Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Sound and Vibration 264 (2003) 983–1004

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Discrete-time norms of flexible structures

W. Gawronski*

Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, M.S. 238-528, Pasadena, CA 91109, USA

Received 22 October 2001; accepted 29 July 2002

Abstract

The paper presents and analyses the H_2, H_∞, and Hankel norms of flexible structures. The analysis is conducted for the discrete-time models of structures and compared with the continuous-time results. The structural state-space models are presented in modal co-ordinates. Closed-form expressions for norms of structural modes are obtained, and norms of a structure are determined from the modal norms. The relationships between the Hankel, H_∞, and H_2 modal norms are derived. In addition, the paper shows that the discrete-time Hankel and H_∞ norms converge to the continuous-time counterparts when the sampling time approaches zero; however, the H_2 norm does not.

1. Introduction

System norms are used to characterize system dynamics, and as such they are applied to model reduction, sensor and actuator placement procedures, or damage detection. This paper develops H_2, H_∞, and Hankel norms for the discrete-time models of flexible structures and compares them to the related continuous-time norms. The norms for the continuous-time models of flexible structures are presented in Ref. [1]. Some variables known from literature are identical with the norms. For example, the component cost of Skelton [2] is equivalent to the H_2 norm. The Hankel singular value of flexible structure, see Refs. [3,4] is equivalent to the Hankel norm. The resonant peak of a structure is equivalent to the H_∞ norm. Nevertheless, it is useful to interpret these quantities as system norms since it expands their applications, by taking advantage of the generic properties of system norms. For example, using norm properties allows combining the norms of structural modes into the related norm of the entire structure.

*Corresponding author. Tel.: +1-818-354-1783; fax: +1-818-393-0207.
E-mail address: wodek.k.gawronski@jpl.nasa.gov (W. Gawronski).

doi:10.1016/S0022-460X(02)01184-7

We pay attention to the discrete-time norms of flexible structures, because of their possible applications. Often structural models need to be represented as discrete-time models, for example in controller design, in hardware-in-the-loop simulations, in system identification procedures. The paper derives discrete-time norms for structural modes in a closed-form. Next, the norm of the entire structure is obtained from modal norms. The H_2, H_∞, and Hankel norms are compared, and their convergence to the continuous-time norms is investigated.

All three norms require the knowledge of controllability (reachability) and observability grammians, therefore in the introductory part of the paper we analyze the grammians. The grammians for flexible structures in continuous time were analyzed by Gregory [3], Jonckheere [4], Gawronski [5], and Williams [6], and in discrete-time by Lim and Gawronski [7].

2. Structural equations

We consider a flexible structure model is represented by the following second order differential equation:

$$M\ddot{q} + D\dot{q} + Kq = B_o u, \qquad y = C_{oq}q + C_{ov}\dot{q}. \tag{1}$$

In this equation q is the $n_d \times 1$ displacement vector; u is the $s \times 1$ input vector, y is the output vector, $r \times 1$; M is the mass matrix, $n_d \times n_d$, D is the damping matrix, $n_d \times n_d$, and K is the stiffness matrix, $n_d \times n_d$. The input matrix B_o is $n_d \times s$, the output displacement matrix C_{oq} is $r \times n_d$, and the output velocity matrix C_{ov} is $r \times n_d$; n_d is the number of degrees of freedom. The mass matrix is positive definite, and the stiffness and damping matrices are positive semidefinite. The damping matrix is assumed proportional, on proportional damping see Ref. [8,9]. The above conditions define a stable and a minimum-phase system.

Let Φ ($n_d \times n$) be a modal matrix, consisting of n natural modes, and $n \leqslant n_d$. Introducing the co-ordinate transformation

$$q = \Phi q_m \tag{2}$$

we obtain Eq. (1) in modal co-ordinates q_m:

$$\begin{aligned} &\ddot{q}_{mi} + 2\zeta_i\omega_i\dot{q}_{mi} + \omega_i^2 q_{mi} = b_{mi}u, \\ &y_i = c_{mqi}q_{mi} + c_{mvi}\dot{q}_{mi} \qquad y = \sum_{i=1}^{n} y_i, \end{aligned} \tag{3}$$

where q_{mi} is a displacement of the ith mode, ω_i is the ith natural frequency, ζ_i is the damping of the ith mode, and b_{mi}, c_{mqi}, c_{mvi} are modal input and output matrices. For details of derivation see for example Ref. [1].

3. State-space modal representation

In this section we derive the continuous- and discrete-time state-space representations in modal co-ordinates that later are used in the derivation of properties of structural norms.

3.1. Continuous-time state-space representation

We begin with state-space representation (A, B, C) of a continuous-time model, which is a short notation of the following differential equation:

$$\dot{x} = Ax + Bu, \qquad y = Cx. \tag{4}$$

In the above equations the N-dimensional vector x is called the state vector, the s-dimensional vector u is the system input, and the r-dimensional vector y is the system output. The A, B, and C matrices are real constant matrices of appropriate dimensions (A is $N \times N$, B is $N \times s$, and C is $r \times N$).

For a flexible structure given by Eq. (3) one can obtain a modal state-space representation, as presented in Ref. [1]. It is a triple (A_m, B_m, C_m) characterized by the block-diagonal state matrix, A_m:

$$A_m = \mathrm{diag}(A_{mi}), \tag{5a}$$

$i = 1, 2, \ldots, m,$ where A_{mi} are 2×2 blocks, and the modal input and output matrices are divided correspondingly:

$$B_m = \begin{bmatrix} B_{m1} \\ B_{m2} \\ \vdots \\ B_{mn} \end{bmatrix}, \qquad C_m = [\, C_{m1} \quad C_{m2} \quad \cdots \quad C_{mn} \,], \tag{5b}$$

where B_{mi} and C_{mi} are $2 \times s$, and $r \times 2$ blocks, respectively.

The state x of the modal representation consists of $n = N/2$ independent components, x_i, and each component consists of two states,

$$x_i = \begin{Bmatrix} x_{i1} \\ x_{i2} \end{Bmatrix}. \tag{6}$$

The ith component, or mode, has the state-space representation (A_{mi}, B_{mi}, C_{mi}). The modal blocks A_{mi}, B_{mi}, and C_{mi} of these models are as follows:

$$A_{mi} = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\omega_i \end{bmatrix}, \qquad B_{mi} = \begin{bmatrix} 0 \\ b_{mi} \end{bmatrix}, \quad C_{mi} = \begin{bmatrix} \dfrac{c_{mqi}}{\omega_i} & c_{mvi} \end{bmatrix}. \tag{7a}$$

The ith state component of the first modal models is

$$x_i = \begin{Bmatrix} \omega_i q_{mi} \\ \dot{q}_{mi} \end{Bmatrix}, \tag{7b}$$

where q_{mi} and $\dot{q}_{mi}$ are the ith modal displacement and velocity.

The output y is obtained as a sum of the modal outputs y_i

$$y = \sum_{i=1}^{n} y_i, \tag{8a}$$

which are independently obtained from the state equations

$$\dot{x}_i = A_{mi}x_i + B_{mi}u, \qquad y_i = C_{mi}x_i. \tag{8b}$$

This decomposition is derived from the block-diagonal form of the matrix A_m.

The transfer function of a structure in modal co-ordinates is

$$G(s) = C_m(sI - A_m)^{-1}B_m \tag{9}$$

and it is a composition of modal transfer functions:

$$G(\omega) = \sum_{i=1}^{n} G_{mi}(\omega) = \sum_{i=1}^{n} \frac{(c_{mqi} + \mathrm{j}\omega c_{mvi})b_{mi}}{\omega_i^2 - \omega^2 + 2\mathrm{j}\zeta_i\omega_i\omega}, \tag{10a}$$

where

$$G_{mi}(\omega) = C_{mi}(\mathrm{j}\omega I - A_{mi})^{-1}B_{mi} = \frac{(c_{mqi} + \mathrm{j}\omega c_{mvi})b_{mi}}{\omega_i^2 - \omega^2 + 2\mathrm{j}\zeta_i\omega_i\omega}, \tag{10b}$$

$i = 1, \ldots, n$ is the transfer function of the ith mode. This decomposition is proved by introduction of A_m, B_m, and C_m as in Eqs. (5a) and (5b) to the definition of the transfer function (9).

3.2. Discrete-time state-space representation

The discrete-time sequences x_k, u_k, and y_k represent sampled continuous-time signals $x(t)$, $u(t)$, and $y(t)$, with sampling time Δt, that is, $x_k = x(k\Delta t)$, $u_k = u(k\Delta t)$, and $y_k = y(k\Delta t)$, $k = 1, 2, 3 \ldots$. The corresponding discrete-time representation for the sampling time Δt is denoted (A_d, B_d, C_d), and represents the discrete-time state-space equations

$$x_{k+1} = A_d x_k + B_d u_k, \qquad y_k = C x_k. \tag{11a}$$

In these equations the matrices A_d B_d and C_d and are obtained as

$$A_d = \mathrm{e}^{A\Delta t}, \qquad B_d = \int_0^{\Delta t} \mathrm{e}^{A\tau} B \, \mathrm{d}\tau, \qquad \text{and} \qquad C_d = C. \tag{11b}$$

We assume that the sampling time is sufficiently small, such that the Nyquist sampling criterion is satisfied, i.e., that

$$\omega_i \Delta t \leqslant \pi \qquad \text{or} \qquad \Delta t \leqslant \frac{\pi}{\omega_i} \quad \text{for all } i. \tag{12}$$

More on the Nyquist criterion see for example Ref. [10 p.111].

Similar to the continuous-time models the discrete-time model can be also presented in modal co-ordinates (assuming small damping). In this case the state matrix in modal co-ordinates A_{dm} is block-diagonal,

$$A_{dm} = \mathrm{diag}(A_{dmi}), \tag{13}$$

$i = 1, \ldots, n$. The 2×2 blocks A_{dmi} are in the form; see Ref. [7]:

$$A_{dmi} = \mathrm{e}^{-\zeta_i\omega_i\Delta t} \begin{bmatrix} \cos(\omega_i\Delta t) & -\sin(\omega_i\Delta t) \\ \sin(\omega_i\Delta t) & \cos(\omega_i\Delta t) \end{bmatrix}, \tag{14}$$

where and ζ_i are the ith natural frequency and the ith modal damping, respectively. The modal input matrix B_{dm} consists of $2 \times s$ blocks B_{dmi}

$$B_{dm} = \begin{bmatrix} B_{dm1} \\ B_{dm2} \\ \vdots \\ B_{dmn} \end{bmatrix}, \tag{15a}$$

where

$$B_{dmi} = S_i B_{mi}, \qquad S_i = \frac{1}{\omega_i} \begin{bmatrix} \sin(\omega_i \Delta t) & -1 + \cos(\omega_i \Delta t) \\ 1 - \cos(\omega_i \Delta t) & \sin(\omega_i \Delta t) \end{bmatrix}, \tag{15b}$$

and B_{mi} is the part of the continuous-time modal representation, see Eq. (7a). The discrete-time modal matrix C_{dm} is the same as the continuous-time modal matrix C_m.

The discrete transfer function is obtained from the state-space representation as in Eq. (11), by introducing the shift operator z such that $x_{i+1} = z x_i$, obtaining

$$G_d(z) = C_d (zI - A_d)^{-1} B_d. \tag{16}$$

We use this equation to obtain the transfer function for the ith mode. Using A_d as in Eqs. (13) and (14), we obtain

$$(zI - A_{dmi})^{-1} = \frac{1}{d} \begin{bmatrix} z - e^{-\zeta\omega_i \Delta t} \cos(\omega_i \Delta t) & e^{-\zeta\omega_i \Delta t} \sin(\omega_i \Delta t) \\ -e^{-\zeta\omega_i \Delta t} \sin(\omega_i \Delta t) & z - e^{-\zeta\omega_i \Delta t} \cos(\omega_i \Delta t) \end{bmatrix}, \tag{17a}$$

where $d = z^2 - 2z e^{-\zeta\omega_i \Delta t} \cos(\omega_i \Delta t) + e^{-2\zeta\omega_i \Delta t}$. Next, using B_d as in Eq. (15b) and noting that $B_{mi} = \begin{bmatrix} 0 \\ b_{mi} \end{bmatrix}$ we arrive at

$$G_{di}(z) = \frac{C_{mi}}{\omega_i d} \begin{bmatrix} (1 - \cos(\omega_i \Delta t))(z + e^{-\zeta\omega_i \Delta t}) \\ \sin(\omega_i \Delta t)(z - e^{-\zeta\omega_i \Delta t}) \end{bmatrix} b_{mi}, \tag{17b}$$

which is the transfer function of the ith mode. Note that C_{mi} and b_{mi} in the above equation are the output and input matrices of the continuous-time model.

4. Controllability/reachability and observability

Controllability and observability properties of a linear time-invariant system can be heuristically described as follows. The system dynamics described by the state variable (x) is excited by the input (u) and measured by the output (y). However, the input may not be able to excite all states, and consequently, cannot fully control the system. Also not all states may be represented at the output, consequently, they cannot be recovered from the system output. However, if the input excites all states, the system is controllable, and if all the states are represented at the output, the system is observable. In terms of modal models, a structure is controllable if the installed actuators excite all its structural modes. It is observable if the installed sensors detect the motions of all the modes. Reachability is a discrete-time equivalent of controllability.

4.1. Continuous-time controllability and observability

Consider a continuous-time system with the state-space representation (A, B, C), where A is $N \times N$. It is controllable if and only if the matrix

$$\mathscr{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{N-1}B] \tag{18a}$$

has rank N. It is observable if and only if the matrix

$$\mathscr{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} \tag{18b}$$

has rank N.

Define the following non-negative matrices

$$W_c = \int_0^\infty \exp(At)BB^{\mathrm{T}} \exp(A^{\mathrm{T}}t)\,\mathrm{d}t, \quad \text{and} \quad W_o = \int_0^\infty \exp(A^{\mathrm{T}}t)C^{\mathrm{T}}C \exp(At)\,\mathrm{d}t, \tag{19}$$

called controllability and observability grammians, respectively. The system is controllable, if the controllability grammian is positive definite. It is observable if the observability grammian is positive definite. The grammians can be obtained from the Lyapunov equations

$$AW_c + W_cA^{\mathrm{T}} + BB^{\mathrm{T}} = 0 \quad \text{and} \quad A^{\mathrm{T}}W_o + W_oA + C^{\mathrm{T}}C = 0. \tag{20}$$

The grammians and their eigenvalues change during co-ordinate transformations; however, the eigenvalues of the grammian product are invariant under co-ordinate transformations. These invariants, denoted γ_i,

$$\gamma_i = \sqrt{\lambda_i(W_cW_o)}, \quad i = 1, \ldots, N. \tag{21}$$

are called the Hankel singular values of the system.

4.2. Discrete-time reachability and observability

Consider now the discrete-time system, as given by Eq. (11). The reachability matrix $\mathscr{C}_k$ is defined similar to the controllability matrix of a continuous-time system:

$$\mathscr{C}_k = [B_d \quad A_dB_d \quad \cdots \quad A_d^{k-1}B_d]. \tag{22}$$

On the other hand, the reachability grammian $W_c(k)$ is defined over the time interval [0, $k\Delta t$] as

$$W_{dc}(k) = \sum_{i=0}^{k} A_d^iB_dB_d^{\mathrm{T}}(A_d^i)^{\mathrm{T}} \tag{23}$$

and unlike the controllability matrix of continuous-time systems the reachability matrix of the discrete-time system can be used to obtain the discrete time reachability grammian $W_{dc}(k)$.

Namely,

$$W_{dc}(k) = \mathscr{C}_k \mathscr{C}_k^{\mathrm{T}}. \tag{24}$$

The stationary grammian (i.e., for $k \to \infty$) satisfies the discrete-time Lyapunov equation

$$W_{dc} - A_d W_{dc} A_d^{\mathrm{T}} = B_d B_d^{\mathrm{T}}. \tag{25}$$

Similarly, the observability matrix $\mathscr{O}_k$ is defined as

$$\mathscr{O}_k = \begin{bmatrix} C \\ CA_d \\ \vdots \\ CA_d^{k-1} \end{bmatrix} \tag{26}$$

and the discrete-time observability grammian $W_{do}(k)$ for the time interval $[0,\ k\Delta t]$ is defined as

$$W_{do}(k) = \sum_{i=0}^{k} (A_d^i)^{\mathrm{T}} C^{\mathrm{T}} C A_d^i, \tag{27}$$

which is obtained from the observability matrix

$$W_{do}(k) = \mathscr{O}_k^{\mathrm{T}} \mathscr{O}_k. \tag{28}$$

For $k \to \infty$ (stationary solution) the observability grammian satisfies the Lyapunov equation

$$W_{do} - A_d^{\mathrm{T}} W_{do} A_d = C^{\mathrm{T}} C \tag{29}$$

Similar to the continuous-time grammians, the eigenvalues of the discrete-time grammian product are invariant under co-ordinate transformation. These invariants are denoted γ_{di},

$$\gamma_{di} = \sqrt{\lambda_i(W_{dc} W_{do})}, \tag{30}$$

where $i = 1, \ldots, N$, and are called the Hankel singular values of the discrete-time system.

4.3. Convergence of the discrete-time grammians

We show that the discrete-time reachability and observability grammians do not converge to the corresponding continuous-time controllability and observability grammians when the sampling time approaches zero. Indeed, consider the continuous time observability grammian as in Eq. (19). It can be approximated in discrete time, at time moments $t = 0,\ \Delta t,\ 2\Delta t,\ \ldots$ as

$$W_o = \sum_{i=0}^{\infty} \mathrm{e}^{iA^{\mathrm{T}}\Delta t} C^{\mathrm{T}} C \mathrm{e}^{iA\Delta t} \Delta t = \sum_{i=0}^{\infty} (A_d^i)^{\mathrm{T}} C^{\mathrm{T}} C A_d^i \Delta t. \tag{31}$$

Introducing Eq. (27) for $k \to \infty$, one obtains

$$W_o = \lim_{\Delta t \to 0} \Delta t\, W_{do}. \tag{32}$$

Similarly, one can obtain this for the controllability/reachability grammians. First note that for small sampling time one obtains

$$B_d \cong \Delta t\, B. \tag{33}$$

Indeed, from the definition of B_d it follows that

$$B_d = \int_0^{\Delta t} e^{A\tau} B \, d\tau = \int_0^{\Delta t} (I + A\tau + \frac{1}{2}A^2\tau^2 + \cdots) B \, d\tau$$
$$= B\Delta t + \tfrac{1}{2}AB\Delta t^2 + \frac{1}{2}A^2 B\Delta t^3 + \cdots \cong \Delta t \, B.$$

Now, from the definition of the continuous-time controllability grammian the following holds:

$$W_c = \int_0^{\infty} e^{A\tau} BB^{\mathrm{T}} e^{A^{\mathrm{T}}\tau} \, d\tau = \lim_{\Delta t \to 0} \sum_{i=0}^{\infty} e^{iA\Delta t} BB^{\mathrm{T}} e^{iA^{\mathrm{T}}\Delta t} \Delta t.$$

Using Eqs. (33) and (23), and $A_d = e^{A\Delta t}$ one obtains

$$W_c = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{i=0}^{\infty} A_d^i B_d B_d^{\mathrm{T}} (A_d^i)^{\mathrm{T}} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} W_{dc},$$

hence

$$W_c = \lim_{\Delta t \to 0} \frac{1}{\Delta t} W_{dc}. \tag{34}$$

Note however from Eq. (32) to (34) that the product of the discrete-time reachability and observability grammians converge to the continuous-time grammians,

$$W_c W_o = \lim_{\Delta t \to 0} (W_{dc} W_{do}), \tag{35}$$

therefore the discrete-time Hankel singular values converge to the continuous-time values, as the sampling time approaches zero:

$$\gamma_i = \lim_{\Delta t \to 0} \gamma_{di}. \tag{36}$$

5. Grammians of flexible structures

For flexible structures in modal co-ordinates the grammians are diagonally dominant, which simplifies the analysis of norms, performed later in this paper.

5.1. Continuous-time grammians of flexible structures

For flexible structures in modal co-ordinates controllability and observability grammians are diagonally dominant, see Refs. [1,5,6], i.e.,

$$W_c \cong \mathrm{diag}(w_{ci} I_2) \quad \text{and} \quad W_o \cong \mathrm{diag}(w_{oi} I_2), \tag{37a}$$

where

$$w_{ci} = \frac{\|B_{mi}\|_2^2}{4\zeta_i \omega_i}, \qquad w_{oi} = \frac{\|C_{mi}\|_2^2}{4\zeta_i \omega_i}, \tag{37b}$$

and the approximate Hankel singular values are obtained from

$$\gamma_i \cong \sqrt{w_{ci} w_{oi}} = \frac{\|B_{mi}\|_2 \|C_{mi}\|_2}{4\zeta_i \omega_i}. \tag{38}$$

5.2. Discrete-time grammians of flexible structures

Consider now a structure in modal co-ordinates. Similar to the continuous-time grammians the discrete-time grammians in modal co-ordinates are diagonally dominant:

$$W_{dc} \cong \mathrm{diag}(W_{dc1}, W_{dc2}, \ldots, W_{dcn}), \qquad W_{do} \cong \mathrm{diag}(W_{do1}, W_{do2}, \ldots, W_{don}), \tag{39}$$

where W_{dci} and W_{doi} are 2×2 blocks, $W_{dci} = w_{dci} I_2$, and $W_{doi} = w_{doi} I_2$, see Ref. [7], where

$$w_{dci} = \frac{\|B_{mi}\|_2^2}{4\zeta_i \omega_i} \frac{2(1 - \cos(\omega_i \Delta t)}{\omega_i^2 \Delta t} = w_{ci} \frac{2(1 - \cos(\omega_i \Delta t)}{\omega_i^2 \Delta t} \tag{40}$$

and

$$w_{doi} = \frac{\|C_{mi}\|_2^2}{4\zeta_i \omega_i} \frac{1}{\Delta t} = w_{oi} \frac{1}{\Delta t}. \tag{41}$$

Again, B_{mi} is the ith block of B_m in the continuous-time modal co-ordinates, see Eq. (7a), and C_{mi} is the ith block of C_m in continuous-time modal co-ordinates. Also w_{ci} and w_{oi} denote the continuous-time controllability and observability grammians, respectively, cf. Eq. (37b).

Note that the discrete-time reachability grammian deviates from the continuous-time controllability grammian by the factor $2(1 - \cos(\omega_i \Delta t))/\omega_i^2 \Delta t$, while the discrete-time observability grammian deviates from the continuous-time observability grammian by factor $1/\Delta t$. Thus, the discrete time grammians do not converge to the continuous-time grammians, but satisfy the conditions

$$\lim_{\Delta t \to 0} \frac{W_{dci}}{\Delta t} = W_{ci}, \qquad \lim_{\Delta t \to 0} W_{doi} \Delta t = W_{oi}, \tag{42a, b}$$

which is consistent with the Moore result; see Ref. [11].

The Hankel singular values are the square roots of the eigenvalues of the grammian products, $\Gamma_d = \lambda^{1/2}(W_{dc} W_{do})$. The approximate values of the Hankel singular values can be obtained from the approximate values of the grammians,

$$\gamma_{di} \cong \sqrt{w_{dci} w_{doi}} = \frac{\|B_{mi}\|_2 \|C_{mi}\|_2}{4\zeta_i \omega_i} \frac{\sqrt{2(1 - \cos(\omega_i \Delta t))}}{\omega_i \Delta t}. \tag{43}$$

Note that the discrete-time Hankel singular values differ from the continuous-time values by a factor k_i,

$$\gamma_{di} \cong k_i \ \gamma_i, \tag{44a}$$

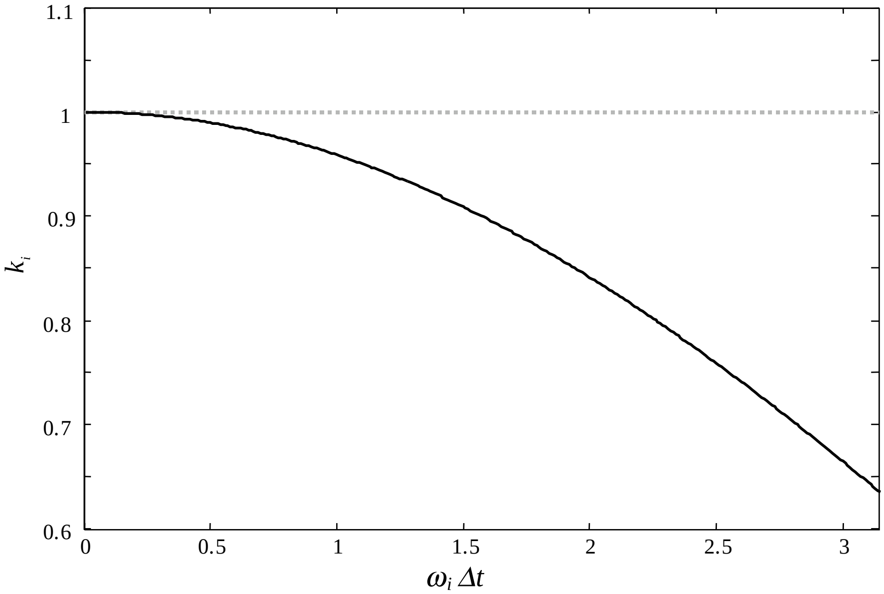

Fig. 1. Plot of k_i versus $\omega_i \Delta t$.

where

$$k_i = \frac{\sqrt{2(1 - \cos \omega_i \Delta t)}}{\omega_i \Delta t}. \tag{44b}$$

The plot of $k_i(\omega_i \Delta t)$ is shown in Fig. 1; it shows that for small sampling time, discrete-time and continuous-time Hankel singular values are almost identical.

It follows from Fig. 1 that if the sampling rate is high enough (or the sampling time small enough), the discrete-time Hankel singular values are very close to the continuous-time Hankel singular values. For example, if $\omega_i \Delta t \leqslant 0.6$ the difference is less than 3%, if $\omega_i \Delta t \leqslant 0.5$ the difference is less than 2%, and if $\omega_i \Delta t \leqslant 0.35$ the difference is less than 1%. Note also that for a given sampling time the discrete-time Hankel singular values corresponding to the lowest natural frequencies are closer to the continuous-time Hankel singular values than the Hankel singular values corresponding to the higher natural frequencies.

Example 1. Consider the discrete-time simple system as in Fig. 2, with $k_1 = k_2 = k_3 = 3$, $k_4 = 0$, and $m_1 = m_2 = m_3 = 1$. The damping is proportional to the stiffness matrix, $D = 0.01\,\mathrm{K}$. Its Hankel singular values are determined for sampling times $\Delta t = 0.7\,\mathrm{s}$ and for $\Delta t = 0.02\,\mathrm{s}$, and compared with the continuous-time Hankel singular values.

The Hankel singular values for the continuous-time structure, and for the discrete-time structure with sampling times $\Delta t = 0.7$ and 0.02 are given in Table 1.

The table shows that for the sampling time $\Delta t = 0.7\,\mathrm{s}$ the discrete-time Hankel singular values are smaller than the continuous-time values, especially for the third mode (note that two Hankel

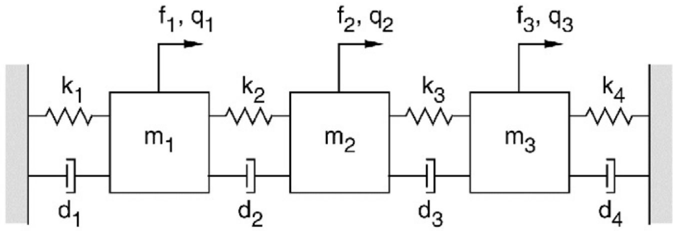

Fig. 2. A simple structure.

Table 1
Hankel singular values

Mode	Continuous time	Discrete time $\Delta t = 0.7\,\mathrm{s}$	Discrete time $\Delta t = 0.02\,\mathrm{s}$
Mode 1	20.342	20.138	20.343
	20.340	20.009	20.340
Mode 2	4.677	4.324	4.679
	4.671	4.225	4.670
Mode 3	0.991	0.848	0.992
	0.986	0.785	0.986

singular values correspond to each mode). In order to explain it, note that the natural frequencies are $\omega_1 = 0.771\,\mathrm{rad/s}$, $\omega_2 = 2.160\mathrm{rad/s}$, and $\omega_3 = 3.121\,\mathrm{rad/s}$. For each mode the sampling time must satisfy the Nyquist condition (12). For the first mode $\pi/\omega_1 = 4.075$, for the second mode $\pi/\omega_2 = 1.454$, and for the third mode $\pi/\omega_3 = 1.007$. Thus, the sampling time satisfies condition (12). However for this sampling time one obtains $\omega_1\Delta t = 0.540$, $\omega_2\Delta t = 1.512$, and $\omega_3\Delta t = 2.185$. It is visible from Fig. 1 that for these values of $\omega_i\Delta t$ the discrete-time Hankel singular values are smaller than the continuous-time ones, especially for the third mode.

The results are different for the sampling time of $\Delta t = 0.02\,\mathrm{s}$. In this case one obtains $\omega_1\Delta t = 0.015$, $\omega_2\Delta t = 0.043$, and $\omega_3\Delta t = 0.062$. One can see from Fig. 1 that for these values of $\omega_i\Delta t$ the discrete-time Hankel singular values are almost equal to the continuous-time ones.

Next, the accuracy of the approximate relationship (44a) between discrete- and continuous-time Hankel singular values is verified. The accuracy is expressed with the coefficient k_i, Eq. (44b). The Hankel singular values were computed for different sampling times, and compared with the continuous-time Hankel singular values. Their ratio determines coefficient k_i. The plot of k_i obtained for all three modes and the plot of the approximate coefficient from Eq. (44b) are shown in Fig. 3. The plot show, that the approximate curve and the actual curves are close, except for $\omega_i\Delta t$ very close to π.

6. Norms

The Hankel, H_∞, and H_2 norms are analyzed in this paper. Their properties are derived and specified for structural applications.

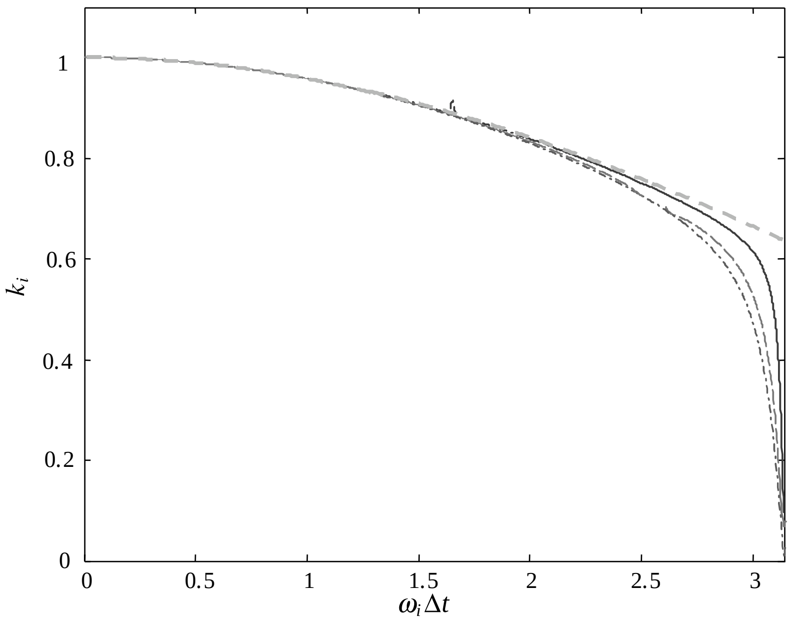

Fig. 3. Exact and approximate coefficient k_i versus $\omega_i \Delta t$: ‒‒‒, approximate; —, k_1; - - -, k_2; - · -, k_3.

6.1. Continuous-time norms

6.1.1. The H_∞ norm

The H_∞ norm is defined as

$$\|G\|_\infty = \sup_{u(t)\neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2} \tag{45a}$$

or alternatively as

$$\|G\|_\infty = \max_{\omega} \ \sigma_{\max}(G(\omega)), \tag{45b}$$

where $\sigma_{\max}(G(\omega))$ is the largest singular value of $G(\omega)$. In particular, the H_∞ norm of a single-input–single-output system is the peak of the transfer function magnitude, $\|G\|_\infty = \max_\omega |G(\omega)|$. This norm is useful in the system analysis and controller design since as the induced norm it can provide the bounds of the r.m.s output errors. Namely, let u and y be the system input and output, respectively, and G its transfer function, then from Eq. (45a) we obtain

$$\|y\|_2 \leqslant \|G\|_\infty \|u\|_2. \tag{46}$$

It is seen from the above inequality and Eq. (45b) that $\|G\|_\infty$ is the worst-case gain for sinusoidal inputs at any frequency.

The H_∞ norm can be computed as a maximal value of ρ such that the solution S of the following algebraic Riccati equation is positive definite:

$$A^{\mathrm{T}}S + SA + \rho^{-1}SBB^{\mathrm{T}}S + \rho^{-1}C^{\mathrm{T}}C = 0. \tag{47}$$

It is an iterative procedure where one starts with a large value of ρ and reduce it until negative eigenvalues of S appear.

6.1.2. The Hankel norm

The Hankel norm of a system is a measure of the effect of its past input on its future output, or the amount of energy stored in, and subsequently retrieved from, the system [12, p. 103]. It is defined as

$$||G||_h = \sup\frac{||y(t)||_2}{||u(t)||_2}, \quad \text{where} \quad \begin{cases} u(t) = 0 & \text{for } t > 0. \\ y(t) = 0 & \text{for } t < 0. \end{cases} \tag{48}$$

Comparing definition (45a) of the H_∞ norm and the above definition of the Hankel norm one can see that the H_∞ norm is the largest output for all possible inputs contained in an unit ball, while the Hankel norm is the largest future output for all past inputs from the unit ball. From these definitions it follows that the Hankel norm never exceeds the H_∞ norm (since the set of outputs used to evaluate the Hankel norm is a subset of outputs used to evaluate the H_∞ norm), thus

$$||G||_h \leqslant ||G||_\infty. \tag{49}$$

The Hankel norm can be determined from the controllability and observability grammians as

$$||G||_h = \sqrt{\lambda_{max}(W_c W_o)} \tag{50}$$

where $\lambda_{max}(.)$ denotes the largest eigenvalue, and W_c, W_o are the controllability and observability grammians, respectively. It follows from (50) that the Hankel norm of the system is the largest Hankel singular value of the system, γ_{max}:

$$||G||_h = \gamma_{max.} \tag{51}$$

6.1.3. The H_2 norm

Let $G(\omega)$ be its transfer function of a stable linear system. The H_2 norm of the system is defined as

$$||G||_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G^*(\omega)G(\omega))\,\mathrm{d}\omega. \tag{52}$$

Note that $\mathrm{tr}(G^*(\omega)G(\omega))$ is the sum of the magnitudes squared of all of the elements of $G(\omega)$, i.e. $\mathrm{tr}(G^*(\omega)G(\omega)) = \sum_{k,l} ||g_{kl}(j\omega)|^2$. Thus, it can be thought to be an average gain of the system, with the average performed over all the elements of the matrix transfer function and over all frequencies.

Since the transfer function $G(\omega)$ is the Fourier transform of the system impulse response $g(t)$, from the Parseval theorem, Eq. (52) can be rewritten as an average of the impulse response,

$$||G||_2^2 = ||g(t)||_2^2 = \int_0^{\infty} \mathrm{tr}(g^{\mathrm{T}}(t)g(t))\,\mathrm{d}t. \tag{53}$$

A convenient way to determine the numerical value of the H_2 norm is to use the formulas

$$||G||_2 = \sqrt{\mathrm{tr}(C^{\mathrm{T}}CW_c)}, \qquad ||G||_2 = \sqrt{\mathrm{tr}(BB^{\mathrm{T}}W_o)}, \tag{54}$$

where W_c and W_o are the controllability and observability grammians.

6.2. Discrete-time norms

In this section we present the Hankel, H_∞, and H_2 norms for the discrete-time systems, and compare them with the norms of the continuous-time systems.

6.2.1. The Hankel norm

The Hankel norm of a discrete-time system is its largest Hankel singular value,

$$\|G_d\|_h = \max_i \gamma_{di}, \tag{55}$$

where subscript d denotes a discrete-time system. Previously, we showed that the discrete-time Hankel singular values converge to the continuous-time Hankel singular values, see Eq. (36), therefore the discrete-time Hankel norms converge to the continuous-time Hankel norms when the sampling time approaches zero:

$$\|G\|_h = \lim_{\Delta t \to 0} \|G_d\|_h. \tag{56}$$

6.2.2. The H_∞ norm

The H_∞ norm of the discrete-time system is defined as (see Ref. [13])

$$\|G_d\|_\infty = \max_{\omega \Delta t} \sigma_{max}(G_d(\mathrm{e}^{\mathrm{j}\omega\Delta t})). \tag{57}$$

Since for small enough sampling time the discrete-time transfer function is approximately equal to the continuous-time transfer function, see Ref. [13], $G_d(\mathrm{e}^{\mathrm{j}\omega\Delta t}) \cong G_c(\mathrm{j}\omega)$, therefore the discrete-time H_∞ norm converges to the continuous-time H_∞ norm

$$\|G_d\|_\infty = \lim_{\Delta t \to 0} \|G\|_\infty \tag{58}$$

when the sampling time approaches zero.

6.2.3. The H_2 norm

The discrete-time H_2 norm is defined as an r.m.s. sum of integrals of the magnitudes of its transfer function, or as an r.m.s. sum of the impulse response

$$\|G_d\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}(G_d^*(\mathrm{e}^{\mathrm{j}\omega\Delta t})G_d(\mathrm{e}^{\mathrm{j}\omega\Delta t})\,\mathrm{d}(\omega\Delta t)\right)^{1/2} = \left(\sum_{i=0}^{\infty} g_d^2(i\Delta t)\right)^{1/2}. \tag{59}$$

In the above equation $g_d(i\Delta t)$ is the impulse response of the discrete-time system at $t = i\Delta t$.

Similar to the continuous-time case the H_2 norm can be calculated using the discrete-time grammians W_{dc} and W_{do}

$$\|G_d\|_2^2 = \mathrm{tr}(C^\mathrm{T} C W_{dc}) = \mathrm{tr}(B_d B_d^\mathrm{T} W_{do}). \tag{60}$$

A relationship between discrete-time H_2 norm and the continuous-time H_2 norm is derived by introducing the relationships between discrete-time grammians and the continuous-time grammians, as in Eqs. (32) and (34) to Eq. (60). In this way we obtained

$$\|G\|_2^2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \|G_d\|_2^2. \tag{61}$$

As we see, unlike the Hankel and H_∞ norm cases, the discrete-time H_2 norm does not converge to the continuous-time H_2 norm when the sampling time approaches zero. It can be explained using the system impulse responses. The continuous-time H_2 norm is obtained from the continuous-time unit impulse response,

$$\|G\|_2^2 = \int_0^\infty g^2(\tau)\,\mathrm{d}\tau$$

which can be approximated as

$$\|G\|_2^2 \cong \sum_{i=0}^{\infty} g^2(i\Delta t)\Delta t. \tag{62}$$

The applied impulse value was equal to 1. Note, however, that for the discrete-time system the impulse response is evaluated for the impulse value different than 1. Indeed, for the discrete-time system the impulse amplitude was 1 and its duration was Δt. Thus the impulse value, as a product of its amplitude and duration, is Δt. For this reason, the relationship between the impulse responses of the continuous- and the discrete-time system is $g_c(i\Delta t) = g_d(i\Delta t)/\Delta t$. Introducing this equation in Eq. (62) one obtains

$$\|G\|_2^2 \cong \frac{1}{\Delta t} \sum_{i=0}^{\infty} g_d^2(i\Delta t) = \frac{1}{\Delta t} \|G_d\|_2^2$$

which is identical to Eq. (61).

7. Norms of flexible structures

In this section we derive the closed-form expressions for the Hankel, H_∞ and H_2 norms of a single mode, and explain how to obtain a norm of an entire structure from the norms of modes. It will be done in both the continuous- and discrete-time cases.

7.1. Continuous time norms of flexible structures

For structures in the modal representation, each mode is independent, thus the norms of a single mode are independent as well.

7.1.1. The H_2 norm

Consider the ith natural mode and its state-space representation (A_{mi}, B_{mi}, C_{mi}); see Eq. (7a). For this representation one obtains the following closed-form expression for the H_2 norm, see Ref. [1]:

$$||G_i||_2 \cong \frac{||B_{mi}||_2 ||C_{mi}||_2}{2\sqrt{\zeta_i \omega_i}}. \tag{63}$$

Note also that $||G_i||_2$ is the modal cost of Skelton [14].

The above represent the H_2 norm of a single mode. The H_2 norm of the entire flexible structure is the r.m.s. sum of the modal norms

$$||G||_2 \cong \sqrt{\sum_{i=1}^{n} ||G_i||_2^2}, \tag{64}$$

where n is the number of modes. It can be shown, by noting that the controllability grammian W_c in modal co-ordinates is diagonally dominant, that

$$||G||_2^2 = \operatorname{tr}(C_m^{\mathrm{T}} C_m W_c) \cong \sum_{i=1}^{n} \operatorname{tr}(C_{mi}^{\mathrm{T}} C_{mi} W_{ci}) = \sum_{i=1}^{n} ||G_i||_2^2.$$

7.1.2. The H_∞ norm

The H_∞ norm of a natural mode can be approximately expressed in closed-form as:

$$||G_i||_\infty \cong \frac{||B_{mi}||_2 ||C_{mi}||_2}{2\zeta_i \omega_i}. \tag{65}$$

In order to show this, note that the largest amplitude of the mode is approximately at the ith natural frequency, thus

$$||G_i||_\infty \cong \sigma_{max}(G_i(\omega_i)) = \frac{\sigma_{max}(C_{mi} B_{mi})}{2\zeta_i \omega_i} = \frac{||B_{mi}||_2 ||C_{mi}||_2}{2\zeta_i \omega_i}.$$

The above represents the H_∞ norm of a single mode. Due to the almost independence of the modes, the H_∞ norm of a flexible structure is the largest of the mode norms, i.e.,

$$||G||_\infty \cong \max_i ||G_i||_\infty, \quad i = 1, \ldots, n.$$

7.1.3. The Hankel norm

For a single mode this norm is approximately evaluated from the following closed-form formula, see Ref. [1]:

$$||G_i||_h = \gamma_i \cong \frac{||B_{mi}||_2 ||C_{mi}||_2}{4\zeta_i \omega_i}. \tag{66}$$

The Hankel norm of the entire structure is the largest norm of its modes, i.e.,

$$||G||_h \cong \max_i ||G_i||_h = \gamma_{max}, \tag{67}$$

where γ_{max} is the largest Hankel singular value of the system.

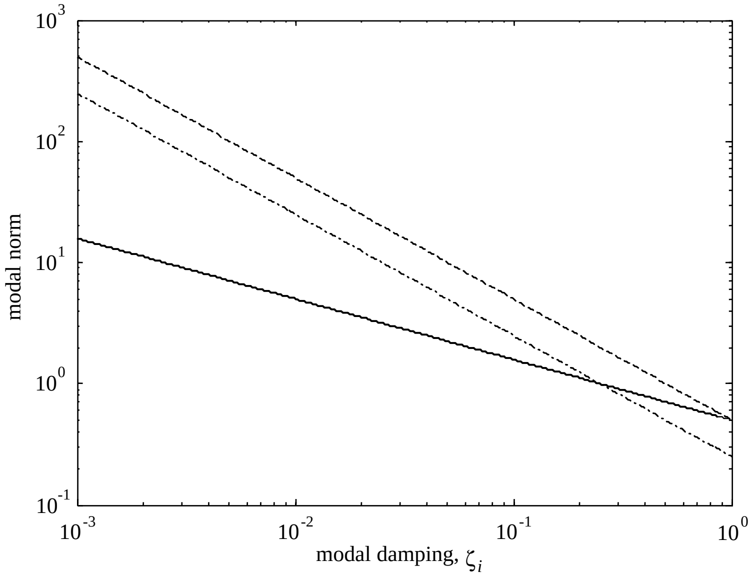

Fig. 4. Modal norms versus modal damping: —, H_2; - - -, H_{inf}; - · - · -, Hankel.

7.1.4. Norm comparison

The Hankel and H_∞ norms are related to the Hankel singular values as follows, see Ref. [15, p. 156]:

$$\|G\|_h = \gamma_{max} \leqslant \|G\|_\infty \leqslant 2 \sum_{i=1}^{n} \gamma_i. \tag{68}$$

It is a very rough estimation, indeed, since it says that $\gamma_{max} \leqslant 2\gamma_{max}$. However, for structures more precise estimation can be obtained. Comparing Eqs. (63), (65) and (66) one obtains the approximate relationships between H_2, H_∞, and Hankel norms,

$$\|G_i\|_\infty \cong 2\|G_i\|_h \cong \sqrt{\zeta_i \omega_i}\|G_i\|_2. \tag{69}$$

The norms depend on modal damping, as illustrated in Fig. 4.

7.2. Discrete-time norms of flexible structures

The norms of discrete-time structures are obtained in a similar way as the norms of the continuous-time structures. First of all, the system matrix A_d in discrete-time modal co-ordinates is block-diagonal, similar to the continuous time case. For a diagonal A_d the structural norms are determined from the norms of structural modes, as described previously.

7.2.1. The Hankel norm

The Hankel norm of the discrete-time system is defined (similar to the continuous-time case) as a "geometric mean" of the discrete-time reachability and observability grammians, i.e.,

$$\|G_d\|_h = \sqrt{\lambda_{max}(W_{dc}W_{do})}. \tag{70}$$

The grammians in modal co-ordinates are diagonally dominant, therefore for a single (ith) mode we obtain from Eqs. (40) and (41) as

$$\|G_{di}\|_h = k_i\Delta t\frac{\|B_{mi}\|_2\|C_{mi}\|_2}{4\zeta_i\omega_{oi}} = k_i\|G_i\|_h, \tag{71}$$

where k_i is given by Eq. (44b), $\omega_{oi} = \omega_i\Delta t$, and $\|G_{di}\|_h$ is the Hankel norm of the ith mode in discrete- time, while $\|G_i\|_h$ is the same norm of the ith mode in continuous- time. For fast sampling, i.e., when $\Delta t \to 0$ one obtains $k_i = 1$, which means that the discrete Hankel norm converges to the continuous one.

Similar to the continuous-time case the Hankel norm of the entire structure is the largest norm of its modes, i.e.,

$$\|G_d\|_h \cong \max_i \|G_{di}\|_h = \gamma_{d\,max}, \tag{72}$$

where γ_{dmax} is the largest Hankel singular value of the discrete-time system.

7.2.2. TheH_∞ norm

The H_∞ norm of a discrete-time system is defined as the peak magnitude over the segment $0 \leqslant \omega\Delta t \leqslant \pi$, i.e.,

$$\|G_d\|_\infty = \sup_{\omega\Delta t} \sigma_{max}(G_d(\mathrm{e}^{\mathrm{j}\omega\Delta t})). \tag{73}$$

For the discrete-time structure it is the largest norm of its modes. However, the H_∞ norm of the ith mode is approximately the magnitude at its resonant frequency, thus for the ith mode, from the above definition,

$$\|G_{di}\|_\infty \cong \sigma_{max}(G_{di}(\mathrm{e}^{\mathrm{j}\omega_i\Delta t})) = \lambda_{max}^{1/2}(G_{di}(\mathrm{e}^{\mathrm{j}\omega_i\Delta t})G_{di}^*(\mathrm{e}^{\mathrm{j}\omega_i\Delta t})), \tag{74}$$

where G_{di} is the discrete-time transfer function of the ith mode, and $\omega_i$is its natural frequency, and λ_{max} denotes the maximal eigenvalue.

In order to obtain its H_∞ norm we use the discrete-time transfer function of the ith mode as in Eq. (17a) and (17b) at $\omega = \omega_i$. First, note that $z = \mathrm{e}^{\mathrm{j}\omega_i\Delta t} = \cos(\omega_i\Delta t) + j\sin(\omega_i\Delta t)$, and that for small ζ_i one can use approximation $\mathrm{e}^{-\zeta_i\omega_i\Delta t} \cong 1 - \zeta_i\omega_i\Delta t$. Now using Eq. (17a) one obtains

$$(zI - A)^{-1}|_{z=\mathrm{e}^{\mathrm{j}\omega_i\Delta t}} = \frac{1}{2\zeta_i\omega_i\Delta t}\begin{bmatrix} j & 1 \\ -1 & j \end{bmatrix}.$$

For B_{dmi} as in Eq. (15b), and $B_{mi} = \begin{bmatrix} 0 \\ b_{mi} \end{bmatrix}$, the modal transfer function at its resonance frequency is therefore

$$G_{dmi}(\omega_i) = \frac{C_{mi}}{2\zeta_i\omega_i^2\Delta t}\begin{bmatrix} 1 - \cos(\omega_i\Delta t) - j\sin(\omega_i\Delta t) \\ \sin(\omega_i\Delta t) + j(1 - \cos(\omega_i\Delta t)) \end{bmatrix} b_{mi}.$$

Introducing the above in Eq. (74) one obtains

$$\|G_{di}\|_\infty = \lambda_{max}^{1/2}(G_{di}(\mathrm{e}^{\mathrm{j}\omega_i \Delta t})G_{di}^*(\mathrm{e}^{\mathrm{j}\omega_i \Delta t})) = \frac{\|C_{mi}\|_2\|b_{mi}\|_2}{\zeta_i\omega_i^2\Delta t}(1-\cos(\omega_i\Delta t)). \tag{75}$$

which can be presented in the form

$$\|G_{di}\|_\infty = k_i\Delta t\frac{\|B_{mi}\|_2\|C_{mi}\|_2}{2\zeta_i\omega_{oi}} = k_i\|G_i\|_\infty, \tag{76}$$

where $\omega_{oi} = \omega_i\Delta t$, $\|G_i\|_\infty$ is the H_∞ norm of the continuous-time mode, c.f. Eq. (65), and k_i is the coefficient given by Eq. (44b). Since $k_i = 1$ for $\Delta t \to 0$, the discrete-time H_∞ norm converges to the continuous-time norm.

The above represents the H_∞ norm of a single mode. Due to the almost independence of the modes, the H_∞ norm of a flexible structure is the largest of the mode norms, i.e.,

$$\|G_d\|_\infty \cong \max_i \|G_{di}\|_\infty, \quad i = 1, \ldots n. \tag{77}$$

7.2.3. The H_2 norm

Similar to the continuous-time case, the H_2 norm of a structure is the r.m.s. sum of the H_2 norms of its modes,

$$\|G\|_2 \cong \sqrt{\sum\nolimits_{i=1}^{n} \|G_i\|_2^2}, \tag{78}$$

where n is the number of modes.

The H_2 norm of the ith mode is obtained as

$$\|G_{di}\|_2 = \sqrt{\mathrm{tr}(B_{di}^{\mathrm{T}} W_{odi} B_{di})} \tag{79a}$$

or, alternatively,

$$\|G_{di}\|_2 = \sqrt{\mathrm{tr}(C_{di} W_{cdi} C_{di}^{\mathrm{T}})}. \tag{79b}$$

Using Eqs. (79a), (15b) and (41) we obtain

$$\|G_{di}\|_2^2 = \frac{1}{\Delta t}\mathrm{tr}(B_i^{\mathrm{T}} S_i^{\mathrm{T}} W_{oi} S_i B_i) = \frac{w_{oi}}{\Delta t}\mathrm{tr}(B_i^{\mathrm{T}} S_i^{\mathrm{T}} S_i B_i),$$

where w_{oi} is the continuous-time grammians given by Eq. (37b). Note also that

$$S_i^{\mathrm{T}} S_i = \frac{2(1-\cos(\omega_i\Delta t))}{\omega_i^2} I_2.$$

Thus

$$\begin{aligned}\|G_{di}\|_2^2 &= \frac{2w_{oi}(1-\cos(\omega_i\Delta t))}{\Delta t\omega_i^2}\mathrm{tr}(B_i^{\mathrm{T}}B_i) = w_{oi}\|B_i\|_2^2\frac{2(1-\cos(\omega_i\Delta t)}{\Delta t\omega_i^2}\\ &= \frac{\|C_i\|_2^2\|B_i\|_2^2}{4\zeta_i\omega_i}\Delta t\frac{2(1-\cos(\omega_i\Delta t)}{\Delta t^2\omega_i^2} = \|G_i\|_2^2\Delta t\, k_i^2,\end{aligned}$$

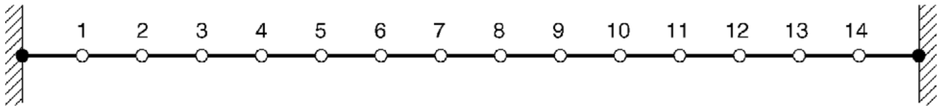

Fig. 5. A beam.

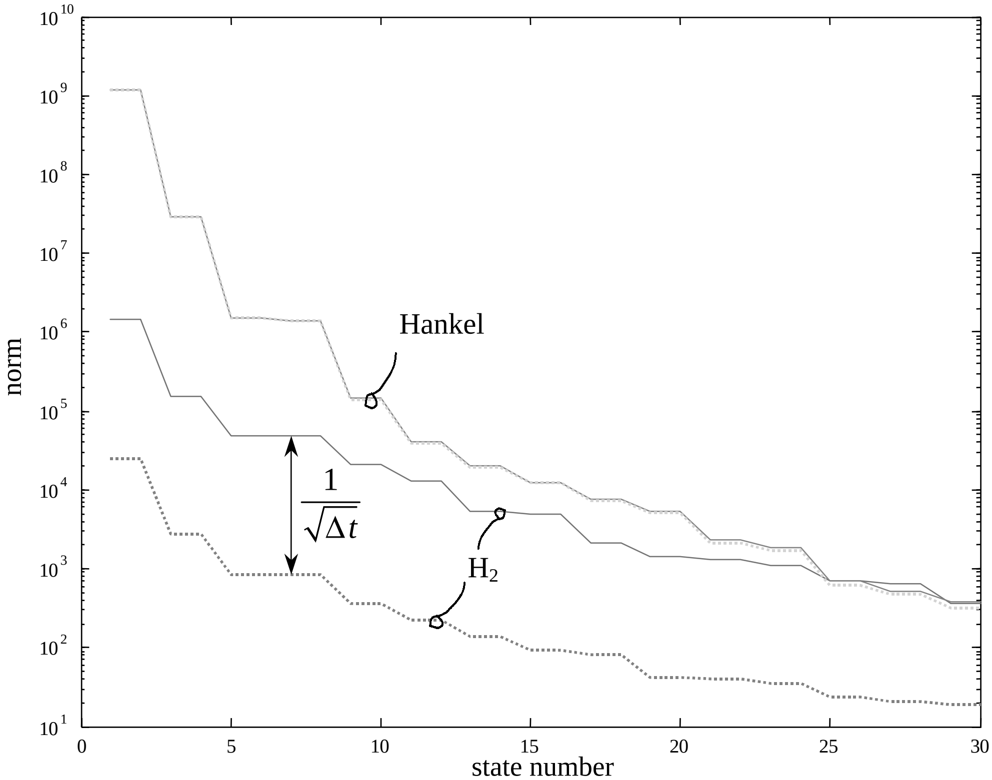

Fig. 6. Hankel and H_2 norms of a beam: continuous-time model (–) discrete-time model (….).

where $||G_i||_2$ is the H_2 norm of the mode in continuous time. Therefore,

$$||G_{di}||_2 = k_i \Delta t \frac{||B_{mi}||_2 ||C_{mi}||_2}{2\sqrt{\zeta_i \omega_{oi}}} = k_i \sqrt{\Delta t} ||G_i||_2, \tag{80}$$

where $\omega_{oi} = \omega_i \Delta t$. For fast sampling $k_i \to 1$, thus

$$\lim_{\Delta t \to 0} \frac{||G_{di}||_2}{\sqrt{\Delta t}} = ||G_i||_2. \tag{81}$$

The above equation indicates that the discrete-time H_2 norm does not converge to the continuous time. This is a consequence of non-convergence of the discrete-time reachability and observability grammians, to the continuous-time grammians, as we showed it previously.

7.2.4. Norm comparison

From Eqs. (71), (76) and (80) the following relationship between the norms of a single mode of a discrete-time system is obtained:

$$\|G_{di}\|_{\infty} \cong 2\|G_{di}\|_{h} \cong \sqrt{\zeta_i \omega_{oi}}\|G_{di}\|_{2}, \tag{82}$$

which is similar to the continuous-time norm, as in Eq. (69).

Example 2. Consider a beam in Fig. 5. Its model consists of the first 15 modes (or 30 states), with a vertical force applied at node 6, and a velocity measured at node 6 in vertical direction. The Hankel and H_2 norms were determined for its continuous- and discrete-time models, and are plotted in Fig. 6. The beam largest natural frequency is 6221 rad/s. We chose sampling time of 0.0003 s. The Nyquist frequency for this sampling time is $\pi/\Delta t = 10472$ rad/s, so that the largest natural frequency is close to the Nyquist frequency. The plots of the norms are shown in Fig. 6, for the continuous-time model in solid line, for the discrete-time model in dotted line. The Hankel norms for the continuous- and discrete-time models are almost identical, except for some discrepancy at higher modes, with natural frequencies close to the Nyquist frequency. The H_2 norms of the continuous- and discrete-time systems are separated by the distance of $1/\sqrt{\Delta t} = 57.74$, as predicted by Eq. (80).

8. Conclusions

The Hankel, H_∞, and H_2 norms were analyzed in this paper. They were obtained for the continuous- and discrete-time systems, and compared. These norms were obtained in closed form for natural modes of flexible structures, and for an entire flexible structure as the rms sum of modal norms (for the H_2 norm), and as the maximal modal norm (in case of the Hankel and H_∞ norms). We also derived the relationship between the continuous and discrete- time norms. We showed that the H_∞, and Hankel discrete-time norms converge to the continuous-time counterparts when the sampling time approaches zero. We show, however, that the H_2 norm does not converge.

References

[1] W. Gawronski, Dynamics and Control of Structures, Springer, New York, 1998.
[2] R.E. Skelton, Dynamic System Control: linear system analysis and synthesis, Wiley, New York, 1988.
[3] C.Z. Gregory Jr., Reduction of large flexible spacecraft models using internal balancing theory, Journal of Guidance, Control, and Dynamics 7 (1984) 725–732.
[4] E.A. Jonckheere, Principal component analysis of flexible systems—open loop case, IEEE Transactions on Automatic Control 27 (1984) 1095–1097.
[5] W. Gawronski, J.N. Juang, Model Reduction for Flexible Structures, in: C.T. Leondes (Ed.), Control and Dynamics Systems, Vol. 36, Academic Press, San Diego, CA, 1990, pp. 143–222.
[6] T. Williams, Closed form grammians and model reduction for flexible space structures, IEEE Transactions on Automatic Control AC-35 (1990) 379–382.

[7] K.B. Lim, W. Gawronski, Hankel singular values of flexible structures in discrete time, AIAA Journal of Guidance, Control, and Dynamics 19 (6) (1996) 1370–1377.
[8] R.W. Clough, J. Penzien, Dynamics of Structures, McGraw-Hill, New York, 1975.
[9] M.R. Hatch, Vibration Simulation Using Matlab and ANSYS, Chapman & Hall/CRC Press, London/Boca Raton, Fl, 2000.
[10] G.F. Franklin, J.D. Powell, M.L. Workman, Digital Control of Dynamic Systems, Addison-Wesley, Reading, MA, 1992.
[11] B.C. Moore, Principal component analysis in linear systems, controllability, observability and model reduction, IEEE Transactions on Automatic Control 26 (1981) 17–32.
[12] S.P. Boyd, C.H. Barratt, Linear Controller Design, Prentice Hall, Englewood Cliffs, NJ, 1991.
[13] T. Chen, B. Francis, Optimal Sampled-Data Control Systems, Springer, London, 1995.
[14] R.E. Skelton, Cost decomposition of linear systems with application to model reduction, International Journal of Control 32 (1980) 1031–1055.
[15] S. Skogestad, I. Postlethwaite, Multivariable Feedback Control, Wiley, Chichester, England, 1996.