial}{\partial x}(\ldots), \quad (\ldots)_t = \frac{\partial}{\partial t}(\ldots).$$

Having a solution to boundary value problem (5)–(7), one can obtain a solution to the discrete problem using the formula

$$\delta_j(t) = \delta(jH,t), \quad j = 1,2,\ldots,n.$$

Taking $f(t) = -1$, it is not difficult to find the exact solution to boundary value problem (3)–(5) [1]:

$$\delta(x,t) = \mathrm{G}\left[nH \arcsin\left|\sin\left(\frac{\pi}{2n}\sqrt{\frac{c}{M}}t\right)\right| - x\right],$$

where G is the Heaviside step function.

Therefore the estimation

$$|\delta(x,t)| \leqslant 1 \tag{8}$$

holds for the whole time interval. From a physical point of view, it means that the force values in various sections of the continuous approximation do not exceed a given force value.

Kurchanov et al. [2–6] have considered a motion of the N-mass oscillator (2)–(4) and its continuous approximation (5)–(7). The authors of Ref. [2] wrote:

> "On the basis of these and some other approximate arguments, some authors (e.g., Refs. [1,7]) have assumed that an analogous inequality (8) holds for all components of the solution of system (3), provided only that n is large enough. However, direct computations for large $n = 40$, 80, 120 have shown that the values of $\delta_j(t)$ at certain times (different for different jth) may considerably exceed 1 (by a few dozen percent)."

In the language of mechanics, what we just said means that when analyzing the so-called "local properties" of a one-dimensional continuous medium, one cannot treat the medium as the limiting case of a linear chain of point masses, obtained when the number of points increases without limit.

In other words, the transition from the N-mass oscillator to the continuous medium can be accompanied by the loss of some subtle effects, and it was named "peaks" in Ref. [3].

The described phenomenon will be further referred to as the Kurchanov–Myskis–Filimonov paradox, which will be further analyzed.

It is worth noting that oscillation of the N-mass oscillator was analyzed in Ref. [8]. It is worthless noticing that authors of Ref. [8], written in 1994, found the sum of some series on the basis of physical consideration and they mentioned: "Despite great efforts, the authors have not been able to find a mathematical way to obtain the sum". However, this sum has rigorously been reported in Ref. [2].

2. Eigenfrequencies

Assuming $f(t) = 0$, the relations between the eigenfrequencies of oscillations of both lumped (2)–(4) and continuous (5)–(7) systems will be investigated.

Lumped system (2)–(4) possesses $(n+1)$ eigenfrequencies of the form

$$\omega_k = 2\sqrt{\frac{c}{M}}\sin\frac{k\pi}{2(n+1)}, \quad k = 1, 2, \ldots, n+1. \tag{9}$$

The continuous system (5)–(7) has an infinite discrete spectrum of the form

$$\omega_k = \pi\sqrt{\frac{c}{M}}\frac{k}{(n+1)}, \quad k = 1, 2, \ldots\,. \tag{10}$$

Formula (10) approximates frequencies (9) sufficiently good only for the first 10 lower frequencies. However, for large k values, this approximation cannot be accepted (for example, ω_{n+1} is estimated with the error being larger than 50%, because one has to deal with the coefficient π instead of 2). Furthermore, the frequencies ω_{n+2}, ω_{n+3}, ... of the continuous system do not have any relations to the discrete system. They are referred as the parasite frequencies and must be omitted while investigating the discrete system.

In the case considered, when the one-dimensional system is investigated, in order to approximate the system with n frequencies, one takes the first n frequencies of a continuous system.

3. Non-autonomous case

Consider boundary value problem (5)–(7). Taking $f(t) = -1$, a solution is sought in the form

$$\delta(x,t) = -1 + \frac{x}{(n+1)H} + u(x,t). \tag{11}$$

Thus, the function $u(x,t)$ is defined by the following boundary value problem:

$$Mu_{tt} = cH^2 u_{xx}, \tag{12}$$

$$u(0,t) = u((n+1)H, t) = 0, \tag{13}$$

$$u(x,0) = 1 - \frac{x}{(n+1)H}, \quad u_t(x,0) = 0. \tag{14}$$

In this case, a solution to (11)–(13) is approximated by the Fourier series

$$u = \frac{(n+1)H}{\pi} \sum_{k=1}^{\infty} \frac{\sin\left(k\pi x/((n+1)H)\right)}{k} \cos\left(\alpha_k t\right),$$

where

$$\alpha_k = \pi \sqrt{\frac{c}{M}} \frac{k}{(n+1)}.$$

Finally one has

$$\delta(x,t) = -1 + \frac{x}{(n+1)H} + \frac{(n+1)H}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi x/((n+1)H))}{k} \cos\left(\alpha_k t\right). \tag{15}$$

The solution obtained governs the motion of a continuous system. Now, if one would like to approximate the N-mass oscillator motion, then in the infinite sum, only the "$n+1$" harmonic must be included. The others have no any relation to the motion of the N-mass chain. In other words, the motion of discrete system (1) can be approximated by the formula

$$\delta(x,t) = \frac{x}{(n+1)H} - 1 + x + \frac{(n+1)H}{\pi} \sum_{k=1}^{n+1} \frac{\sin\left(k\pi x/((n+1)H)\right)}{k} \cos(\alpha_k t). \tag{16}$$

The numerical calculations for $n = 60,\ 80,\ 120$ show that δ can exceed 1, which is in agreement with the comments in Refs. [2–6].

It is worth noting that the method involves finding the sum of a finite number of terms of the Fourier series, and does not approximate an arbitrary function by an infinite Fourier series. This excludes an occurrence of the Gibbs phenomenon.

4. Higher accuracy of continuous approximation

Although the obtained solution (16) properly qualitatively approximates the motion of the N-mass chain, its accuracy is rather low. The eigenforms in a neighbourhood of "$n+1$" governed by approximation (3) are not accurate enough. In order to improve the accuracy, the following approach is proposed.

Formally a discrete-difference operator D is replaced with a high-order differential operator, using the following identity [9]:

$$D = \exp\left(\frac{\partial}{\partial x}\right) - 2 + \exp\left(-\frac{\partial}{\partial x}\right)$$

or the pseudo-differential operator $\sin^2(-(\mathrm{i}H/2)(\partial/\partial x))$ [10].

System (2) can be presented in the form [9–13]:

$$M\frac{\partial^2\delta}{\partial t^2}+4c\sin^2\left(-\frac{\mathrm{i}H}{2}\frac{\partial}{\partial x}\right)\delta=0. \tag{17}$$

The pseudo-differential operator $\sin^2(-(\mathrm{i}H/2)(\partial/\partial x))$ can be developed into the Taylor series in a neighbourhood of zero:

$$\sin^2\left(-\frac{\mathrm{i}H}{2}\frac{\partial}{\partial x}\right)=-\frac{H^2}{4}\frac{\partial^2}{\partial x^2}+\frac{H^4}{48}\frac{\partial^4}{\partial x^4}-\frac{H^6}{1440}\frac{\partial^6}{\partial x^6}+\cdots. \tag{18}$$

Taking into account only the first term in Eq. (18), one gets the classical continuous approximation (5). However, taking into account the three first terms, the approximation obtained reads

$$M\frac{\partial^2\delta}{\partial t^2}=cH^2\left[\frac{\partial^2\delta}{\partial x^2}-\frac{H^2}{12}\frac{\partial^4\delta}{\partial x^4}+\frac{H^4}{360}\frac{\partial^6\delta}{\partial x^6}\right]. \tag{19}$$

Comparing the $(n+1)$th frequency of the continuous system (19) with the $(n+1)$th frequency of the N-mass oscillator indicates a sufficient increase of the accuracy (we get the coefficient 2.11 instead of 2 in the exact solution). Therefore, solution (16) serves as a good approximation to the motion of the N-mass chain, where

$$\alpha_k=\pi\sqrt{\frac{c}{M}}\frac{k}{n+1}\sqrt{1-\frac{\pi^2k^2}{12(n+1)^2}+\frac{\pi^4k^4}{360(n+1)^4}}.$$

The proposed approximation has been already used within the frame of the theory of elasticity [11] and in the method of a differential approximation [12].

5. "High-frequency averaging" and the composite equations

The operator occurring in Eq. (17) can be also approximated as follows:

$$\sin^2\left(-\frac{\mathrm{i}H}{2}\frac{\partial}{\partial x}\right)=1+\frac{H^2}{4}\frac{\partial^2}{\partial x^2}+\cdots \tag{20}$$

and the continuous approximation can be presented in the form (see also Refs. [13,14])

$$M\frac{\partial^2\delta}{\partial t^2}+4c\delta+cH^2\frac{\partial^2\delta}{\partial x^2}=0. \tag{21}$$

The equation

$$M\frac{\partial^2\delta}{\partial t^2}+4c\delta=0$$

describes the "saw-tooth" oscillations of the N-mass oscillator, $\delta_i=-\delta_{i-1}$.

Eq. (21) governs the N-mass oscillator oscillations of the close to the "saw-tooth" shape form.

The existence of continuous approximations (5) and (21) give the possibility to construct the composite equation [15,16], which is uniformly suitable in the whole interval of the frequencies and the oscillation forms of the chain of masses. Let us emphasize that the composite equations, due to Van Dyke [17], can be obtained as a result of synthesizing the limiting cases.

The principal idea of the method of the composite equations can be formulated in the following way [17, p. 195]:

(i) Identify the terms in the differential equations, whose neglect in the straightforward approximation is responsible for the non-uniformity.
(ii) Approximate those terms insofar as possible while retaining their essential character in the region of non-uniformity.

In the present case, the composite equation will be constructed in order to overlap (approximately) with Eq. (5) for $k \ll (n+1)$, whereas for $k = (n+1)$, this "new" equation should yield the exact frequency value. As a result of the procedure described, one obtains

$$M\left(1 - \alpha^2 H^2 \frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 \delta}{\partial t^2} - cH^2 \frac{\partial^2 \delta}{\partial x^2} = 0, \tag{22}$$

where $\alpha^2 = (\pi^2 - 4)/16$.

The kth oscillation frequency is obtained from the formula

$$\omega_k = \pi\sqrt{\frac{c}{M}}\frac{k}{n+1}\frac{1}{\sqrt{1 + \dfrac{\alpha^2 H^2 \pi^2 k^2}{(n+1)^2}}}. \tag{23}$$

It is not difficult to check that Eq. (23) yields the frequency values close to the frequencies of the N-mass oscillator for $k \in [1, n+1]$. The largest error is achieved for $[k/(n+1)] = 0.5$, where $[(\ldots)]$ is the integral part of $(\ldots)$.

The multiplier appearing in the exact solution is equal to $\sqrt{2}$, whereas the multiplier appearing in Eq. (23) is equal to 1.34 (the error is 5%).

Therefore, Eq. (22) can be used for the motion analysis of the N-mass chain. In this case, the solution has the form of Eq. (16), where

$$\alpha_k = \pi\sqrt{\frac{c}{M}}\frac{k}{\sqrt{(n+1)^2 + \pi^2\alpha^2 H^2 k^2}}. \tag{24}$$

The numerical calculations using solutions (16) and (24) are in agreement with the theoretically predicted peaks [2–4].

6. Concluding remarks

Ulam [18, pp. 89, 90] wrote: "The simplest problems involving an actual infinity of particles in the distributions of matter appear already in classical mechanics. A discussion of these will permit us to introduce more general schemes which may possibly be useful in future physical theories.

Strictly speaking, one has to consider a true infinity in the distribution of matter in all problems of the physics of continua. In the classical treatment, as usually given in textbooks of hydrodynamics and field theory, this is, however, not really essential, and in most theories serves merely as a convenient limiting model of *finite* systems enabling one to use the algorithms of the calculus. The usual introduction of the continuum leaves much to be discussed and examined critically. The derivation of the equations of motion for fluids, for example, runs somewhat as follows. One images a very large number N of particles, say with equal masses, constituting a net approximating the continuum which is to be studied. The forces between these particles are assumed to be given, and one writes the Lagrange equations for the motion of N particles. The finite system of ordinary differential equations "becomes" in the limit $N = \infty$ one or several *partial* differential equations. The Newtonian laws of conservation of energy and momentum are seemingly correctly formulated for the limiting case of the continuum. There appears immediately, however, at least one possible objection to the unrestricted validity of this formulation. For the very fact that the limiting equations imply tacitly the continuity and differentiability of the functions describing the motion of the continuum seems to impose various *constraints* on the possible motions of the approximating finite systems. Indeed, at any stage of the limiting process, it is quite conceivable for two neighbouring particles to be moving in opposite directions with a relative velocity which need not tend to zero as N becomes infinite, whereas the continuity imposed on the solution of the limiting continuum excludes such a situation. There are, therefore, constraints on the class of possible motions which are not explicitly recognized. This means that a viscosity or other type of constraint must be introduced initially, singling out the "smooth" motions from the totality of all possible ones. In some cases, therefore, the usual differential equations of hydrodynamics may constitute a misleading description of the physical process."

It happened that the problem described by Ulam have been numerically observed and mathematically investigated by Kurchanov et al. [2–6]. It is rather to be expected that the averaged relations do not allow getting reliable results for all possible potential behaviours. However, the problem can be formulated in the following manner: Is it possible to construct a partial differential equation governing the dynamics of a lumped system?

This paper, and the results included in Refs. [5,6] indicate that this is possible, but many problems of mathematical nature must be still clearly stated and appropriately solved.

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