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Theoretical analysis of sound radiation of an elastically supported circular plate

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Abstract

This study focuses on the analysis of the active and reactive sound power of the axisymmetric modes of free vibrations of elastically supported circular plates embedded in a rigid baffle. Some linear and time-harmonic processes have been considered. It is assumed that the plate radiates some acoustic waves into a hemisphere filled with a lossless gaseous medium. The integral formulations for the active and reactive sound power have been derived and expressed in their Hankel representations. They have been used to derive some elementary formulations in the form of some high-frequency asymptotes valid for frequencies higher than the successive coincidence frequencies of the plate. Therefore, the discussion on some sample numerical results mostly covers the sound power radiated at those frequencies. The asymptotes are easy to express in a computer code and they do not need great processor capacity. They are therefore useful for engineering use.

The main benefit of the analysis presented in this paper is that the sound power for all the possible boundary configurations of the boundary stiffnesses, i.e., classical clamped, guided, simply supported or completely free boundaries as well as all the intermediate situations, has been described using the same formulae. This is possible simply by changing the two values of stiffnesses associated with the boundary conditions, whose influence on the radiated sound power has been discussed. The solution of the problem of sound power radiated by a vibrating elastically supported circular plate presented herein is essentially more general than the solutions presented earlier for the classical boundary configurations, such as clamped, simply supported, guided or completely free circular plates.

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1. Introduction

A theoretical analysis of the energetic aspect of sound radiation very seldom leads to results in the form of some elementary formulations. Obtaining some exact analytical expressions valid for some planar sound sources seems to be very difficult or impossible. Therefore, there are rather few studies dealing with the theoretical analysis of some approximation methods to investigate the problem of sound radiation. However, the methods are convenient for engineering computations, because the approximated formulations obtained are easy to express in a computer code and they do not too much processor capacity.

Some specific techniques of a closed path integral were used to derive some high-frequency asymptotes for the sound radiation power of a clamped circular plate [1,2], clamped annular plate [3] or simply supported circular plate [4].

The Rayleigh–Ritz method was applied for the sound radiation analysis of a stationary or rotating computer disk [5]. The author investigated the influence of the specific clamped–free boundaries of the disk on the efficiency of its sound radiation with the multimodal excitation of the disk. The analogy method was used for theoretical considerations on the sound radiated by an elastically supported rectangular plate by Berry et al. [6]. The well-known idea of modelling any intermediate boundary configurations of a plate by some elastic forces counteracting any deflections and rotations of the plate's edge was applied by the authors to analyze the influence of several boundary configurations on the sound radiation efficiency of the plate.

Czarnecki et al. [7] proposed a method of equivalent area of a circular plate to estimate its sound power radiated, and also presented some analytical formulae convenient for engineering computations. Their theoretical results show a good agreement with those based on measurements.

The authors of this paper have not come across any theoretical analysis of magnitudes useful to describe the sound radiation of an elastically supported circular plate. The influence of the boundary conditions of the plate on the sound field generated is still unknown from the theoretical viewpoint. This paper is therefore aimed at filling this literature gap by deriving and presenting the integral formulations expressed in their Hankel representations useful for both numerical and analytical computations. On this basis some efficient high-frequency asymptotes formulae for the radiated and lost sound power of an elastically supported circular plate have been derived. The influence of the elastic stiffnesses associated with the boundary forces on the sound field generated has also been discussed.

2. Free vibrations

A thin circular plate is elastically supported by a rigid baffle (cf., Fig. 1). The plate's radius, density and thickness are a, ϱ and h, respectively, and the amplitude of the plate's transverse deflection for some time-harmonic and axisymmetric processes is given by $\eta(r,t)=\eta(r)\exp(-\mathrm{i}\omega t)$ and the nth mode shape can be expressed in the form of

$$\eta_n(r) = A_n\,[\mathrm{J}_0(k_n r) - B_n \mathrm{I}_0(k_n r)], \tag{1}$$

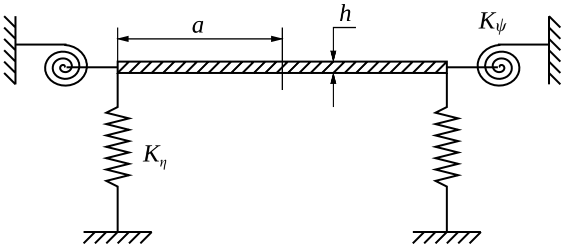

Fig. 1. The boundary configuration of an elastically supported circular plate (cf., Refs. [8–10]).

where $r \in [0, a]$ is the distance from the plate's point to its center, $n = 0, 1, 2, \ldots$ is the mode number, ω_n is nth eigenfrequency, $k_n^4 = \omega_n^2 \varrho h / B$ is the nth structural wavenumber raised to 4th power, $B = Eh^3/[12(1-\nu^2)]$ is the plate's bending stiffness, E is Young's modulus, ν is the Poisson ratio, J_n and I_n are the nth order Bessel and modified Bessel functions, respectively.

The boundary conditions at the plate's edge are (cf., Refs. [8–10])

$$M(a) = K_\psi \left. \frac{\mathrm{d}\eta(r)}{\mathrm{d}r} \right|_{r=a}, \quad V(a) = -K_\eta \eta(a), \tag{2}$$

where K_η [N/m^2] is the stiffness constant associated with the force counteracting the transverse deflection of the plate's boundary and K_ψ [N] is the stiffness constant associated with the rotary moment of the plate's boundary. The plate's material satisfies the following relations:

$$M(r) = -B\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\eta(r), \quad V(r) = -B\frac{\mathrm{d}}{\mathrm{d}r}\nabla_r^2 \eta(r), \quad r \in [0, a]. \tag{3}$$

Inserting mode shape (1) into boundary conditions (2) and using relations (3) leads to constant B_n representing also the plate's frequency equation

$$B_n = -\frac{\mathrm{J}_0(x_n) + p\mathrm{J}_1(x_n)}{\mathrm{I}_0(x_n) + p\mathrm{I}_1(x_n)} = \frac{\mathrm{J}_1(x_n) - q\mathrm{J}_0(x_n)}{\mathrm{I}_1(x_n) - q\mathrm{I}_0(x_n)}, \tag{4}$$

where $x_n = k_n a$ is nth eigenvalue of the plate,

$$x_n^3 q = K_\eta a^3 / B, \quad x_n p = (K_\psi a / B) - (1 - \nu). \tag{5}$$

The frequency equation (4) can also be presented in the form:

$$\left(1 - \frac{1}{pq}\right)(\alpha_n + \beta_n) = 2\left(\frac{\alpha_n \beta_n}{q} - \frac{1}{p}\right) \tag{6}$$

with the following notation introduced:

$$\alpha_n = \frac{\mathrm{J}_1(x_n)}{\mathrm{J}_0(x_n)}, \quad \beta_n = \frac{\mathrm{I}_1(x_n)}{\mathrm{I}_0(x_n)}. \tag{7}$$

The value of constant A_n has been derived by means of the standardization condition $(1/S)\int_S [\eta_n(\vec{r})/A_n]^2 \, \mathrm{d}S = (2/a^2)\int_0^a [\eta_n(r)/A_n]^2 r \, \mathrm{d}r = 1$, and its value is given by

$$A_n^{-2} = \mathrm{J}_1^2(x_n) + \mathrm{J}_0^2(x_n) + B_n^2 [\mathrm{I}_0^2(x_n) - \mathrm{I}_1^2(x_n)] - 2B_n S(x_n)/x_n, \tag{8}$$

where $S(x_n) = \mathrm{J}_1(x_n)\mathrm{I}_0(x_n) + \mathrm{J}_0(x_n)\mathrm{I}_1(x_n)$.

3. The sound power in the integral form

Some time-harmonic vibrations of the plate are the source of acoustic waves radiated into the half-space $z \geqslant 0$ filled with the lossless gaseous medium of rest density ϱ_0 and sound velocity c. The sound radiation power of the nth mode of the plate can be expressed in the Hankel form (cf., Fig. 2 and Refs. [4,10,11])

$$\Pi_n = \pi\varrho_0 ck^2 \int_0^\infty W_n(x)W_n^*(x)\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{9}$$

where $k = 2\pi/\lambda$ is the radiated wavenumber, λ is the radiated wavelength, and

$$W_n(x) = \mathrm{i}\omega_n \int_0^a \eta_n(r)\mathrm{J}_0(krx)r\,\mathrm{d}r \tag{10}$$

is the characteristic function of sound radiation of the nth in-vacuo mode of the plate.

The nth reference sound power $\Pi_n^{(\infty)} = \lim_{k\to\infty}\Pi_n = \pi\varrho_0 c\int_0^a \omega_n^2\eta_n^2(r)r\,\mathrm{d}r$ represents the active sound power for the infinite value of the acoustic wavenumber k. Its choice is motivated by the well-known fact that the active sound power with growing k tends to a non-zero value, while the reactive sound power reaches zero. Eq. (8) makes it possible to simplify the reference sound power and present its elementary form as

$$\Pi_n^{(\infty)} = (\pi a^2/2)\varrho_0 c\omega_n^2. \tag{11}$$

Integral (10) has been computed for the mode shape (1) and then inserted into integral (9), which results in the standardized sound power in the form of integral

$$\mathscr{P}_n = 4\delta_n^4 q_n \int_0^\infty \left[\frac{\alpha_n\delta_n\mathrm{J}_0(kax) - x\mathrm{J}_1(kax)}{(x^4-\delta_n^4)} + \frac{1}{2\delta_n^2}\frac{\gamma_n\delta_n\mathrm{J}_0(kax) - b_n x\mathrm{J}_1(kax)}{(x^2+\delta_n^2)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{12}$$

where the following notations are introduced $\mathscr{P}_n = \mathscr{P}_{a,n} - \mathrm{i}\mathscr{P}_{r,n} = \Pi_n/\Pi_n^{(\infty)}$. The standardized active and reactive sound power are denoted by $\mathscr{P}_{a,n}$ and $\mathscr{P}_{r,n}$, respectively, $q_n = 2A_n^2\mathrm{J}_0^2(x_n)$, and

$$\gamma_n = \alpha_n + B_n\mathrm{I}_1(x_n)/\mathrm{J}_0(x_n), \quad b_n = 1 - B_n\mathrm{I}_0(x_n)/\mathrm{J}_0(x_n). \tag{13}$$

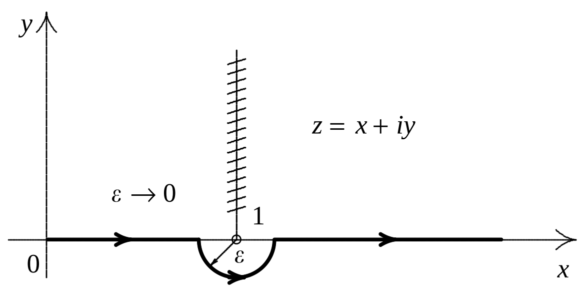

Fig. 2. The integration path used in Eqs. (9), (12), (15), (26), (28) and (32) in a plane of complex variables z (cf., Refs. [4,10,11]).

The integral within the limits $(0, \infty)$ in Eq. (12) should be understood as the Cauchy principal value (cf., Fig. 2 and Refs. [4,11]). The integral within the limits $(0, 1)$ represents the active sound power, whereas, in the case of the reactive sound power the integration is performed within the limits $(1, \infty)$ and the following transformation is used $\sqrt{1-x^2} \Rightarrow \mathrm{i}\sqrt{x^2-1}$. The form of integral (12) is useful to analyze the sound radiation power for some other boundary conditions of a circular plate being clamped, simply supported and completely free.

4. Asymptotic formulae

An asymptotic method of computing an integral in the plane of complex variable has been used to transform the integral formulae expressed in their Hankel representations for the sound power to the formulae in their elementary form. This method was applied earlier to derive the active and reactive sound power of a clamped or simply supported circular plate (cf., Refs. [1,2,4]). However, the method was not applied to deal with the radiated and lost sound power of an elastically supported circular plate. All the analytical computation presented below is based on the exact solution of the plate's motion given by Leissa et al. [8,9].

4.1. The active sound power

The standardized active sound power is computed from Eq. (12) within the limits $(0, 1)$. It is then convenient to express this integral by the sum of three partial integrals

$$\mathscr{P}_{a,n} = 4\delta_n^4 q_n [\mathscr{P}_{a,n}^{(1)} + \mathscr{P}_{a,n}^{(2)} + \mathscr{P}_{a,n}^{(3)}], \tag{14}$$

where

$$\mathscr{P}_{a,n}^{(1)} = \int_0^1 \left[\frac{\alpha_n \delta_n \mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{(x^4 - \delta_n^4)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{15a}$$

$$\mathscr{P}_{a,n}^{(2)} = \frac{1}{\delta_n^2} \int_0^1 \frac{\alpha_n \delta_n \mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{(x^4 - \delta_n^4)} \frac{\gamma_n \delta_n \mathrm{J}_0(\beta x) - b_n x\mathrm{J}_1(\beta x)}{(x^2 + \delta_n^2)} \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{15b}$$

$$\mathscr{P}_{a,n}^{(3)} = \frac{1}{4\delta_n^4} \int_0^1 \left[\frac{\gamma_n \delta_n \mathrm{J}_0(\beta x) - b_n x\mathrm{J}_1(\beta x)}{(x^2 + \delta_n^2)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}. \tag{15c}$$

The method, given by Levine and Leppington (cf., Refs. [1,2,4]), is further used to compute the asymptotic values of integrals (15) satisfying the set of conditions $ka > 1$ and $\delta_n = k_n/k = x_n/\beta < 1$. Each of the integrals has been expressed by the appropriate path integral in the plane of complex variable z and the Cauchy theorem on residues has been used to deal with them (cf., Appendix A and Fig. 3). The non-oscillating $\bar{\mathscr{P}}_{a,n}$ and oscillating $\tilde{\mathscr{P}}_{a,n}$ parts have been separated in the integrals with a new notation introduced $\mathscr{P}_{a,n}^{(\mu)} = \bar{\mathscr{P}}_{a,n}^{(\mu)} + \tilde{\mathscr{P}}_{a,n}^{(\mu)}$. Therefore, the non-oscillating active sound power can be written as

$$\bar{\mathscr{P}}_{a,n} = 4\delta_n^4 q_n [\bar{\mathscr{P}}_{a,n}^{(1)} + \bar{\mathscr{P}}_{a,n}^{(2)} + \bar{\mathscr{P}}_{a,n}^{(3)}], \tag{16}$$

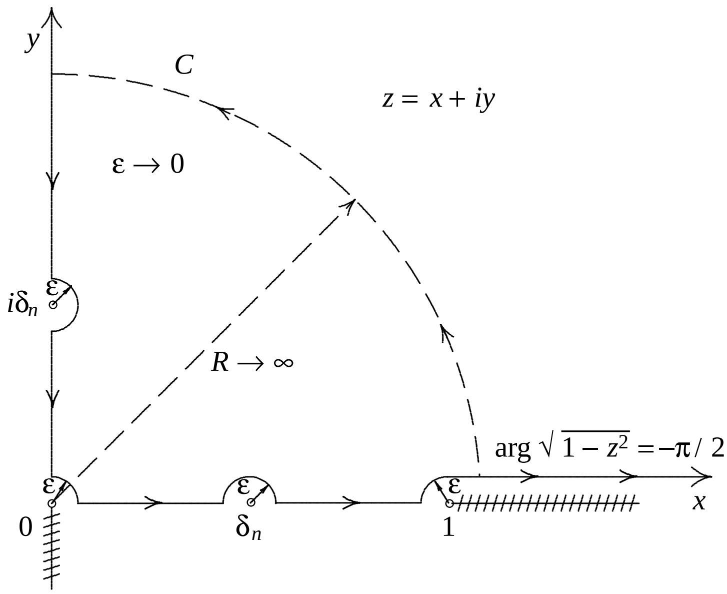

Fig. 3. The integration path C used in Eqs. (A.4) and (A.5) to derive the asymptotic formula for the active sound power (cf., Refs. [1,3,4]).

where magnitudes $\bar{\mathscr{P}}_{a,n}^{(\mu)}$ are equal to the sum of residues in the singular points of integrands in Eq. (A.4) (cf., Appendix A). Summing up all the residues (cf., Eq. (A.11)) by Eq. (16) produces

$$\begin{aligned}\bar{\mathscr{P}}_{a,n} &= 4\delta_n^4 q_n \,\mathrm{Re}\{\pi\mathrm{i}\,[\mathscr{F}_1^{(1)}(\delta_n) + \mathscr{F}_1^{(2)}(\mathrm{i}\delta_n) + \mathscr{F}_2^{(1)}(\delta_n) + \mathscr{F}_2^{(2)}(\mathrm{i}\delta_n) + \mathscr{F}_3'(\mathrm{i}\delta_n)]\} \\ &= \left(\frac{u_n}{\sqrt{1-\delta_n^2}} + \frac{w_n}{\sqrt{1+\delta_n^2}}\right)\Bigg/(u_n + w_n),\end{aligned} \tag{17}$$

where

$$u_n = 1 + \alpha_n^2 - \frac{(\alpha_n + \beta_n)}{x_n}(1 - b_n), \tag{18a}$$

$$w_n = (1 - b_n)\left[(1 - \beta_n^2)(1 - b_n) - \frac{(\alpha_n + \beta_n)}{x_n}\right], \tag{18b}$$

$$q_n = 2/(u_n + w_n). \tag{18c}$$

Given that $w_n/(u_n + w_n) = 1 - q_n u_n/2$, the elementary form of Eq. (17) can also be expressed by

$$\bar{\mathscr{P}}_{a,n} = \frac{1}{\sqrt{1+\delta_n^2}} + \frac{q_n u_n}{2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right), \tag{19}$$

which represents the non-oscillating part of the standardized active sound power of the nth axisymmetric mode of an elastically supported circular plate.

The oscillating part of the active sound power is expressed by (cf., Eq. (A.13)

$$\tilde{\mathscr{P}}_{a,n} = 4\delta_n^4 q_n \,[\tilde{\mathscr{P}}_{a,n}^{(1)} + \tilde{\mathscr{P}}_{a,n}^{(2)} + \tilde{\mathscr{P}}_{a,n}^{(3)}]. \tag{20}$$

The integration is carried out using the stationary phase method (cf., Eq. (A.14) leading to the elementary expression for the oscillating part of the standardized active sound power, i.e.,

$$\tilde{\mathscr{P}}_{a,n} = X_n \cos(2\beta + \pi/4) + Y_n \sin(2\beta + \pi/4) \tag{21}$$

with the following notations:

$$d_n = \frac{1}{(1-\delta_n^2)} + \frac{b_n}{2\delta_n^2}, \quad h_n = \frac{\alpha_n \delta_n}{(1-\delta_n^2)} + \frac{\gamma_n}{2\delta_n}, \tag{22}$$

$$X_n = \frac{2q_n}{\beta\sqrt{\pi\beta}}\left(\frac{\delta_n^2}{1+\delta_n^2}\right)^2 (d_n^2 - h_n^2), \quad Y_n = \frac{2q_n}{\beta\sqrt{\pi\beta}}\left(\frac{\delta_n^2}{1+\delta_n^2}\right)^2 (2d_n h_n). \tag{23}$$

Finally, the sum of the oscillating and non-oscillating parts leads to the elementary expression for the standardized active sound power of the nth axisymmetric mode of an elastically supported circular plate

$$\mathscr{P}_{a,n} = \bar{\mathscr{P}}_{a,n} + \tilde{\mathscr{P}}_{a,n} + \mathscr{O}(\delta_n^4\,\beta^{-3/2}), \tag{24}$$

where term $\mathscr{O}(\delta_n^4\,\beta^{-3/2})$ represents the approximation error.

4.2. The reactive sound power

Eq. (12) is the basis for computing the asymptotic value of the reactive sound radiation power. The integral computed within the limits $(1, \infty)$ has been expressed as a sum of three partial integrals

$$\mathscr{P}_{r,n} = 4\delta_n^4 q_n [\mathscr{P}_{r,n}^{(1)} + \mathscr{P}_{r,n}^{(2)} + \mathscr{P}_{r,n}^{(3)}], \tag{25}$$

where

$$\mathscr{P}_{r,n}^{(1)} = \int_1^\infty \left[\frac{\alpha_n \delta_n \mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{(x^4 - \delta_n^4)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}, \tag{26a}$$

$$\mathscr{P}_{r,n}^{(2)} = \frac{1}{\delta_n^2}\int_1^\infty \frac{\alpha_n \delta_n \mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{(x^4 - \delta_n^4)} \frac{\gamma_n \delta_n \mathrm{J}_0(\beta x) - b_n x\mathrm{J}_1(\beta x)}{(x^2 + \delta_n^2)} \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}, \tag{26b}$$

$$\mathscr{P}_{r,n}^{(3)} = \frac{1}{4\delta_n^4}\int_1^\infty \left[\frac{\gamma_n \delta_n \mathrm{J}_0(\beta x) - b_n x\mathrm{J}_1(\beta x)}{(x^2 + \delta_n^2)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}. \tag{26c}$$

The products of Bessel functions for $\beta = ka > 10$ have been expressed by the asymptotic formulae

$$\mathrm{J}_0^2(u) \sim (\pi u)^{-1}\,(1 + \sin 2u), \tag{27a}$$

$$\mathrm{J}_1^2(u) \sim (\pi u)^{-1}\,(1 - \sin 2u), \tag{27b}$$

$$\mathrm{J}_0(u)\mathrm{J}_1(u) \sim -(\pi u)^{-1} \cos 2u. \tag{27c}$$

The non-oscillating $\bar{\mathscr{P}}_{r,n}$ and oscillating $\tilde{\mathscr{P}}_{r,n}$ parts have been separated in integrals (26) with the following notation $\mathscr{P}_{r,n}^{(\mu)} = \bar{\mathscr{P}}_{r,n}^{(\mu)} + \tilde{\mathscr{P}}_{r,n}^{(\mu)}$. The non-oscillating parts are

$$\bar{\mathscr{P}}_{r,n}^{(1)} = \frac{1}{\pi\beta}\int_1^\infty \frac{(\alpha_n^2\delta_n^2 + x^2)}{(x^4 - \delta_n^4)^2}\frac{\mathrm{d}x}{\sqrt{x^2-1}}, \tag{28a}$$

$$\bar{\mathscr{P}}_{r,n}^{(2)} = \frac{1}{\pi\beta\delta_n^2}\int_1^\infty \frac{(\alpha_n\gamma_n\delta_n^2 + b_n x^2)}{(x^4 - \delta_n^4)(x^2+\delta_n^2)}\frac{\mathrm{d}x}{\sqrt{x^2-1}}, \tag{28b}$$

$$\bar{\mathscr{P}}_{r,n}^{(3)} = \frac{1}{\pi\beta\delta_n^4}\int_1^\infty \frac{(\gamma_n^2\delta_n^2 + b_n^2 x^2)}{(x^2 + \delta_n^2)^2}\frac{\mathrm{d}x}{\sqrt{x^2-1}}. \tag{28c}$$

Some elementary computations lead to

$$\bar{\mathscr{P}}_{r,n} = \frac{q_n}{\pi\beta}\left[\frac{\hat{a}_n^{(1)}}{(1+\delta_n^2)} + \frac{\hat{a}_n^{(2)}\arcsin\delta_n}{2\delta_n(1-\delta_n^2)^{3/2}} + \frac{\hat{a}_n^{(3)}\,\mathrm{arcsinh}\,\delta_n}{2\delta_n(1+\delta_n^2)^{3/2}}\right], \tag{29}$$

where

$$\hat{a}_n^{(1)} = \frac{(1+\alpha_n^2\delta_n^2)}{(1-\delta_n^2)} - \frac{1}{2}[1 - \alpha_n^2 - (1-\beta_n^2)(1-b_n)^2], \tag{30a}$$

$$\hat{a}_n^{(2)} = 2(1-\delta_n^2)[1 + \alpha_n^2 - (1-\alpha_n\beta_n)(1-b_n)] - [1 - 2\delta_n^2 + (3-4\delta_n^2)\alpha_n^2], \tag{30b}$$

$$\begin{aligned}\hat{a}_n^{(3)} = {} & [\alpha_n + \beta_n(1-b_n)]\{-2\alpha_n(2+3\delta_n^2) + (1+2\delta_n^2)[\alpha_n + \beta_n(1-b_n)]\} \\ & - (1+2\delta_n^2) + (3+4\delta_n^2)\alpha_n^2 + b_n(2\delta_n^2 + b_n).\end{aligned} \tag{30c}$$

Eq. (29) represents the asymptotic value of the non-oscillating part of the reactive sound power of an elastically supported circular plate for its nth axisymmetric mode. The equation is valid if the condition $\delta_n = k_n/k = x_n/\beta < 1$ is satisfied, i.e., for the acoustically fast waves.

The oscillating part of the standardized reactive sound power is

$$\tilde{\mathscr{P}}_{r,n} = 4\delta_n^4 q_n[\tilde{\mathscr{P}}_{r,n}^{(1)} + \tilde{\mathscr{P}}_{r,n}^{(2)} + \tilde{\mathscr{P}}_{r,n}^{(3)}]. \tag{31}$$

Expressions $\tilde{\mathscr{P}}_{r,n}^{(\mu)}$ have been separated from Eq. (26) by considering the oscillating terms of the asymptotic approximations of the products of the Bessel functions only. As a result the following integrals have been obtained:

$$\tilde{\mathscr{P}}_{r,n}^{(1)} = \frac{1}{\pi\beta}\int_1^\infty \frac{-(x^2 - \alpha_n^2\delta_n^2)\sin 2\beta x + 2\alpha_n\delta_n x\cos 2\beta x}{(x^4 - \delta_n^4)^2}\frac{\mathrm{d}x}{\sqrt{x^2-1}}, \tag{32a}$$

$$\tilde{\mathscr{P}}^{(2)}_{r,n} = \frac{1}{\pi\beta\delta_n^2}\int_1^\infty \frac{-(b_n x^2 - \alpha_n\gamma_n\delta_n^2)\sin 2\beta x + (\gamma_n + \alpha_n b_n)x\delta_n \cos 2\beta x}{(x^4 - \delta_n^4)(x^2 + \delta_n^2)}\frac{\mathrm{d}x}{\sqrt{x^2-1}}, \tag{32b}$$

$$\tilde{\mathscr{P}}^{(3)}_{r,n} = \frac{1}{4\pi\beta\delta_n^4}\int_1^\infty \frac{-(b_n^2 x^2 - \gamma_n^2\delta_n^2)\sin 2\beta x + 2\gamma_n b_n \delta_n x \cos 2\beta x}{(x^2 + \delta_n^2)^2}\frac{\mathrm{d}x}{\sqrt{x^2-1}}. \tag{32c}$$

The integrals appear while computing the active sound power (cf., Eqs. (A.13) and (A.14)). If summed up by Eq. (31) they give

$$\tilde{\mathscr{P}}_{r,n} = Y_n \cos(2\beta + \pi/4) - X_n \sin(2\beta + \pi/4) \tag{33}$$

together with notation (23).

The sum of the oscillating and non-oscillating parts

$$\mathscr{P}_{r,n} = \bar{\mathscr{P}}_{r,n} + \tilde{\mathscr{P}}_{r,n} + \mathscr{O}(\delta_n^4 \beta^{-3/2}) \tag{34}$$

represents an elementary expression for the standardized reactive sound radiation power of an elastically supported circular plate for its nth axisymmetric mode of the free vibrations.

It is worth noting that the same amplitudes X_n, Y_n are used in Eqs. (21) and (33) representing the oscillating parts of the active and reactive sound power, respectively.

5. Specific cases

Some classical boundary configurations occur, when the boundary stiffnesses of the vibrating plate reach their specific values of zero or infinity. The sound radiation behavior of the plate was analyzed and reported in the literature for some of the classical configurations (cf., Refs. [1,2] or [4] for clamped or simply supported circular plates, respectively). The asymptotic and integral formulae presented in this paper can be used to describe the sound radiation of some free vibrations of an elastically supported circular plate for any values of the boundary stiffnesses K_η and K_ψ within the range of $[0, +\infty]$. The formulae are still valid and useful for any specific values of the boundary stiffnesses analyzed earlier.

The numerical analysis performed herein leads to the question if the agreement seen in the numerical results for the specific boundary configurations with the literature data implies the analogy agreement between the corresponding asymptotic and integral formulations for the sound power. In other words, the question is if some appropriate limiting transitions to carry out make it possible to transform the formulations presented in this paper to the corresponding formulations reported in the literature for a clamped or simply supported circular plate.

The limiting case, when $K_\eta = \infty$, implies that the plate's edge cannot make any transverse deflections, but still can make rotations all around the edge if $0 \leqslant K_\psi \ll \infty$. In this specific case the following relations are valid $b_n = 0$ and $\alpha_n + \beta_n = -2/p$.

5.1. Clamped plates

The integral for the standardized complex sound power (12) gets reduced to

$$\mathscr{P}_n = 4\delta_n^4 \int_0^\infty \left[\frac{\alpha_n\delta_n \mathrm{J}_0(kax) - x\mathrm{J}_1(kax)}{(x^4-\delta_n^4)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{35}$$

when both boundary stiffnesses reach their specific values of infinity, i.e., integral (35) is valid for a clamped circular plate. For the acoustically fast waves, i.e., when $\beta = ka > 1$ and $\delta_n = x_n/\beta = k_n/k < 1$, Eqs. (18), (22) and (30) get transformed to

$$u_n = 1+\alpha_n^2, \quad w_n = 1-\alpha_n^2, \quad q_n = 2/(u_n+w_n) = 1, \tag{36a}$$

$$d_n = 1/(1-\delta_n^2), \quad h_n = \alpha_n\delta_n d_n, \tag{36b}$$

$$\hat{a}_n^{(1)} = \frac{(1+\alpha_n^2\delta_n^2)}{(1-\delta_n^2)}, \quad \hat{a}_n^{(2)} = -(1-2\delta_n^2)-(3-4\delta_n^2)\alpha_n^2, \tag{36c}$$

$$\hat{a}_n^{(3)} = -(1+2\delta_n^2)+(3+4\delta_n^2)\alpha_n^2. \tag{36d}$$

The frequency equation of the plate and all the remaining magnitudes, necessary to define the high-frequency asymptotics for the radiated and lost sound power obtained by some appropriate

Table 1
Some sample magnitudes appearing in the asymptotics representing the active and reactive sound power of an elastically supported circular plate for the specific cases under discussion

Plate's edge	$K_\psi,\ p$	$K_\eta,\ q$	γ_n	b_n	d_n	h_n
Clamped	∞	∞	0	0	$\frac{1}{1-\delta_n^2}$	$\alpha_n\delta_n d_n$
Simply supported	$0, -\frac{1-v}{x_n}$	∞	$\frac{2x_n}{1-v}$	0	$\frac{1}{1-\delta_n^2}$	$\alpha_n\delta_n d_n + \frac{\beta}{1-v}$
Completely free	$0, -\frac{1-v}{x_n}$	0	$2\alpha_n$	$1-\frac{\alpha_n}{\beta_n}$	$\frac{1}{1-\delta_n^2} + \frac{1-\alpha_n(1-v)/x_n}{\delta_n^2}$	$\frac{\alpha_n}{\delta_n(1-\delta_n^2)}$
Guided	∞	0	0	1	$\frac{1}{1-\delta_n^2} + \frac{1}{2\delta_n^2}$	0

Plate's edge	u_n	w_n	Frequency equation
Clamped	$1+\alpha_n^2$	$1-\alpha_n^2$	$\alpha_n+\beta_n=0$
Simply supported	$1+\alpha_n^2-\frac{2}{1-v}$	$1-\beta_n^2-\frac{2}{1-v}$	$\alpha_n+\beta_n=\frac{2x_n}{1-v}$
Completely free	$1+\alpha_n^2\left(1-2\frac{1-v}{x_n^2}\right)$	$1-\alpha_n^2-4\alpha_n\frac{1-v}{x_n}$ $\left[1-\frac{\alpha_n}{x_n}(\frac{1}{2}-v)\right]$	$\frac{1}{\alpha_n}+\frac{1}{\beta_n}=2\frac{1-v}{x_n}$
Guided	1	0	$\alpha_n=0$

limiting transitions, are gathered in Table 1. The asymptotes for the active and reactive sound power together with the corresponding approximation errors are

$$\mathscr{P}_{a,n} = \frac{(1+\alpha_n^2)}{2\sqrt{1-\delta_n^2}} + \frac{(1-\alpha_n^2)}{2\sqrt{1+\delta_n^2}} + \frac{2\delta_n^4}{\beta\sqrt{\pi\beta}(1-\delta_n^4)^2} \times [(1-\alpha_n^2\delta_n^2)\cos(2\beta+\pi/4) + 2\alpha_n\delta_n\sin(2\beta+\pi/4)] + \mathscr{O}(\delta_n^4\beta^{-3/2}), \tag{37}$$

and

$$\mathscr{P}_{r,n} = \frac{1}{\pi\beta}\left\{\frac{(1+\alpha_n^2\delta_n^2)}{(1-\delta_n^4)} - [1-2\delta_n^2+(3-4\delta_n^2)\alpha_n^2]\frac{\arcsin\delta_n}{2\delta_n(1-\delta_n^2)^{3/2}} - [1+2\delta_n^2-(3+4\delta_n^2)\,\alpha_n^2]\frac{\operatorname{arcsinh}\delta_n}{2\delta_n(1+\delta_n^2)^{3/2}}\right\} + \frac{2\delta_n^4}{\beta\sqrt{\pi\beta}(1-\delta_n^4)^2}[(\alpha_n^2\delta_n^2-1)\sin(2\beta+\pi/4) + 2\alpha_n\delta_n\cos(2\beta+\pi/4)] + \mathscr{O}(\delta_n^4\beta^{-3/2}) \tag{38}$$

and they were reported earlier in Refs. [1,2,4], respectively.

5.2. Simply supported plates

In the limiting case, i.e., when $K_\eta = \infty$, the plate's edge cannot make any displacements. If, additionally, the stiffness constant $K_\psi = 0$, then the edge can make rotations counteracting no resisting forces. This is the configuration of a simply supported circular plate and the corresponding high-frequency asymptotics for the sound power are known from the literature (cf., Ref. [4]). The value of constant B_n is determined by Eq. (13) and $b_n = 0$. When the dimensionless parameter $p = -(1-v)/x_n$ the frequency equation becomes

$$\alpha_n + \beta_n = \frac{2x_n}{(1-v)}. \tag{39}$$

Constant $q_n = 2A_n^2\mathrm{J}_0^2(x_n)$ can be simplified by applying Eqs. (8) and (39) giving

$$q_n = \frac{(1-v)}{[2x_n\alpha_n - (1+v+\frac{2x_n^2}{1-v})]}, \tag{40}$$

which agrees with that presented in Ref. [4]. Constants (13) get transformed to $\gamma_n = \alpha_n + \beta_n$, $b_n = 0$ and the following integral formulation is reached instead of Eq. (12) (cf., Ref. [4])

$$\mathscr{P}_n = 4\delta_n^4 q_n \int_0^\infty \left[\frac{\alpha_n\delta_n\mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{(x^4-\delta_n^4)} + \frac{\beta}{(1-v)}\frac{\mathrm{J}_0(\beta x)}{(x^2+\delta_n^2)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}. \tag{41}$$

The high-frequency asymptotics (18), (22) and (30) get reduced to the following forms:

$$u_n = 1 + \alpha_n^2 - \frac{2}{(1-v)}, \quad w_n = 1 - \beta_n^2 - \frac{2}{(1-v)}, \tag{42a}$$

$$d_n = \frac{1}{(1-\delta_n^2)}, \quad h_n = \frac{\beta}{(1-\nu)} + \alpha_n \delta_n d_n. \tag{42b}$$

$$\hat{a}_n^{(1)} = \frac{(1+\alpha_n^2\delta_n^2)}{(1-\delta_n^2)} + \frac{2x_n}{(1-\nu)}\left(\alpha_n - \frac{x_n}{(1-\nu)}\right), \tag{43a}$$

$$\hat{a}_n^{(2)} = 4x_n(1-\delta_n^2)\frac{\alpha_n}{(1-\nu)} - (1-2\delta_n^2) - (3-4\delta_n^2)\alpha_n^2, \tag{43b}$$

$$\hat{a}_n^{(3)} = 2x_n\left\{\frac{2}{(1-\nu)}\left[-(2+3\delta_n^2)\alpha_n + (1+2\delta_n^2)\frac{x_n}{(1-\nu)}\right] - \frac{1+2\delta_n^2-(3+4\delta_n^2)\alpha_n^2}{2x_n}\right\}, \tag{43c}$$

and the magnitudes X_n and Y_n are still defined by Eq. (23) with reduced formulations defining factors d_n and h_n given by Eq. (42b).

All the equations given in this subsection have been obtained from corresponding equations representing the integral formulations and the high-frequency asymptotics for the sound power of an elastically supported circular plate given in this paper. The equations from this subsection together with Eqs. (12), (24) and (34) make the complete equation set necessary to describe the sound power of a simply supported circular plate. Obviously, the equations are not new and they were reported in the literature earlier (cf., Ref. [4]). The literature data agree with those obtained in this subsection, which shows that the literature results can be generalized analytically by the formulations describing the sound power of an elastically supported circular plate given in this paper.

5.3. Completely free plates

Equations describing sound radiation of an elastically supported circular plate strictly determine the analogy equations valid for the limiting case when the plate's edge is completely free. This boundary configuration makes a good approximation of the situation when the constraint forces are weak enough to be neglected in practical use. This is equivalent with the situation, when both stiffness constants K_η and K_ψ reach their specific values of zero. Parameter $p = -(1-\nu)/x_n$ gets reduced to the form identical with that appearing for the simply supported boundaries (cf., Table 1) but assumes different values because their eigenvalues x_n satisfy different frequency equation, i.e.,

$$\frac{1}{\alpha_n} + \frac{1}{\beta_n} = 2\frac{(1-\nu)}{x_n}, \tag{44}$$

which is obtained from Eq. (6) for $K_\eta, K_\psi = 0$. Eqs. (4), (8) and (13) determine the following magnitudes by means of the appropriate limiting transitions (cf., Table 1)

$$\gamma_n = 2\alpha_n, \quad b_n = 1 - \frac{\alpha_n}{\beta_n}, \tag{45a}$$

$$q_n = 2A_n^2 \mathrm{J}_0^2(x_n) = \left[1 - 2(1-\nu)\left(1 + \nu\frac{\alpha_n}{x_n}\right)\frac{\alpha_n}{x_n}\right]^{-1}, \tag{45b}$$

and Eq. (30) gets transformed to

$$\hat{a}_n^{(1)} = \frac{(1+\alpha_n^2\delta_n^2)}{(1-\delta_n^2)} + \frac{1}{2}\left[-1 + \alpha_n\left(\alpha_n - \beta_n + \frac{1}{\beta_n}\right)\right], \tag{46a}$$

$$\hat{a}_n^{(2)} = 1 + \alpha_n^2 + 2(1-\delta_n^2)\left[1 - 2(1-\nu)\frac{\alpha_n}{x_n}\right], \tag{46b}$$

$$\hat{a}_n^{(3)} = -(1+\delta_n^2) - 4(1-\delta_n^2)\alpha_n^2 + (3+4\delta_n^2)\left(1 - \frac{\alpha_n}{\beta_n}\right)\left(1 + 2\delta_n^2 - \frac{\alpha_n}{\beta_n}\right). \tag{46c}$$

Finally, expressions representing the high-frequency asymptotes for the standardized active and reactive sound power are determined by Eqs. (24) and (34) together with values of constants given by Eqs. (45), (46) and magnitudes u_n, w_n, d_n, h_n presented in Table 1. All the results presented in this subsection were not reported in the literature earlier.

5.4. Guided plates

The last specific case to be considered here occurs, when the boundary stiffnesses K_η and K_ψ reach their specific values of zero and infinity, respectively, which determines the boundary configuration of a guided circular plate. The asymptotics together with all the necessary magnitudes can be obtained from the asymptotics presented in this paper by some appropriate limiting transitions. The frequency equation gets reduced to

$$\alpha_n = 0, \tag{47}$$

and the following relations are valid:

$$\beta_n = \frac{\mathrm{I}_1(x_n)}{\mathrm{I}_0(x_n)} > 0, \quad q_n = 2, \quad B_n = 0, \tag{48a}$$

$$X_n = \frac{1}{\beta\sqrt{\pi\beta}}\frac{1}{(1-\delta_n^2)^2}, \quad Y_n = 0, \tag{48b}$$

$$\hat{a}_n^{(1)} = \frac{1}{(1-\delta_n^2)} - \frac{1}{2}, \quad \hat{a}_n^{(2)} = 1, \quad \hat{a}_n^{(3)} = 0. \tag{48c}$$

Finally, the high-frequency asymptotics for the standardized active and reactive sound power of the nth individual mode can be formulated as

$$\mathscr{P}_{a,n} = \frac{1}{\sqrt{1-\delta_n^2}} + \frac{1}{\beta\sqrt{\pi\beta}}\frac{1}{(1-\delta_n^2)^2}\cos(2\beta + \pi/4) + \mathcal{O}(\delta_n^4\,\beta^{-3/2}) \tag{49}$$

and

$$\mathscr{P}_{r,n} = \frac{1}{\pi\beta}\frac{1}{(1-\delta_n^2)}\left(1+\frac{\arcsin\delta_n}{\delta_n\sqrt{1-\delta_n^2}}\right) - \frac{1}{\beta\sqrt{\pi\beta}}\frac{1}{(1-\delta_n^2)^2}\sin(2\beta+\pi/4) + \mathcal{O}(\delta_n^4\beta^{-3/2}), \qquad (50)$$

respectively. The results presented in this subsection have not been reported in the literature earlier.

It is worth noting, that the simplicity of the asymptotes achieved in this subsection is analogous to the simplicity of the asymptotes describing the complex sound power of a circular membrane reported in Ref. [12]. However, a membrane is a system described by an equation of motion of a lower order than that for a plate.

Obviously, the idea of modelling any intermediate configurations of the plate's boundaries between the classical ones by the elastic forces counteracting any deflections and rotations of the plate's edge is well known. The influence of the boundary conditions of a rectangular plate on its acoustic field generated was investigated earlier by Berry et al. (cf., Ref. [6]). However, the influence of the boundary conditions on the radiated and lost sound power of a vibrating circular plate was not analyzed in the literature. The formulations presented herein are to fill this literature gap. Moreover, they generalize the magnitudes describing sound radiation of a vibrating circular plate with any classical homogeneous boundary conditions. The sound radiation for some of the classical boundaries were analyzed earlier, i.e., in the cases of the clamped or simply supported circular plates (cf., Refs. [1,2,4]). It has been shown that the formulations for the radiated and lost sound power of completely free or guided circular plates can also be derived from the generalized formulations presented in this paper. To the best of the authors' knowledge, the asymptotes valid for those boundary configurations have not been presented in the literature before.

6. Numerical analysis

One of the most important benefits of the theoretical analysis performed in this study is the possibility to express the sound power radiated by a circular plate for any pair of the boundary stiffness values K_η and K_ψ using the same elementary formulae. The asymptotic expressions make it possible to perform a numerical analysis of the influence of the stiffness constants on the active and reactive sound power radiated for the high frequencies.

Based on the plots prepared, it is easy to notice some general regularities. First of all, the active and reactive sound power of the zero elastic mode shows a strong dependence on constant K_η associated with the force counteracting any transverse deflections of the plate's edge. This strong dependence occurs only within a finite range of the lower values of K_η (cf., Fig. 4). For some higher values of K_η the sound power assumes some constant values. For the mode number $n = 1$ the dependence is still very clear, but for the higher mode numbers the dependence is rather weak (cf., Fig. 4(b)). Some curves representing the sound power of the zero mode for some sample values of the standardized acoustic wavenumber k/k_0 are presented in Fig. 4(a). They show that an increase in the wavenumber determines the extension of the range of the strong dependence of the sound power on K_η. A decrease in the value of K_η determines a considerable decrease in the values assumed by the sound power. Indeed, the plots in Fig. 4 are prepared only for two different

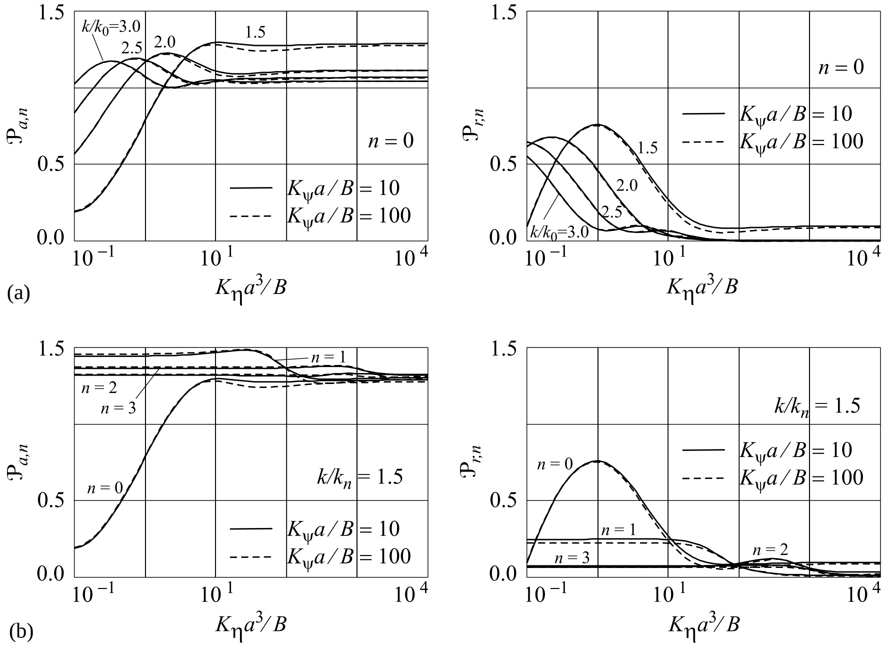

Fig. 4. The active and reactive sound power, $\mathscr{P}_{a,n}$ and $\mathscr{P}_{r,n}$, respectively, plotted in terms of $K_\eta a^3/B$ for some sample values of $K_\psi a/B$, k/k_n and n.

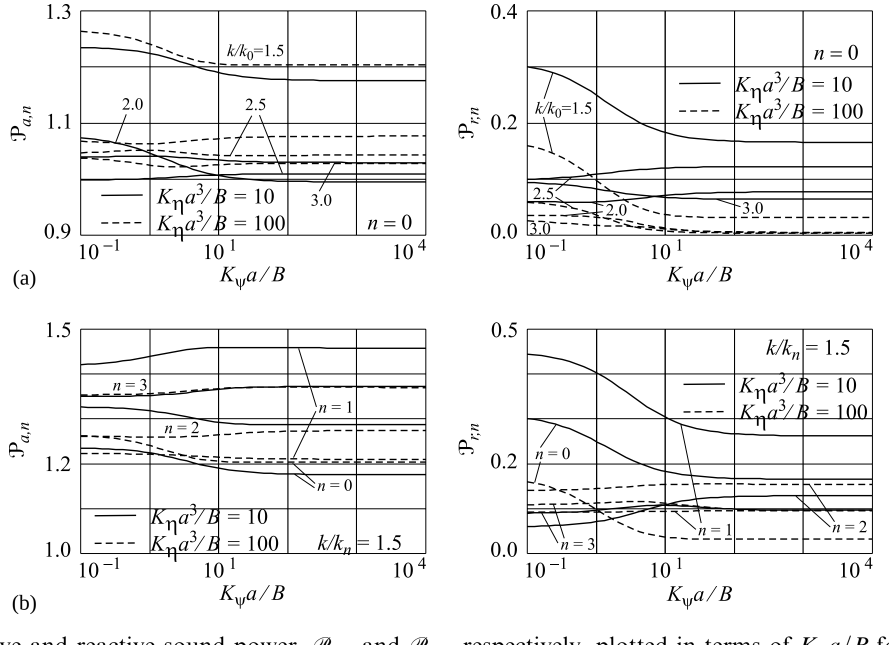

Fig. 5. The active and reactive sound power, $\mathscr{P}_{a,n}$ and $\mathscr{P}_{r,n}$, respectively, plotted in terms of $K_\psi a/B$ for some sample values of $K_\eta a^3/B$, k/k_n and n.

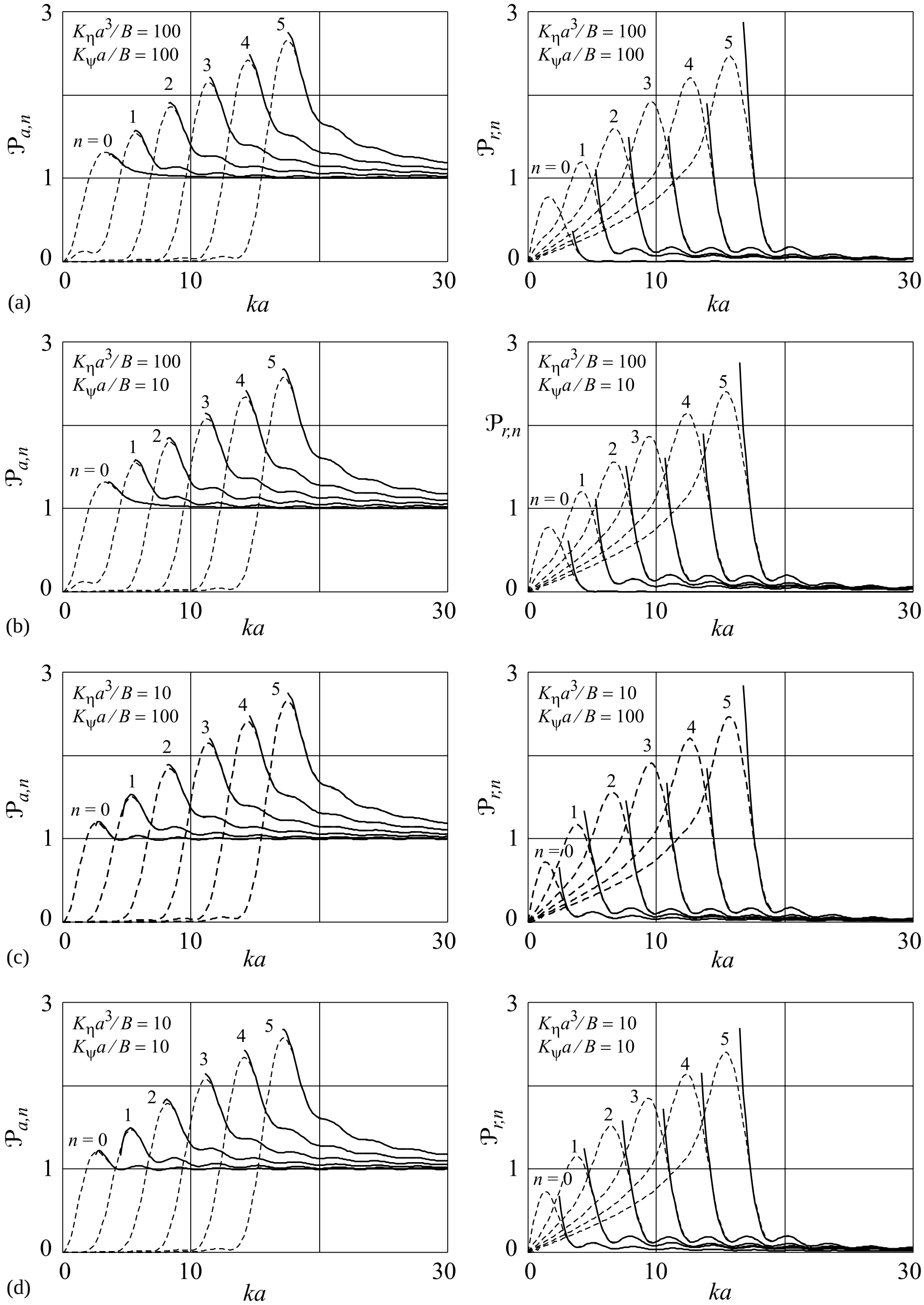

Fig. 6. The active and reactive sound power, $\mathscr{P}_{a,n}$ and $\mathscr{P}_{r,n}$, respectively, plotted in terms of ka for some sample values of the boundary stiffness constants. The solid lines are derived from the asymptotic formulae and the dashed lines from the integral formulae.

Table 2
Some sample eigenvalues x_n of the plate for some sample values of the plate's boundary stiffness K_η and K_ψ standardized by the material stiffness B and radius a

		Mode number n					
$K_\eta a^3/B$	$K_\psi a/B$	0	1	2	3	4	5
100	100	2.9475	4.7054	7.1527	10.1798	13.2859	16.4080
100	10	2.7911	4.7041	7.0181	9.9369	12.9846	16.0690
10	100	2.0559	3.9097	6.9976	10.1305	13.2639	16.3962
10	10	2.0316	3.8076	6.7886	9.8593	12.9492	16.0499

values of the constant K_ψ associated with the force counteracting any rotations of the plate's edge. Nevertheless, it is easy to notice in Fig. 4 that the curves prepared for $K_\psi a/B = 10$ or 100 almost reproduce each other for any value assumed by K_η. Thus, the dependence of the sound power on K_ψ seems to be rather weak. This fact can be easily noticed in Fig. 5, where the dependence of the sound power on K_ψ is illustrated. It is clearly visible that the dependence is much weaker in the case of constant K_ψ than in the case of constant K_η. The dependence is most clearly seen for mode zero and for mode one. An increase in the values of quotient k/k_n implies a weaker dependence of the sound power on the boundary stiffness K_ψ. On the other hand, the mode number as well as the value of quotient k/k_n do not practically have any influence on the range width of $K_\psi a/B$ within which the sound power varies considerably with the varying value of K_ψ.

All the curves presented in Figs. 4 and 5 are prepared using the asymptotic formulae only, and they are valid for the high frequencies, whereas, Fig. 6 shows the results obtained from the integral formulae together with those obtained from the asymptotes. It is assumed that the results obtained from the integral formulae are exact and therefore they are used to check the accuracy of the high-frequency asymptotics. The curves in Fig. 6 are prepared for some sample values assumed by K_η and K_ψ. By analyzing the curves and the corresponding eigenvalues given in Table 2, it can be noticed that the integral formulae and the asymptotes show a good agreement for any mode number n and for the values of k/k_n being slightly greater than 1. If the constants K_η and K_ψ assume their limiting values of zero or infinity the results obtained are identical with the literature data valid for the classical boundaries (cf., Refs. [1,2,4,7,10]).

7. Conclusions

It has been proved analytically that the formulae derived for the active and reactive sound power of radiation of the vibrating elastically supported circular plate can be used to generalize some of the earlier investigated formulae valid for the classical boundary conditions as well as the two boundary configurations discussed in this study.

Since the two boundary stiffness values K_η and K_ψ, used in the high-frequency asymptotes, can vary within the infinite range of $[0, +\infty]$, the asymptotes can be used to describe the sound radiation of a circular plate with clamped, simply supported, free or guided boundaries, and with any other intermediate boundary configurations as well. The influence of the stiffness constants varying within the infinite range of the sound field generated by an individual elastic mode of the plate has also been presented.

The agreement shown by the results obtained from the asymptotic and integral formulae shows that the asymptotes valid for the high frequencies can be used for some fast engineering computations. The asymptotes are easy to express in terms of a computer code and they do not require too much processor capacity, unlike any other analytical formulations expressed in the form of integral or expansion series (cf., Ref. [5]).

The integral and asymptotic formulae presented in this paper can make the basis for some further computations of the total sound power radiated or lost by the plate excited by any axisymmetric and time-harmonic force in an acoustic fluid, which falls outside the scope of this paper.

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Appendix A. An asymptotic method of integral computing

An asymptotic method of integral computing was used in this project. The method was presented earlier in Refs. [1,2,4]. The first integral computed by the method is (cf., Eq. (15a))

$$\mathscr{P}_{a,n}^{(1)} = \int_0^1 \left[\frac{\alpha_n\delta_n \mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{x^4-\delta_n^4}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{A.1}$$

with the following conditions satisfied $k > k_n$ and $\beta = ka > 1$, where $\delta_n = x_n/\beta < 1$. Auxiliary function F_1, in a plane of complex variable $z = x + \mathrm{i}y$, was introduced

$$\begin{aligned} F_1(z) = {} & \alpha_n^2\delta_n^2 \mathrm{J}_0(\beta z)\mathrm{H}_0^{(1)}(\beta z) + z^2\mathrm{J}_1(\beta z)\mathrm{H}_1^{(1)}(\beta z) \\ & - \alpha_n\delta_n z[\mathrm{J}_0(\beta z)\mathrm{H}_1^{(1)}(\beta z) + \mathrm{J}_1(\beta z)\mathrm{H}_0^{(1)}(\beta z)], \end{aligned} \tag{A.2}$$

such that

$$\mathrm{Re}\, F_1(x) = [\alpha_n\delta_n\mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)]^2, \tag{A.3}$$

where $\mathrm{H}_n^{(1)}(x)$ is first kind Hankel function of nth order. The following integral was computed along path C (cf., Fig. 3)

$$\oint_C \frac{zF_1(z)\,\mathrm{d}z}{\sqrt{1-z^2}(z^4-\delta_n^4)^2} = 0. \tag{A.4}$$

The integrand was regular analytical and unique along and within the path. The branch point of term $\sqrt{1-z^2}$ is $z = 1$ and the branch point of Hankel function of zero order is $z = 0$. The Hankel function has a logarithmic singularity for $|z| \to 0$. The integrand has second order poles for $z = \delta_n$ and $z = \mathrm{i}\delta_n$. Based on the Cauchy theorem on residues it is possible to express integral (A.1) by

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another integral computed along a closed path C (cf., Fig. 3)

$$\mathrm{Re}\int_0^1 \frac{xF_1(x)\,\mathrm{d}x}{\sqrt{1-x^2}\,(x^4-\delta_n^4)^2} = \int_0^1 \left[\frac{\alpha_n\delta_n \mathrm{J}_0(\beta x) - x\mathrm{J}_1(\beta x)}{(x^4-\delta_n^4)}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$$
$$= \mathrm{Re}\{\pi\mathrm{i}[\mathscr{F}_1^{(1)}(\delta_n) + \mathscr{F}_1^{(2)}(\mathrm{i}\delta_n)]\} + \int_1^\infty \{\alpha_n^2\delta_n^2 \mathrm{J}_0(\beta x)N_0(\beta x) + x^2\mathrm{J}_1(\beta x)N_1(\beta x)$$
$$-\ \alpha_n\delta_n x[\mathrm{J}_0(\beta x)N_1(\beta x) + \mathrm{J}_1(\beta x)N_0(\beta x)]\}\,\frac{x\,\mathrm{d}x}{\sqrt{x^2-1}(x^4-\delta_n^4)^2}, \tag{A.5}$$

where $\mathrm{Re}\,F_1(\mathrm{i}y) = 0,\ \mathscr{F}_1'(z) = \dfrac{\mathrm{d}\mathscr{F}_1(z)}{\mathrm{d}z}$

and

$$\mathscr{F}_1^{(1)}(z) = \frac{zF_1(z)}{\sqrt{1-z^2}(z+\delta_n)^2(z^2+\delta_n^2)^2} \quad \text{for} \quad z = \delta_n, \tag{A.6a}$$

$$\mathscr{F}_1^{(2)}(z) = \frac{zF_1(z)}{\sqrt{1-z^2}\,(z+\mathrm{i}\delta_n)^2(z^2-\delta_n^2)^2} \quad \text{for} \quad z = \mathrm{i}\delta_n. \tag{A.6b}$$

The following magnitudes were computed $F_1(\delta_n) = 0,\ \mathrm{Re}\,F_1'(\delta_n) = 0,$

$$\mathrm{Im}\,F_1(\mathrm{i}\,\delta_n) = -\frac{2}{\pi}\frac{\delta_n^2}{\mathrm{J}_0^2(x_n)}\,S(x_n)N(x_n) \tag{A.7}$$

with notations $S(x_n) = J_1(x_n)I_0(x_n) + J_0(x_n)I_1(x_n)$ and $N(x_n) = J_1(x_n)K_0(x_n) - J_0(x_n)K_1(x_n)$, where K_0 and K_1 are the McDonald functions of the zero and first order, respectively. Moreover,

$$\mathrm{Im}\,F_1'(\delta_n) = \frac{2}{\pi}\,\delta_n(1+\alpha_n^2), \tag{A.8}$$

$$F_1'(\mathrm{i}\delta_n) = \mathrm{Re}\,F_1'(\mathrm{i}\delta_n) = \frac{2}{\pi}\frac{\delta_n^2\beta}{\mathrm{J}_0^2(x_n)}[U(x_n)S(x_n) - V(x_n)N(x_n)], \tag{A.9}$$

with the following notation introduced

$$V(x_n) = \mathrm{J}_1(x_n)\mathrm{I}_1(x_n) + \mathrm{J}_0(x_n)\mathrm{I}_0(x_n), \tag{A.10a}$$

$$U(x_n) = \mathrm{J}_1(x_n)K_1(x_n) - \mathrm{J}_0(x_n)K_0(x_n). \tag{A.10b}$$

Some basic transformations led to

$$\mathrm{Re}\,\{\pi\mathrm{i}\,[\mathscr{F}_1^{(1)}(\delta_n) + \mathscr{F}_1^{(2)}(\mathrm{i}\delta_n)]\} = \frac{1}{8\delta_n^4}\left\{\frac{(1+\alpha_n^2)}{\sqrt{1-\delta_n^2}} + \frac{1}{\sqrt{1+\delta_n^2}}\frac{1}{\mathrm{J}_0^2(x_n)}\right.$$
$$\left.\times\left[x_n\,(U(x_n)S(x_n) - V(x_n)N(x_n)) + \left(2 + \frac{\delta_n^2}{(1+\delta_n^2)}\right)S(x_n)N(x_n)\right]\right\}, \tag{A.11}$$

which represents the non-oscillating part of integral (A.1), whereas, the term appearing in Eq. (A.5), being the integral computed within the limits (1, ∞), was computed by the stationary phase method and by the asymptotic formulae for the products of Bessel and Neumann functions

$$\mathrm{J}_0(u)N_0(u) \sim -\mathrm{J}_1(u)N_1(u) \sim -\frac{\cos 2u}{\pi u}, \tag{A.12a}$$

$$\mathrm{J}_0(u)N_1(u) + \mathrm{J}_1(u)N_0(u) \sim -\frac{2\sin 2u}{\pi u}. \tag{A.12b}$$

The computation led to

$$\begin{aligned}\tilde{\mathscr{P}}_{a,n}^{(1)} \equiv \int_1^\infty &\{\alpha_n^2\delta_n^2 \mathrm{J}_0(\beta x)N_0(\beta x) + x^2 \mathrm{J}_1(\beta x)N_1(\beta x) \\ &- \alpha_n\delta_n x[\mathrm{J}_0(\beta x)N_1(\beta x) + \mathrm{J}_1(\beta x)N_0(\beta x)]\}\frac{x\,\mathrm{d}x}{\sqrt{x^2-1}(x^4-\delta_n^4)^2} \\ = \frac{1}{\pi\beta}\int_1^\infty &[(x^2-\alpha_n^2\delta_n^2)\cos 2\beta x + 2\alpha_n\delta_n x\sin 2\beta x]\frac{\mathrm{d}x}{\sqrt{x^2-1}(x^4-\delta_n^4)^2}\end{aligned} \tag{A.13}$$

while the stationary phase method produced

$$\tilde{\mathscr{P}}_{a,n}^{(1)} = \frac{1}{2\beta\sqrt{\pi\beta}}\frac{(1-\alpha_n^2\delta_n^2)\cos(2\beta+\pi/4) + 2\alpha_n\delta_n\sin(2\beta+\pi/4)}{(1-\delta_n^4)^2}, \tag{A.14}$$

which represents the oscillating part of integral (A.1). The other two integrals given in Eq. (15) may be computed in an analogous way.

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