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Journal of Sound and Vibration 265 (2003) 231–234

JOURNAL OF SOUND AND VIBRATION

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Letter to the Editor

Comments on "Non-linear vibration analysis and sub-harmonic whirl frequencies of the Jeffcott rotor model"

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Received 30 January 2002; accepted 25 November 2002

1. Fundamentals in dynamics of the Jeffcott rotor [1]

The linear differential equation of motion of the Jeffcott rotor in the *stationary* (inertial) co-ordinates is simply given as

$$\ddot{z} + 2\gamma\omega_n\dot{z} + \omega_n^2 z = e\omega^2 \exp(\mathrm{j}\omega t), \tag{1}$$

where $z(t)$ is the complex position vector representing the co-ordinate of the geometrical rotor center, e is the mass eccentricity, ω is the rotational speed, and ω_n and γ correspond to the natural frequency and damping ratio. The characteristic equation is obtained as the second order polynomial equation

$$p(\lambda) = \lambda^2 + 2\gamma\omega_n\lambda + \omega_n^2 = 0 \tag{2}$$

having the characteristic roots

$$\lambda^F = -\gamma\omega_n + \mathrm{j}\omega_n\sqrt{1-\gamma^2}, \quad \lambda^B = -\gamma\omega_n - \mathrm{j}\omega_n\sqrt{1-\gamma^2}. \tag{3}$$

Here, $\mathrm{Im}(\lambda^F) > 0$, $\mathrm{Im}(\lambda^B) < 0$ are referred to as the forward and backward modal frequencies defined in the *stationary* co-ordinates, respectively.

The solution of Eq. (1) reduces to

$$z(t) = \frac{e\omega^2}{(\omega_n^2 - \omega^2) + \mathrm{j}2\gamma\omega_n\omega}\exp(\mathrm{j}\omega t) + C_1\exp(-\gamma\omega_n t)\exp(\mathrm{j}\omega_n\sqrt{1-\gamma^2}) + C_2\exp(-\gamma\omega_n t)\exp(-\mathrm{j}\omega_n\sqrt{1-\gamma^2}), \tag{4}$$

where C_1, C_2 are the constants to be determined from initial conditions. Note that the decay rate $\gamma\omega_n$ and the apparent damping ratio, which may be defined as $\gamma_{ap} \cong |\mathrm{Re}(\lambda)/\mathrm{Im}(\lambda)| = \gamma$ for small γ,

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0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01576-6

are the same for the forward and backward modes. Thus, to observers sitting in the *stationary* co-ordinate frame, the transient response of the Jeffcott rotor subject to unbalance will consist of the synchronous sustained unbalance response and the decaying (at a rate of $\gamma\omega_n$) transient response associated with the natural modes of frequency $\pm\omega_n\sqrt{1-\gamma^2}$ and damping ratio γ. The equation of motion (1) defined in the *stationary* co-ordinates can be expressed in the *rotating* co-ordinates as

$$\ddot{\varsigma} + 2(\mathrm{j}\omega + \gamma\omega_n)\dot{\varsigma} + (\omega_n^2 - \omega^2 + \mathrm{j}2\gamma\omega_n\omega)\varsigma = e\omega^2 \tag{5}$$

since it holds

$$z = \varsigma \exp(\mathrm{j}\omega t), \tag{6}$$

where $\varsigma(t)$ is the complex position vector representing the co-ordinate of the geometrical rotor center defined in the *rotating* co-ordinates. Note that the co-ordinate transformation (6) yields a differential equation of motion (5), which is still linear. The characteristic equation becomes then

$$p(\mu) = \mu^2 + 2(\mathrm{j}\omega + \gamma\omega_n)\mu + (\omega_n^2 - \omega^2 + \mathrm{j}2\gamma\omega_n\omega) = 0 \tag{7}$$

resulting in the characteristic roots given by

$$\mu^F = -\gamma\omega_n + \mathrm{j}(\omega_n\sqrt{1-\gamma^2} - \omega), \quad \mu^B = -\gamma\omega_n + \mathrm{j}(-\omega_n\sqrt{1-\gamma^2} - \omega). \tag{8}$$

Here, $\mathrm{Im}(\mu^F)$, $\mathrm{Im}(\mu^B)$ are referred to as the forward and backward modal frequencies defined in the *rotating* co-ordinates, respectively. It is straightforward to obtain the relation

$$\lambda^k = \mu^k + \mathrm{j}\omega, \quad k = F, B. \tag{9}$$

The solution of Eq. (5) is obtained to be

$$\begin{aligned}\varsigma = {} & \frac{e\omega^2}{(\omega_n^2 - \omega^2) + \mathrm{j}2\gamma\omega_n\omega} + C_1 \exp(-\gamma\omega_n t)\exp\{\mathrm{j}(\omega_n\sqrt{1-\gamma^2} - \omega)\} \\ & + C_2 \exp(-\gamma\omega_n t)\exp\{\mathrm{j}(-\omega_n\sqrt{1-\gamma^2} - \omega)\}.\end{aligned} \tag{10}$$

Note that whereas the decay rate $\gamma\omega_n$ still remains the same for the forward and backward modes, the apparent damping ratios defined by $\gamma_{ap} \cong |\mathrm{Re}(\mu)/\mathrm{Im}(\mu)| = \gamma$ for small γ will be different unlike the modes defined in the stationary co-ordinates, i.e.,

$$\gamma_{ap}^F = \gamma\left[\frac{\omega_n}{\left|\omega_n\sqrt{1-\gamma^2} - \omega\right|}\right] \geqslant \gamma_{ap}^B = \gamma\left[\frac{\omega_n}{\left|\omega_n\sqrt{1-\gamma^2} + \omega\right|}\right]. \tag{11}$$

In other words, the apparent damping ratio of the backward mode is always less, irrespective of the rotational speed ω, than that of the forward mode, if defined in the rotating co-ordinates. To observers rotating together with the rotating (rotor-body fixed) co-ordinates, the transient response of the Jeffcott rotor subject to unbalance will consist of the static unbalance response and the two different decaying transient responses associated with the forward and backward modes. Although both modes have the same decay rate $\gamma\omega_n$ like in the stationary co-ordinates, the apparent damping ratios are different as mentioned above. The forward and backward damped natural frequencies, $\omega_n\sqrt{1-\gamma^2} - \omega$, $-\omega_n\sqrt{1-\gamma^2} - \omega$, that strongly depend upon the rotational speed ω and are directly observed in transient responses, may normally be 'asynchronous', as phrased in Ref. [2]. However, the precise definition of 'synchronous' or asynchronous response should be made based on the response defined in the *stationary* co-ordinates.

2. False claims made in Ref. [2]

Diken [2] used the non-linear dynamic equations of the Jeffcott rotor derived in Ref. [3], which can be converted back to the classical time-invariant linear differential equation (1) by using the co-ordinate transformation of $z(t) = r(t)e^{j\theta(t)}$. Note that the unusual non-linear co-ordinate transformation, from the stationary Cartesian co-ordinates to the polar co-ordinates defined with respect to the *rotating* body-fixed frame, leads to parametrically excited non-linear differential equations in which the solution technique becomes, if not impossible, unnecessarily complicated. Then he used an inappropriate perturbation technique to derive approximate linearized equations from the non-linear equations, in order to understand the transient behavior of the Jeffcott rotor. He claimed that the two observed (in the rotating co-ordinates) asynchronous frequency components, $(\omega + \omega_n)$ and $(\omega - \omega_n)$, from the transient response are due to the non-linearity of the system itself and the supersynchronous component becomes unstable when $\omega/\omega_n \geqslant 2$. And the damping ratios associated with the two asynchronous frequency components are different, again due to the system non-linearity.

It turns out that the first order perturbation is not accurate enough to properly predict the dynamic behavior of the rotor system (the first order approximation results do not match well with the numerical integrations), on the contrary to the author's claim, as clearly seen in Figs. 2–5 of Ref. [2]. In particular, the first order approximated characteristic Eq. (10) in Ref. [2] is far off from the correct Eq. (7).[1] For example, the real parts of the characteristic roots shown in Table 1 of Ref. [2] are far apart from the correct value $-0.01(= -\gamma\omega_n)$. Thus, any critical observations made based on the inaccurate characteristic equation are erroneous, including discussions with the instability. The system of interest is inherently stable, regardless of the co-ordinate transformation. For comparison with Table 1 of Ref. [2], a part of the correct results are summarized in Table 1. It clearly manifests that the asynchronous responses shown in Figs. 2–5 of Ref. [2] are not due to non-linearity of the system, but to the natural modes and the system is inherently stable and linear. The original damping ratios associated with the forward and backward modes defined in the *stationary* co-ordinates are the same, but they become different when they are defined in the *rotating* co-ordinates as shown in Eq. (11). It is due to the co-ordinate transformation, shifting the natural frequencies by ω_n, or the imaginary part of the characteristic roots, as in Eq. (9). However, the real part of the characteristic roots remains unchanged and thus the relative stability is independent of the co-ordinate transformation.

3. Recommendations

The approach and the analysis results taken in Ref. [2] are neither appropriate nor correct. The sub-harmonic components in the transient response of the Jeffcott rotor system are due to the

[1] For direct comparison, Eq. (7) should be modified as $p(\mu)\bar{p}(\mu) = 0$ to account for the complex conjugate roots.

Table 1
Roots of the characteristic Eq. (8): $\gamma = 0.01,\ \ \omega = 1$

$p = \omega_n/\omega$	$\mu^F = -\gamma\omega_n + \mathrm{j}(\omega_n\sqrt{1-\gamma^2} - \omega)$ $\mu^B = -\gamma\omega_n + \mathrm{j}(-\omega_n\sqrt{1-\gamma^2} - \omega)$	$\gamma^F_{ap},\ \gamma^B_{ap}$	Remark
1.4	$\mu^F = -0.01 + 0.4\mathrm{j}$ $\mu^B = -0.01 - 2.4\mathrm{j}$	$\gamma^F_{ap} = 0.0250$ $\gamma^B_{ap} = 0.0042$	Figs. 2 and 3 of Ref. [2]
0.8	$\mu^F = -0.01 - 0.2\mathrm{j}$ $\mu^B = -0.01 - 1.8\mathrm{j}$	$\gamma^F_{ap} = 0.0500$ $\gamma^B_{ap} = 0.0056$	Figs. 4 and 5 and Table 1 of Ref. [2]

natural modes of the system. If the presence of sub-harmonics, other than the natural vibrations, in the rotor response is an inherent nature, its existence should be proven rigorously, irrespective of the choice of co-ordinate system.

Choice of co-ordinate system is very important in rotordynamic analysis. However, co-ordinate transformation does not alter the physics of the rotor system of interest. And we should also be aware of the pitfalls in the process of co-ordinate transform, from the stationary to the rotating co-ordinates and vice versa. All observations in Ref. [2] are based on the rotating co-ordinates, while, in practice, the response measurements and the interpretations are taken with respect to the stationary co-ordinates. Serious problems may arise when one tries to convert the information taken with respect to the *rotating* co-ordinates to that with respect to the *stationary* co-ordinates [1,4]. Co-ordinate transformation is merely a mathematical convenience, never creating any unseen physical phenomena.

It is not an efficient, if not incorrect, approach to convert the originally time-invariant linear system, where a complete analytical solution is readily available, to a complicated parametrically excited non-linear system, which is then linearized to obtain an approximate solution. Approximation techniques such as the perturbation method may lead to false results, particularly when the order of approximation is not high enough.

References

[1] C.W. Lee, Vibration Analysis of Rotors, Kluwer Academic Publishers, Dordrecht, 1993.
[2] H. Diken, Non-linear vibration analysis and subharmonic whirl frequencies of the Jeffcott rotor model, Journal of Sound and Vibration 243 (1) (2001) 117–125.
[3] W.T. Thomson, Theory of Vibration with Applications,, 3rd Edition, Prentice-Hall, Englewood Cliffs, NJ, 1988 Section 3.4.
[4] C.W. Lee, Rotor dynamics and control in complex modal space, Keynote paper, The First International Conference on Motion and Vibration Control, Yokohama, Japan, 1992, pp. 22–31.