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Model updating via weighted reference basis with connectivity constraints

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Abstract

The paper considers the problem of updating an analytical model from experimental data using the reference basis approach. In the general framework of the reference basis method, certain quantities, e.g., natural frequencies or modeshapes, are considered to be completely accurate and the others are updated by solving a constrained optimization problem. However, the underlying structure, known as connectivity, existing in the model is not preserved, and the method is not suited for parametric updating. In this paper, a method for introducing connectivity constraints into reference basis, while maintaining its advantages, is presented. It brings the reference basis method closer to a broad class of updating methods that use parametric updating. The notions of "connectivity cost" and "parameterization cost" are defined and used to obtain the best model for a given parameterization and to compare the outcomes of different parameterizations.

1. Introduction

The model-updating problem has received much attention over the years and many approaches to it have been suggested. The sensitivity-based methods [1–5] use the dependence of quantities such as natural frequencies, antiresonances, and modeshapes on physical parameters, calculated either numerically or analytically, to find the required change in a set of parameters to match the experimental modal data. In Ref. [6], the rank of the error matrix, rather than the norm as in all other methods, is minimized. Another group of methods is that which uses as its data the complete frequency response function, e.g., Ref. [7]. Mathematically, the model-updating problem is closely

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related to that of structural modification [8]. An extensive survey of model-updating methods can be found in Ref. [9].

In the "reference basis" methods [10–12], certain quantities are assumed to be accurate and those that are free (generally stiffness and mass) should satisfy the relationships that must hold in a passive vibrating structure, while their deviation from the analytical model is minimal. The method possesses some clear advantages. The updated system eigendata coincides with the measured one, an obvious requirement that many other methods fail to achieve. It is based on a reasonable, and well-defined, optimization criterion, and presents a closed-form solution of the problem. Moreover, and perhaps most importantly, the dimension that dominates the required computational effort is the number of measured modes, compared to the number of degrees of freedom (d.o.f.) in other methods. Since the two usually differ by orders of magnitude, the reference basis method is much more efficient and can handle models having a high dimension. The standard reference basis method has also a major disadvantage, which prevents it from becoming a leading and vastly applied method. The updated model has no relationship with physical parameters, nor does it preserve the connectivity of the system. Another problem with the method in its standard form, is that it cannot incorporate any prior knowledge or engineering consideration into the process. Such user's intervention, using some tuning parameters, is often essential for successful algorithms.

The generalized reference basis method [13] is aimed at amending these problems, thus enabling the use of this elegant and efficient method. The key element is the introduction of general weighting matrices into the optimization criterion rather than the mass matrix, which is used in the standard reference basis. This required a more complex derivation, yet the solution maintained the closed-form and efficiency properties. By appropriate choice of those weights, the corrections can be directed into desired areas.

In this paper, an automatic method for determining the weighting matrix is presented. It is based on a single free parameter, which may be iterated to obtain best results. This weighting selection enables the procedure of a posteriori connectivity assignment, in which the assumed connectivity is enforced. Introducing connectivity considerations to the reference basis method brings it closer to well-established methods, in particular sensitivity methods. In the first stage of the suggested algorithm, the amount of connectivity violation is used in the adjustment of the weighting matrices. In the second, the connectivity is assigned a posteriori by imposing it on the updated model.

2. Problem statement

One starts by setting up the reference basis model-updating problem. Let the "true" equations of the system be

$$\mathbf{M}_T\ddot{\mathbf{x}}(t) + \mathbf{K}_T\mathbf{x}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{x} \in R^n$ is a vector of generalized displacements. Clearly the absolutely accurate model of the system is non-linear, infinite dimensional, etc., so "true" means here accurate enough for all practical purposes. The analytic model of the system, obtained by finite elements or any other

method is given by

$$\mathbf{M}_A\ddot{\mathbf{x}}(t) + \mathbf{K}_A\mathbf{x}(t) = \mathbf{f}(t), \tag{2}$$

where in general $\mathbf{M}_A \neq \mathbf{M}_T$ and $\mathbf{K}_A \neq \mathbf{K}_T$. The result of a modal test of the system are m ($m<n$) natural frequencies, ω_i, and m modeshapes, $\tilde{\boldsymbol{\varphi}}_{Ei}$. In general, the measured modeshapes contain only a partial set of the d.o.f. The problem is how to combine the analytic information $\mathbf{M}_A$, $\mathbf{K}_A$ and the experimental results to obtain a model which is more accurate, i.e., closer to $\mathbf{M}_T$, $\mathbf{K}_T$.

An assumption that is made throughout this paper is that the measured natural frequencies are correct. The physical justification for this is that they are global, relatively easy to obtain from measured data, variables, which are common to all measurements in the system and therefore after processing their error is small. The measured modeshapes, in addition to being incomplete, cannot be regarded as accurate. A preliminary step is processing them to yield a set of full-dimension modeshapes satisfying the orthogonality condition

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}_T\boldsymbol{\Phi} = I, \tag{3}$$

where the columns of $\boldsymbol{\Phi}(n \times m)$ are the modeshapes. This step is not part of the discussion in this paper and is described by a generic function of all the analytical and experimental data:

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\tilde{\boldsymbol{\Phi}}_E, \mathbf{M}_A, \mathbf{K}_A, \boldsymbol{\Omega}). \tag{4}$$

Specific algorithms can be found in Refs. [14,15].

3. Stiffness matrix updating

Updating of the stiffness matrix is based on three accurate quantities which constitute the reference basis in this case. They are: (1) the incomplete natural frequencies matrix $\boldsymbol{\Omega}(m \times m)$, (2) the mass matrix $\mathbf{M}$, and (3) the incomplete, yet orthogonal and normalized modeshapes matrix $\boldsymbol{\Phi}(n \times m)$. While the first assumption has been discussed in Section 2, the other two require some explanation. There are three possibilities regarding the mass matrix. First, it can be assumed to be accurate, and therefore left unchanged in the updating process. Since the mass matrix is usually better known than the stiffness matrix, $\mathbf{M}_A = \mathbf{M}_T = \mathbf{M}$ is a reasonable assumption. Secondly, one can adopt a sequential updating procedure and in that case $\mathbf{M}$ represents the updated mass matrix. Finally, the mass and stiffness matrices can be updated simultaneously. The solution of this problem using the reference basis is given in Ref. [13]. It was shown here that despite the simultaneous setting, the solution is such that stiffness updating follows the mass updating. Hence it is similar to sequential updating, though with different updated mass matrix. The assumption regarding $\boldsymbol{\Phi}$ is clearly inaccurate and errors in this matrix will propagate to errors in the updated stiffness matrix. However, this is inherently the case in any updating method. Combining the expanding and orthogonalization process in Eq. (4) with the reference basis stiffness updating that will follow results in $\mathbf{K} = \mathbf{K}(\tilde{\boldsymbol{\Phi}}_E, \mathbf{M}_A, \mathbf{K}_A, \boldsymbol{\Omega})$ as in any other method.

The rest of this section is a brief summary of the results in Ref. [13], which are required for the algorithm suggested in the next section and the examples which will follow. To enforce the symmetry and positive semi-definiteness requirements, the stiffness matrix is defined as $\mathbf{K} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$. Clearly $\mathbf{K} \geqslant 0$ and if, in addition, $\mathbf{L}$ has full row rank, then $\mathbf{K} > 0$. This is a deviation from previous

works where only the symmetry of $\mathbf{K}$ was enforced, and via a formal constraint. The minimization problem is thus given by

$$\min J = \|\mathbf{W}^{-1/2}(\mathbf{L}\mathbf{L}^{\mathrm{T}} - \mathbf{K}_A)\mathbf{W}^{-1/2}\|_F^2, \quad \text{s.t.} \quad \mathbf{L}\mathbf{L}^{\mathrm{T}}\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2, \tag{5}$$

where $\|\ \|_F$ is the Frobenius norm defined by

$$\|A\|_F^2 = \sum_i \sum_j |a_{ij}|^2 = \mathrm{tr}(A^* A). \tag{6}$$

$\mathbf{W}$ is a symmetric but otherwise general weighting matrix. This is the main distinction between the work in Ref. [13] and the results of Baruch and Bar-Itzhack [10] (and also those of Refs. [11,12]) where $\mathbf{W} = \mathbf{M}$ was selected. The judicious application of the flexibility offered by using a general $\mathbf{W}$ is a major issue of this paper.

The solution for optimization problem (5) can be expressed in several forms. One of them is

$$\mathbf{K} = \mathbf{K}_A - (\mathbf{K}_A\boldsymbol{\Phi} - \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2)\mathbf{R}^{\mathrm{T}} - \mathbf{R}(\mathbf{K}_A\boldsymbol{\Phi} - \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2)^{\mathrm{T}} + \mathbf{R}(\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}_A\boldsymbol{\Phi} - \boldsymbol{\Omega}^2)\mathbf{R}^{\mathrm{T}}, \tag{7}$$

where

$$\mathbf{R} = \mathbf{W}\boldsymbol{\Phi}(\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{W}\boldsymbol{\Phi})^{-1}. \tag{8}$$

A second form is

$$\mathbf{K} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M} + (\mathbf{I} - \boldsymbol{\Phi}\mathbf{R}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{K}_A - \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M})(\mathbf{I} - \boldsymbol{\Phi}\mathbf{R}^{\mathrm{T}}). \tag{9}$$

To see its geometrical interpretation, let $\boldsymbol{\Phi}_F = [\,\boldsymbol{\Phi} \quad \tilde{\boldsymbol{\Phi}}\,]$ and $\boldsymbol{\Omega}_F = \mathrm{diag}\{\boldsymbol{\Omega}, \tilde{\boldsymbol{\Omega}}\}$ be the complete set of natural frequencies and modeshapes. Then

$$\mathbf{K}_T = \mathbf{M}\boldsymbol{\Phi}_F\boldsymbol{\Omega}_F^2\boldsymbol{\Phi}_F^{\mathrm{T}}\mathbf{M} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M} + \mathbf{M}\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Omega}}^2\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}\mathbf{M}. \tag{10}$$

$(\mathbf{I} - \boldsymbol{\Phi}\mathbf{R}^{\mathrm{T}})$ is a projection matrix into a subspace orthogonal to $\boldsymbol{\Phi}$. Thus $\mathbf{K}$ has an accurate part (the first term), in a subspace defined by the measured modeshapes $\boldsymbol{\Phi}$, and a correction part consisting of the difference between the model stiffness matrix and the accurate part, projected into a subspace orthogonal to $\boldsymbol{\Phi}$. In case $\mathbf{K}_A$ is accurate, i.e., $\mathbf{K}_A = \mathbf{K}_T$, the difference already belongs to that subspace and therefore remains unchanged by the projection. $\mathbf{W}$ affects only the "angle" of this projection but not its image. However, as will be demonstrated in the sequel, this has a strong effect on the results.

4. Selection of the weighting matrix

A key issue in the application of the generalized reference basis method is the selection of the weighting matrix $\mathbf{W}$. As was already mentioned, in earlier works $\mathbf{W}$ was selected as the mass matrix. While this is a plausible choice, it does not utilize the opportunities offered by the ability to use a general weighting matrix. An intuitive analysis can be made by considering a diagonal weighting matrix $\mathbf{W} = \mathrm{diag}\{w_i\}$. The cost function in Eq. (5) is a weighted sum of the deviations

of $\mathbf{K}$ from $\mathbf{K}_A$. More specifically,

$$J = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i^{-1} w_j^{-1} (K_{ij} - K_{A,ij})^2. \tag{11}$$

Since the inverse of $\mathbf{W}$ appears in Eq. (11), if w_i is large, a unit change in the ith d.o.f. results in a small contribution to J. Similarly, if w_i is small, a unit change in the ith d.o.f. results in a large contribution to J. The optimization machinery will therefore automatically try to direct the changes to areas with large w_i (small w_i^{-1}), and to avoid changes in areas where w_i is small. To see it in a more formal way, assume that the d.o.f.s are rearranged into two groups, $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ such that $\mathbf{W} = \mathrm{diag}\{\mathbf{W}_1, \mathbf{W}_2\}$ where $\mathbf{W}_1 \gg \mathbf{W}_2$, which is equivalent to saying that $\mathbf{W}_1$ is $o(1)$ and $\mathbf{W}_2 \cong 0$. The modeshapes matrix is partitioned accordingly as $\mathbf{\Phi} = [\mathbf{\Phi}_1^{\mathrm{T}} \mathbf{\Phi}_2^{\mathrm{T}}]^{\mathrm{T}}$. Using $\mathbf{W}_2 = 0$, Eq. (7) leads to

$$\mathbf{K} = \mathbf{K}_A + \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^{\mathrm{T}} & 0_{n_2 \times n_2} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} \Delta_{11} = {} & \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}} (\mathbf{\Phi}_1 \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}})^{-1} (\mathbf{\Phi}^{\mathrm{T}} \mathbf{K}_{\mathrm{A}} \mathbf{\Phi} - \mathbf{\Omega}^2)(\mathbf{\Phi}_1 \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}})^{-1} \mathbf{\Phi}_1 \mathbf{W} \\ & - \mathrm{Y}_1 (\mathbf{\Phi}_1 \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}})^{-1} \mathbf{\Phi}_1 \mathrm{W} - \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}} (\mathbf{\Phi}_1 \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}})^{-1} \mathbf{Y}_1^{\mathrm{T}}, \end{aligned} \tag{13}$$

$$\Delta_{12} = -\mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}} (\mathbf{\Phi}_1 \mathbf{W}_1 \mathbf{\Phi}_1^{\mathrm{T}})^{-1} \mathbf{Y}_2^{\mathrm{T}} \tag{14}$$

and

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \mathbf{K}_A \mathbf{\Phi} - \mathbf{M} \mathbf{\Phi} \mathbf{\Omega}^2.$$

Eq. (12) implies that the sub-block (2, 2), describing the internal stiffness of x_2 remains unchanged. Furthermore, as will be shown later, the small weight $\mathbf{W}_2$ is used for d.o.f.s where the errors in the eigenvalue equation are small; hence Y_2 is small and consequently Δ_{12} is small too. Therefore, as expected from the intuitive explanation, changes will occur mainly in the (1, 1) sub-block, which correspond to x_1. The conclusion from this analysis is that the weighting matrix $\mathbf{W}$ provides means of restricting the updating to the desired d.o.f.s in a soft manner. This set of d.o.f. can be a result of either a priori or a posteriori knowledge. Sometimes it is clear that certain areas are better modelled than others. In such cases, w_i^{-1} is a confidence measure of the model of that d.o.f. In this paper, we consider the other possibility, i.e., a priori equal confidence in all d.o.f. The selection of the weighting matrix is then based on detection, rather than expectation, and the specific tool which is going to be used is error localization. There is a variety of error localization methods, e.g., Refs. [16,17], and the output of all of them is, or can be translated to, a number giving the magnitude of the error in that d.o.f. In the sequel, the eigenvalue equation error is considered as such an indicator, but other methods can be applied as well, without any effect on the general scheme of using the generalized reference basis approach. Let $\mathbf{d}$ be a vector proportional to the norms of the rows of the eigenvalue equation error matrix:

$$\mathbf{d}_i = c_i \| (\mathbf{K}_A \mathbf{\Phi} - \mathbf{M} \mathbf{\Phi} \mathbf{\Omega}^2)_{i,*} \|, \tag{15}$$

where the c_i are constants that can be used as normalization factors for non-identical d.o.f., such as displacements and angles or for containing a priori knowledge. A specific choice of c_i, used in some of the examples is

$$c_i = 1/\|(\mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2)_{i,*}\|, \tag{16}$$

which makes $\mathbf{d}_i$ non-dimensional. $\mathbf{W}$ is selected as

$$\mathbf{W} = \mathrm{diag}\{\mathbf{d}_i^{\beta}\}, \tag{17}$$

where β is a parameter for adjusting the updating procedure. For $\beta = 0$, $\mathbf{W}$ becomes the identity matrix and no d.o.f. are emphasized. As β increases, areas with larger errors in the eigenvalue equation are more and more emphasized. In case the modeshapes contain some noise, large values of β tend to increase it. Hence there is a trade-off between these conflicting effects and in general there is a finite optimal β.

5. A posteriori connectivity assignment

The stiffness matrix has usually inherent connectivity properties. In their simplest form, called "zero–nonzero" connectivity, some of the entries should be identically zero. Let I_0 be the set of entries of the stiffness matrix which are known to be identically zero. Then this type of connectivity is defined formally by

$$\mathbf{K}_{ij} = 0, \quad \forall i,j \in I_0. \tag{18}$$

Further connectivity requirements exist in the structured connectivity case, where certain relationships between the non-zero elements of the stiffness matrix should hold. Hence the model is actually determined by a smaller set of parameters. As an example, in the case of n masses connected serially by springs in a fixed–fixed form, the $3n-2$ ($2n-1$ when symmetry is invoked) elements of the stiffness matrix are determined by the $n+1$ spring constants with the requirement that each diagonal element is the sum of its two neighboring off-diagonal elements. A more realistic situation is the finite element structure. Following the framework suggested in many works, e.g., Refs. [1–3], it is assumed that the stiffness matrix depends on the free parameters linearly as

$$\mathbf{K} = \mathbf{K}_A + \sum_{i=1}^{p} \alpha_i \mathbf{K}_i, \tag{19}$$

where K_i are given matrices and α_i are scalar parameters, with nominal value of zero. In the above-cited works, as well as in others, the problem was solved using sensitivity methods. In this work, the error-modelling philosophy has been adopted, but different means are used to solve the problem.

Remark 1. The "zero–nonzero" connectivity can be regarded as a special case of the structured connectivity where the free parameters are all the non-zero entries, subject to symmetry. One therefore continues with the formulation in Eq. (19).

5.1. A priori connectivity assignment

The mathematically correct way to incorporate the connectivity into the optimization problem is via a formal constraint

$$\min_{K,\alpha} \quad J = \|\mathbf{W}^{-1/2}(\mathbf{K}-\mathbf{K}_A)\mathbf{W}^{-1/2}\|_F^2$$

$$\text{s.t.} \quad \mathbf{K}\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2, \quad \mathbf{K} = \mathbf{K}_A + \sum_{i=1}^{p} \alpha_i \mathbf{K}_i. \tag{20}$$

Or equivalently,

$$\min_{\alpha} \quad J = \left\| \mathbf{W}^{-1/2} \left(\sum_{i=1}^{p} \alpha_i \mathbf{K}_i \right) \mathbf{W}^{-1/2} \right\|_F^2$$

$$\text{s.t.} \quad \left(\mathbf{K}_A + \sum_{i=1}^{p} \alpha_i \mathbf{K}_i \right) \boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2. \tag{21}$$

The main problem with this approach is that a minimum number of parameters, which is in the order of the number of the d.o.f., is required to satisfy the constraint equation. This is always satisfied for "zero–nonzero" connectivity and this case was considered in Refs. [18,19]. However, the important properties of a closed-form solution, and more importantly, the need to invert only an $m \times m$ matrix, are no longer valid. Rather than that, a high-order least-square problem, with the non-zero elements as the unknowns has to be solved. In the case of structured connectivity, in general this option does not exist at all, since the constraint cannot be met.

5.2. A posteriori connectivity assignment

To circumvent the difficulties outlined in the previous sub-section, it is suggested in this work that the connectivity constraints are applied *after* the basic reference basis updating procedure. Clearly, this is wrong from pure mathematical considerations; however, from practical point of view, the procedure combines the best features of both reference basis and parametric approaches. The starting point is a reference basis update $\mathbf{K}$, given by Eqs. (7) or (9), that does not satisfy the connectivity constraints. This update will be replaced by $\mathbf{K}_{con}$, that satisfies the connectivity constraints, and is closest to $\mathbf{K}$ in a Frobenius norm sense.

In the case of "zero–nonzero" connectivity, one can enforce the correct structure on $\mathbf{K}$ by simply setting all the elements of I_0 to zero:

$$(\mathbf{K}_{con})_{ij} = \left\{ \begin{array}{ll} K_{ij}, & i,j \notin I_0 \\ 0, & i,j \in I_0 \end{array} \right\}. \tag{22}$$

In the case of structured connectivity, the following optimization problem is set up:

$$\min_{\alpha} \quad J = \left\| \sum_{i=1}^{p} \alpha_i \mathbf{K}_i - (\mathbf{K}-\mathbf{K}_A) \right\|_F^2. \tag{23}$$

This problem is easily transformed into the standard least squares

$$\min_{\alpha} \quad J = ||\mathbf{A}\boldsymbol{\alpha} - \mathbf{b}||_2^2, \tag{24}$$

where the ith column of $\mathbf{A}(n^2 \times p)$ is the matrix $\mathbf{K}_i$ stacked into a single column and $\mathbf{b}(n^2 \times 1)$ is constructed similarly from $\mathbf{K} - \mathbf{K}_A$. Alternatively, the derivation in Appendix A results in an explicit expression for the normal equations of the problem of a smaller dimension:

$$\begin{bmatrix} \mathrm{tr}(\mathbf{K}_1\mathbf{K}_1) & \cdots & \mathrm{tr}(\mathbf{K}_1\mathbf{K}_p) \\ \vdots & \ddots & \vdots \\ \mathrm{tr}(\mathbf{K}_p\mathbf{K}_1) & \cdots & \mathrm{tr}(\mathbf{K}_p\mathbf{K}_p) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\mathbf{K}_1(\mathbf{K} - \mathbf{K}_A)) \\ \vdots \\ \mathrm{tr}(\mathbf{K}_p(\mathbf{K} - \mathbf{K}_A)) \end{bmatrix}. \tag{25}$$

Eq. (24) is better conditioned, since actual forming of the normal equations, as in Eq. (25), means squaring of the condition number [20]. But the matrix $\mathbf{A}$ can be prohibitively large, and one is compelled to use Eq. (25). Notice that the coefficient matrix is independent of both $\mathbf{K}$ and $\mathbf{K}_A$. In the iterative scheme that will follow, this matrix has to be inverted (or the equivalent mathematical procedure that is used, e.g., QR factorization) only once.

Remark 2. Treating the "zero–non-zero" case formally as a structured connectivity, it is clear that Eq. (22) is the solution of Eq. (23) for that case.

The first advantage of the suggested method has already been revealed. By breaking the global optimization problem into two local ones, there is no direct relationship between the measured modal data and the updated parameters, therefore bypassing the constraint that the number of parameters cannot exceed the number of data. If $\mathbf{K}_{con}$ is not too far from $\mathbf{K}$, then one can safely use it as the final updated model. The next sub-section presents a method of obtaining, or at least approaching, such a case.

5.3. The cost of connectivity

The distance between $\mathbf{K}$, that satisfies all the updating requirements, except for connectivity, and $\mathbf{K}_{con}$ can be regarded as "payment" for invoking connectivity. It is termed the "connectivity cost". Several normalized criteria can be defined for that purpose. The first one is a measure of how much the assumed connectivity was violated in the updated $\mathbf{K}$:

$$J_{con} = ||\mathbf{K} - \mathbf{K}_{con}||_F / ||\mathbf{K}_{con}||_F. \tag{26}$$

This is actually the minimum value of J in Eq. (23), normalized to obtain a meaningful result. A variation of this criterion can be using the max norm, or additional normalization with respect to the number of parameters.

Unlike $\mathbf{K}$, $\mathbf{K}_{con}$ is not completely compatible with the measured eigendata. The natural frequencies and modeshapes of the system ($\mathbf{K}_{con}$, $\mathbf{M}$), corresponding to the m measured values, are given by

$$\boldsymbol{\Omega}_{con} = \boldsymbol{\Omega}(\mathbf{M}, \mathbf{K}_{con}), \quad \boldsymbol{\Phi}_{con} = \boldsymbol{\Phi}(\mathbf{M}, \mathbf{K}_{con}). \tag{27a, b}$$

A second criterion is the deviation of the natural frequencies from the measured ones (which are also the natural frequencies of $\mathbf{K}$):

$$\Delta\omega_i = (\omega_{i,con} - \omega_i)/\omega_i. \tag{28}$$

Another possible criterion is the deviation of the modeshapes from the measured ones, which is expressed by the deviation of $\mathbf{MAC}(\mathbf{\Phi}_{com}, \mathbf{\Phi})$ from the identity matrix $\mathbf{I}_m$. However, this criterion is very insensitive to even large deviations from the true stiffness matrix, and therefore has not been pursued further. Simulations have found that the deviation from the assumed connectivity, given in Eq. (26), is not only the easiest to calculate, but also the most robust indicator. Therefore, it was selected to be the connectivity cost for the rest of the paper.

The entire updating procedure still has one free parameter, the power β in the weighting matrix $\mathbf{W}$ in Eq. (17). Since $\mathbf{K}$ depends on $\mathbf{W}$, and $\mathbf{K}_{con}$ on $\mathbf{K}$, it follows that with the given analytical model and measurement data, the cost of connectivity is a function of β:

$$J_{con}(\beta) = J_{con}(\mathbf{W} = \mathrm{diag}\{\mathbf{d}_i^{\beta}\}). \tag{29}$$

To decide what value of β to use, consider first the ideal scenario: the correct connectivity is known but not imposed, and the unconstrained updated model happens to conform with it. That leads to the fundamental premise of the suggested algorithm.

Unconstrained updated models with smaller connectivity cost are closer to the true model.

The meaning of that idea is that a weighting matrix is better if it leads to an unconstrained updated stiffness matrix which is closer to the assumed connectivity. In other words, the amount of connectivity violation of $\mathbf{K}$, or more generally, the discrepancy between $\mathbf{K}$ and $\mathbf{K}_{con}$, serves as an indicator for the quality of the selected β. From a slightly different point of view, connectivity is treated in the unconstrained part of the algorithm as a soft constraint rather than a hard one. The algorithm therefore includes a search for β which minimizes J_{con}. With perfect measurements, larger β always yields better results. When the measurements contain some noise, larger β tends to increase it, and usually there is a finite optimal β. The examples in the next section show that there is an excellent correspondence between J_{con} and the deviation from the true stiffness matrix. They also verify the qualitative description of the effect of noise; however, a detailed analysis for that phenomenon is still required, and is a matter of current investigation.

With β determined, one can move to an optional final step, which is an iterative model reference updating with connectivity constraints. It is identical with the steps taken so far, except that in its kth iteration, $\mathbf{K}_{con}^{(k)}$ replaces $\mathbf{K}_A$. The motivation in that step is to reduce the violation of connectivity, using the same tools.

The algorithm can be summarized as:

Step 1: Obtain from the measurements, and the analytic model a full-dimension, orthogonal, modeshape matrix $\mathbf{\Phi} = \mathbf{\Phi}(\tilde{\mathbf{\Phi}}_E, \mathbf{M}_A, \mathbf{K}_A, \mathbf{\Omega})$.

Step 2: Search for β^* which minimizes J_{con} (β). For each value of β,

(1) find the unconstrained update $\mathbf{K}$ using Eq. (7);
(2) find the constrained update $\mathbf{K}_{con}$ by solving the LS problem in Eq. (23);
(3) calculate J_{con}.

Step 3: Final iteration (optional). Starting from $\mathbf{K}_{con}(\beta^*)$, at each iteration, execute 1–2 of Step 2 where the current $\mathbf{K}_{con}$ plays the role of $\mathbf{K}_A$. Stop when $\mathbf{K}_{con}$ converges.

Due to the small computational requirement in each updating, the overall algorithm is very efficient even if many loops are performed.

5.4. The cost of a parameterization

While connectivity assignment is generally regarded as a positive, perhaps even necessary, step in the process of model updating, care must be taken in its application. When used "blindly", this approach actually ignores the possibility that, as with any other aspect of the model, the assumed connectivity itself may be inaccurate. Such structural modelling errors will be corrected by varying the parameters in the wrong configuration, which may lead to gross mistakes. A quantitative criterion for the quality of a certain parameterization, as compared to other possibilities, is therefore a valuable tool. The cost of connectivity can serve this need as well. Let P be a certain parameterization, and define

$$J_p(P) = J_{con}(P, \beta = \beta^*). \tag{30}$$

Then if $J_p(P_1) > J_p(P_2)$, one can say that the parameterization P_1 imposes less changes on the unconstrained optimization than P_2, hence, following the premise in the previous sub-section, superior.

6. Examples

Example 1. Consider the system shown in Fig. 1. This technical example will be used to demonstrate the various aspects of the suggested updating algorithm. The true numerical values are $m = 1$, $k_1 = k_3 = k_4 = k_5 = k_6 = 50$, $k_2 = 70$, while in the analytical model, $m = 1$, and $k_i = 50$, $i = 1, \ldots 6$. The true and analytic stiffness matrices are then

$$\mathbf{K}_T = \begin{bmatrix} 120 & -70 & 0 & 0 & 0 \\ -70 & 120 & -50 & 0 & 0 \\ 0 & -50 & 100 & -50 & 0 \\ 0 & 0 & -50 & 100 & -50 \\ 0 & 0 & 0 & -50 & 100 \end{bmatrix}, \quad \mathbf{K}_A = \begin{bmatrix} 100 & -50 & 0 & 0 & 0 \\ -50 & 100 & -50 & 0 & 0 \\ 0 & -50 & 100 & -50 & 0 \\ 0 & 0 & -50 & 100 & -50 \\ 0 & 0 & 0 & -50 & 100 \end{bmatrix}.$$

Their corresponding natural frequencies are given by

$$\mathbf{\Omega}_T = \{3.7532,\ 7.0711,\ 10.2090,\ 12.8027,\ 14.4146\},$$
$$\mathbf{\Omega}_A = \{3.6603,\ 7.0711,\ 10.0000,\ 12.2474,\ 13.6603\}.$$

The experimental data consist of the first two natural frequencies and modeshapes. First assume that the modeshapes are accurate. The reference basis algorithm with $\beta = 0$ (since all the masses are identical, this is also the classical reference basis weighting [10]), without connectivity

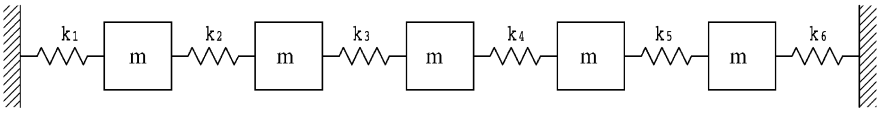

Fig. 1. The system in Example 1.

Table 1
Summary of the various updated models in Example 1

Case	Spring constants	Frequencies
True system	50.0000, 70.0000, 50.0000, 50.0000, 50.0000, 50.0000	3.7532, 7.0711
Analytic model	50.0000, 50.0000, 50.0000, 50.0000, 50.0000, 50.0000	3.6603, 7.0711
No noise, $\beta = 0$	46.2716, 51.6074, 49.4063, 50.1592, 49.9543, 50.0000	3.6236, 6.9948
No noise, $\beta = 0.5$	50.0000, 70.0000, 50.0000, 50.0000, 50.0000, 50.0000	3.7532, 7.0711
0.5% noise	49.8083, 69.8768, 50.0459, 49.7715, 50.5400, 50.0887	3.7542, 7.0660
1% noise	49.1912, 70.2701, 49.0485, 50.0744, 49.9774, 50.0328	3.7423, 7.0413
Sensitivity, $k_1 \div k_6$	50.0000, 70.0000, 50.0000, 50.0000, 50.0000, 50.0000	3.7532, 7.0711
Sensitivity, $k_4 \div k_6$	50.0000, 50.0000, 50.0000, 49.1045, 64.3060, 51.1192	3.7511, 7.0837
Ref. basis, $k_4 \div k_6$	50.0000, 50.0000, 50.0000, 50.0000, 50.0000, 50.0000	3.6603, 7.0711

assignment, resulted in

$$\mathbf{K} = \begin{bmatrix} 97.8790 & -50.6112 & -1.9625 & -1.7271 & -1.0051 \\ -50.6112 & 103.0060 & -48.2799 & 1.5138 & 0.8810 \\ -1.9625 & -48.2799 & 99.8259 & -50.1532 & -0.0892 \\ -1.7271 & 1.5138 & -50.1532 & 99.8652 & -50.0785 \\ -1.0051 & 0.8810 & -0.0892 & -50.0785 & 99.9543 \end{bmatrix}.$$

As can be seen from Table 1, the change from the analytical model is minimal, yet sufficient to match exactly the first two natural frequencies and modeshapes. Applying the a posteriori connectivity assignment with the six spring constants as parameters, leads to the values in the third row of Table 1, which are clearly unsatisfactory results, actually worse than $\mathbf{K}_A$. With $\beta = 0.5$, on the other hand, the updated model becomes

$$\mathbf{K} = \begin{bmatrix} 120.0000 & -70.0000 & 0.0000 & 0.0000 & 0.0000 \\ -70.0000 & 120.0000 & -50.0000 & 0.0000 & 0.0000 \\ 0.0000 & -50.0000 & 100.0000 & -50.0000 & 0.0000 \\ 0.0000 & 0.0000 & -50.0000 & 100.0000 & -50.0000 \\ 0.0000 & 0.0000 & 0.0000 & -50.0000 & 100.0000 \end{bmatrix}.$$

Hence no connectivity assignment or iterative loops are required, and the spring constants assume their accurate values. Fig. 2 shows how this value of β was obtained. The cost $J_{con}(\beta)$ is shown in the solid line. The dashed line represents the normalized true error, $\|\mathbf{K}_T - \mathbf{K}_{con}\|_F / \|\mathbf{K}_{con}\|_F$, which is of course unavailable in real application. It is seen that for $\beta > 0.25$ weighted model updating gives the correct connectivity in a natural manner.

Next consider the case where the modeshapes contain noise with a standard deviation of 0.5%. Fig. 3 shows the same two costs, for that case, which for ease of viewing were normalized with respect to their values at $\beta = 0$. Again there is good correspondence between the available

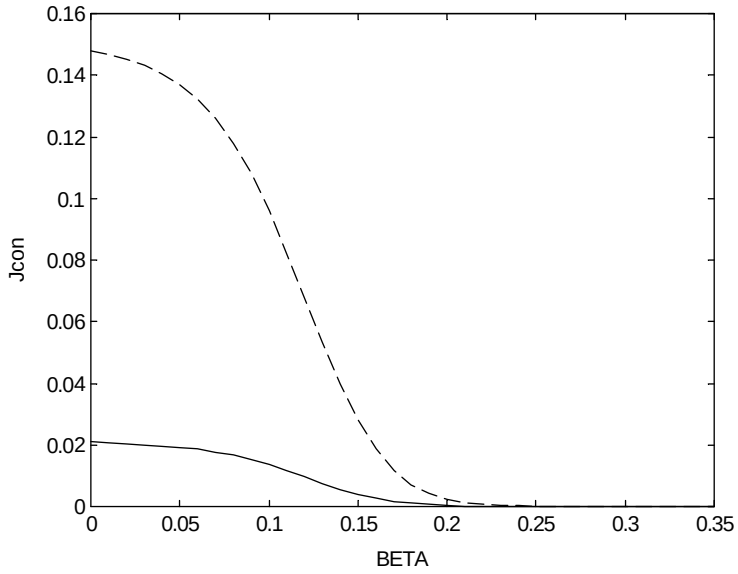

Fig. 2. Connectivity (solid) and updating (dashed) errors for measurements without noise.

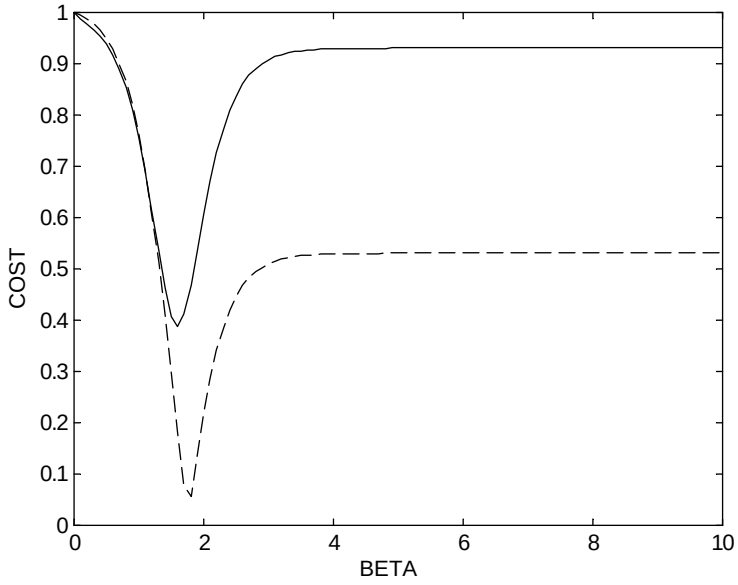

Fig. 3. Normalized connectivity (solid) and updating (dashed) errors for noisy measurements.

connectivity cost, and the unavailable true error. The minimum is obtained at $\beta^* = 1.6$, and the parameters and natural frequencies corresponding to it are given in the fifth row of Table 1. The iterative procedure in step 3 of the algorithm converged, before connectivity

assignment, to

$$\mathbf{K} = \begin{bmatrix} 119.6851 & -70.0047 & 0.0347 & 0.0578 & -0.0020 \\ -70.0047 & 119.6670 & -50.0789 & 0.0677 & 0.0602 \\ 0.0347 & -50.0789 & 100.0072 & -49.8091 & 0.0385 \\ 0.0578 & 0.0677 & -49.8091 & 100.0466 & -50.6725 \\ -0.0020 & 0.0602 & 0.0385 & -50.6725 & 100.6288 \end{bmatrix}.$$

i.e., almost perfect connectivity. Consequently, the identified parameters are very close to the true ones with an average error of 0.4%. Increasing the noise level to 1% leads to an average of 2% error in the parameters.

The problem was solved also using the sensitivity method. Without noise it yields, as expected, the true values of the parameters. Suppose that the parameterization is wrong, and only k_4, k_5, and k_6 are allowed to change. The sensitivity method was able to find a set of parameters, which lead to a fairly good approximation of the frequencies; yet they do not represent the actual change in the system. The proposed algorithm, under the same circumstances, was not fooled by the data and did not move at all, indicating that the parameterization should be modified.

Example 2. The system, shown in Fig. 4, is the one reported in Ref. [3]. As in that work, the mass matrix is assumed to be accurate, and is given by

$$\mathbf{M} = \begin{bmatrix} 0.5423 & 0.0042 & 0.0525 & 0 & 0 \\ 0.0042 & 0.5906 & 0.0042 & 0 & 0 \\ 0.0525 & 0.0042 & 0.7916 & 0.0042 & 0.0525 \\ 0 & 0 & 0.0042 & 0.5906 & 0.0042 \\ 0 & 0 & 0.0525 & 0.0042 & 0.8423 \end{bmatrix}.$$

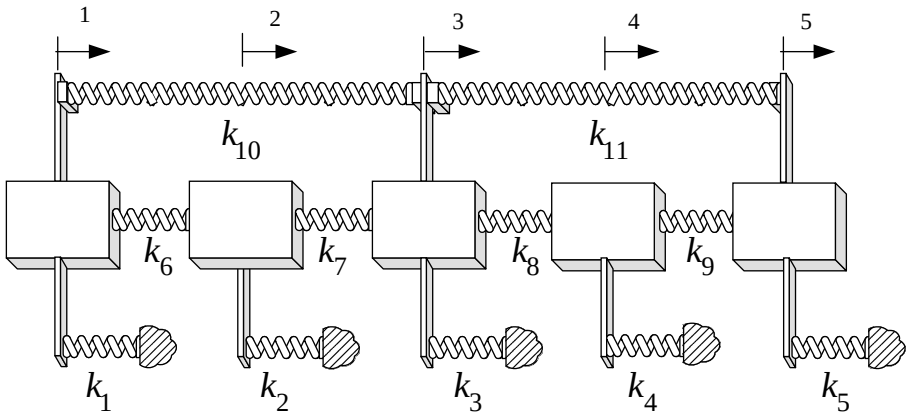

Fig. 4. Lumped model (springs only) of the system in Example 2.

The stiffness matrix has the structure

$$\mathbf{K}_A = \begin{bmatrix} k_1+k_6+k_{10} & -k_6 & -k_{10} & 0 & 0 \\ -k_6 & k_2+k_6+k_7 & -k_7 & 0 & 0 \\ -k_{10} & -k_7 & k_3+k_7+k_8+k_{10}+k_{11} & -k_8 & -k_{11} \\ 0 & 0 & -k_8 & k_4+k_8+k_9 & -k_9 \\ 0 & 0 & k_{11} & -k9 & k_5+k_9+k_{11} \end{bmatrix},$$

where the nominal values of the parameters are

$$k_{1-11} = \{27216,\ 27216,\ 39191,\ 27216,\ 27216,\ 17901,\ 17901,\ 17901,\ 17901,\ 15312,\ 15312\}.$$

All the five natural frequencies and modeshapes were measured. The measured and nominal natural frequencies (in Hz), and the measured modeshapes, are given by

$$\mathbf{\Omega} = \{33.50\ 43.53\ 57.28\ 58.60\ 66.33\}, \quad \mathbf{\Omega}_A = \{31.82,\ 41.48,\ 54.88,\ 59.70,\ 66.08\},$$

$$\mathbf{\Phi} = \begin{bmatrix} 0.9972 & -0.9734 & -0.0657 & -0.9165 & -0.6639 \\ 0.9934 & -0.9736 & 0.2565 & 1.0000 & -0.1130 \\ 1.0000 & 0.0309 & -0.1664 & -0.0963 & 1.0000 \\ 0.8513 & 0.9907 & 1.0000 & -0.3683 & -0.1593 \\ 0.9064 & 1.0000 & -0.7634 & 0.4106 & -0.5599 \end{bmatrix}.$$

The results of applying the proposed updating algorithm, assuming one to five measured natural frequencies and modeshapes are given in Tables 2 and 3. Values that correspond to measured data appear in bold. The changes in the parameters are shown in Fig. 5. These values were compared to the results in Ref. [3], which were obtained using the sensitivity method with regularization. In the case of full information (five modes), the frequencies were matched almost identically (less than 0.05% error). The modeshapes were also matched with great accuracy (the diagonal **MAC** values deviate from 1 by less than 0.03). The largest error is in the third modeshape. However, this modeshape violates the orthogonality with respect to the mass matrix; hence the source of the inability to match it is the data itself, or equivalently, in the assumption that the mass matrix is accurate. In the case of five measured modes, the following changes of the

Table 2
Deviations between measured and updated natural frequencies in Example 2

Case	Deviations (%)				
Analytical	−5.0094	−4.6912	−4.1767	1.8847	−0.3683
1 mode	**0.0000**	0.3187	−3.4468	1.9173	0.4624
2 modes	**−0.6368**	**0.5466**	−1.0396	1.9261	1.6067
3 modes	**−0.1431**	**0.1184**	**0.0139**	−0.5012	0.4480
4 modes	**−0.1072**	**0.0879**	**0.2387**	**−0.2451**	0.2064
5 modes	**−0.0065**	**0.0212**	**0.0001**	**0.0109**	**0.0015**
Sensitivity	**0.0049**	**-0.0581**	**−0.0037**	**−0.0542**	**−0.0306**

Table 3
MAC values of measured and updated modeshapes in Example 2

Case	**MAC** (diagonal)				
Analytical	0.9436	0.8896	0.8079	0.8083	0.8991
1 mode	**1.0000**	0.9882	0.8950	0.8396	0.9447
2 modes	**0.9990**	**0.9986**	0.9578	0.8809	0.9770
3 modes	**0.9989**	**0.9987**	**0.9875**	0.9429	0.9833
4 modes	**0.9986**	**0.9987**	**0.9895**	**0.9529**	0.9880
5 modes	**0.9990**	**0.9952**	**0.9722**	**0.9875**	**0.9974**
Sensitivity	**0.9965**	**0.9854**	**0.9855**	**0.9786**	**0.9858**

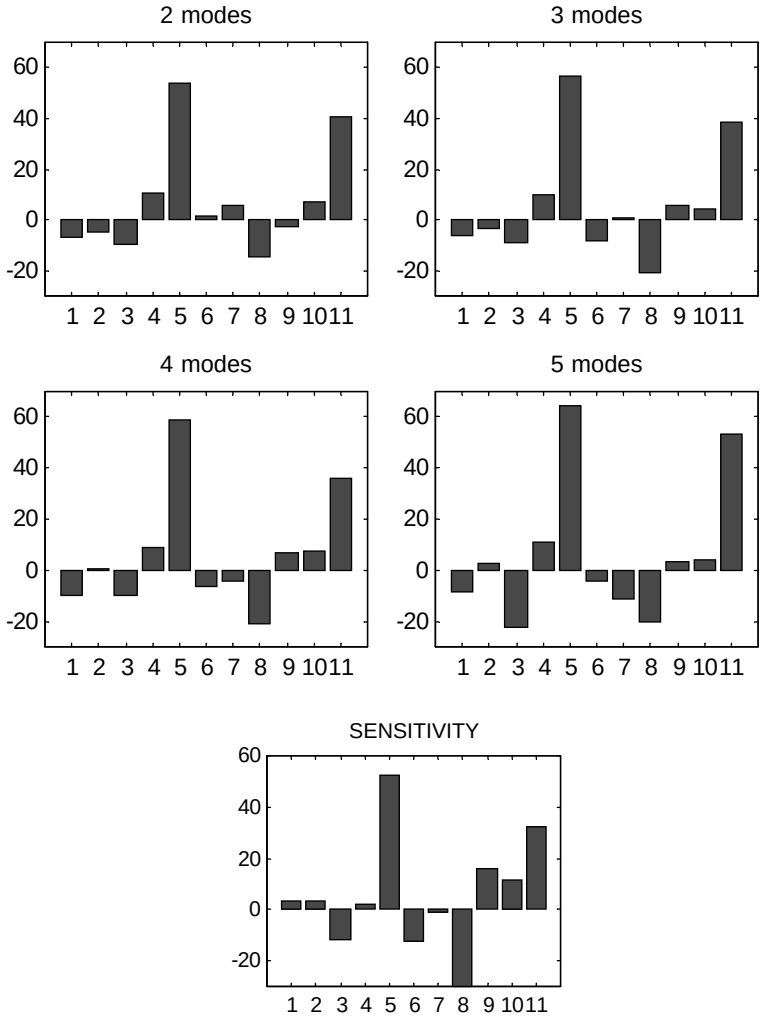

Fig. 5. Parameters change (in %) for different number of measurements and for sensitivity method with five modes.

spring constants (in %) were obtained:

$$\Delta k = \{-8.3,\ 2.1,\ -12.0,\ 8.2,\ 61.6,\ -2.2,\ -14.7,\ -22.8,\ 6.1,\ 1.9,\ 51.5\}.$$

The correction in Δk is somewhat larger than the results in Ref. [3], where the natural frequencies were also exactly matched. However, in the solution presented here, the modeshapes of the updated model are slightly closer to the measured ones.

Fig. 6 demonstrates the algorithm for the case where only the first two natural frequencies and modeshapes are measured. The minimum value of the connectivity cost occurs at $\beta^* = 4.4$. This value was then used to obtain the results in the third row of Tables 2 and 3.

The example was used also to study the effect of different parameterizations. In Ref. [3], the following reduced parameterization is suggested:

$$\alpha_1 = \Delta k_1 = \Delta k_2 = \Delta k_4,\ \alpha_2 = \Delta k_3,\ \alpha_3 = \Delta k_5,\ \alpha_4 = \Delta k_6 = \Delta k_7 = \Delta k_8,\ \alpha_5 = \Delta k_9,$$
$$\alpha_6 = \Delta k_{10},\ \alpha_7 = \Delta k_{11}, \tag{31}$$

where Δ represents relative change. This parameterization was selected based on sensitivity considerations and yielded good results. It is compared with the arbitrary parameterization,

$$\alpha_1 = \Delta k_1 = \Delta k_2 = \Delta k_3,\ \alpha_2 = \Delta k_4,\ \alpha_3 = \Delta k_5 = \Delta k_7,\ \alpha_4 = \Delta k_6,\ \alpha_5 = \Delta k_8,\ \alpha_6 = \Delta k_9,$$
$$\alpha_7 = \Delta k_{10} = \Delta k_{11}. \tag{32}$$

Since the parameters define relative change, lumping together springs with different constants is legitimate. The two plots of the connectivity costs for two measured modes are shown in Fig. 7. The superiority of parameterization (31) is evident, and indeed, when it was used, the deviation

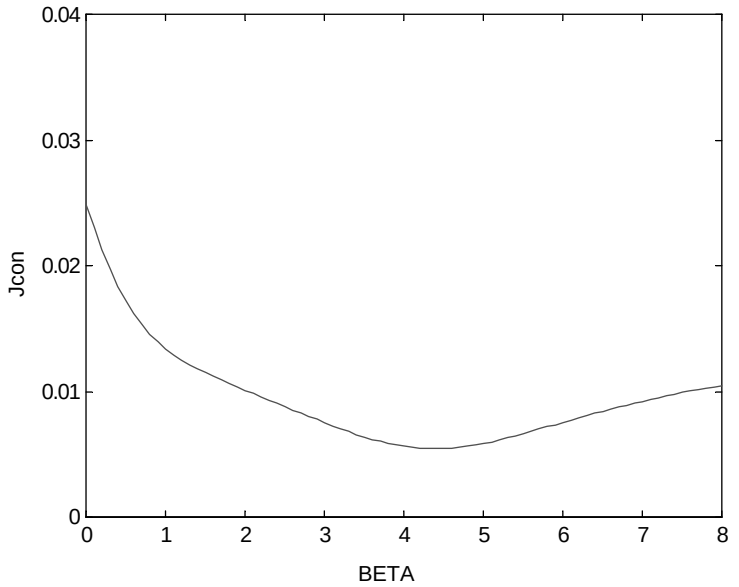

Fig. 6. $J_{con}(\beta)$ with measurements of two modes and full parameterization.

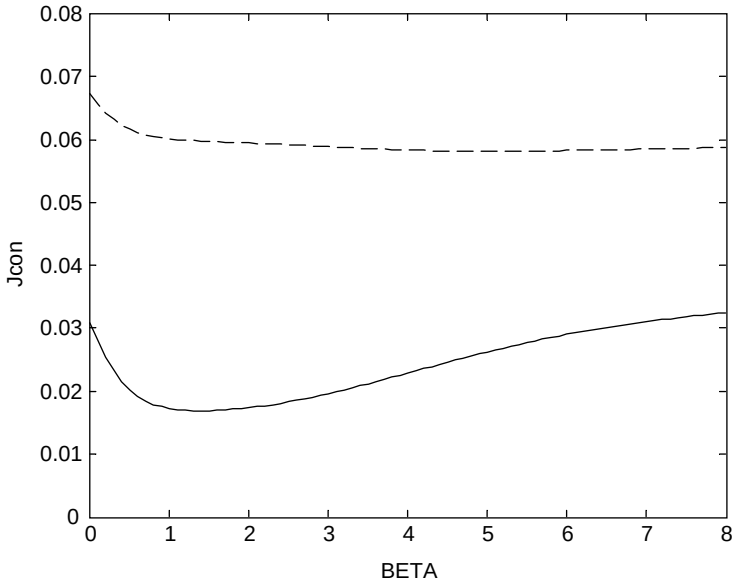

Fig. 7. $J_{con}(\beta)$ with measurements of two modes and the reduced parameterizations. Solid—(31), dashed—(32).

(in %) in the natural frequencies is

$$\Delta\omega = \{-0.0410,\ 0.3529,\ -1.8122,\ 1.7915,\ 0.3391\},$$

as compared to close to 10% deviation using parameterization (32). It should be noted though, that the **MAC** values are slightly worser than in the case of 11 parameters, especially in the third and fourth modeshapes. The changes in the parameters (in %) were

$$\alpha = \{-0.03,\ -9.86,\ 57.46,\ -0.46,\ -4.49,\ 0.96,\ 21.77\}.$$

The small change in α_1, and to a lesser degree in α_4 and α_6, indicates that similar results can be obtained with even a smaller number of parameters. Using the parameterization

$$\alpha_1 = \Delta k_3,\ \ \alpha_2 = \Delta k_5,\ \ \alpha_3 = \Delta k_9,\ \ \alpha_{11} = \Delta k_{11}, \tag{33}$$

one obtains

$$\Delta\omega = \{-0.0021,\ 0.3495,\ -1.8476,\ 1.9068,\ 0.2724\}$$

with

$$\alpha = \{-9.75,\ 57.70,\ -4.78,\ 21.08\}.$$

Example 3. The system is the truss shown in Fig. 8, which was reported in Ref. [5]. The finite element model consist of 34 d.o.f.s. In the first case, the only change is that the Young's modulus of member A was increased by 50%. Using six parameters, the method found their correct value (changed or unchanged) with an accuracy of 1%. Plots of the connectivity cost, with a minimal amount of noise to avoid singularities, and the true cost (using the true model) are shown in Fig. 9. In this figure, both costs are visibly similar. The effect of the noise level

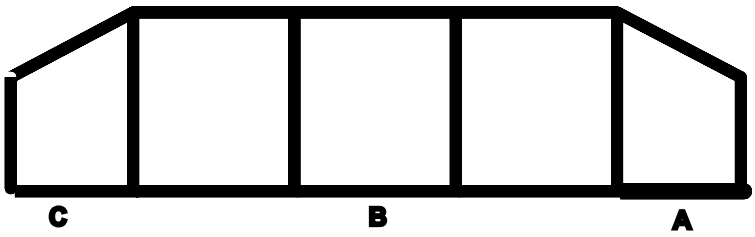

Fig. 8. The truss in Example 3.

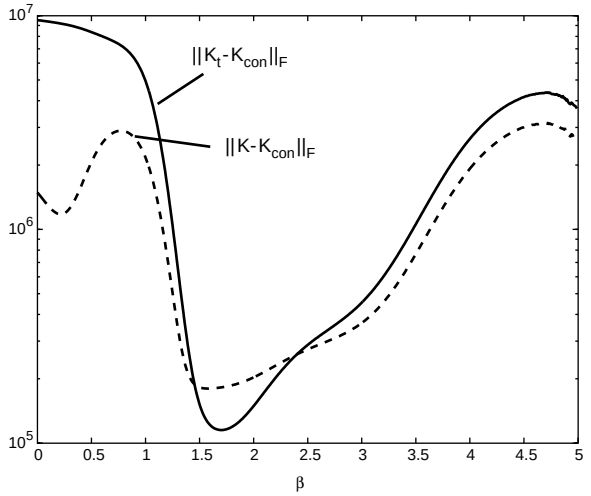

Fig. 9. Connectivity cost and updating cost, noise level = 0.01%.

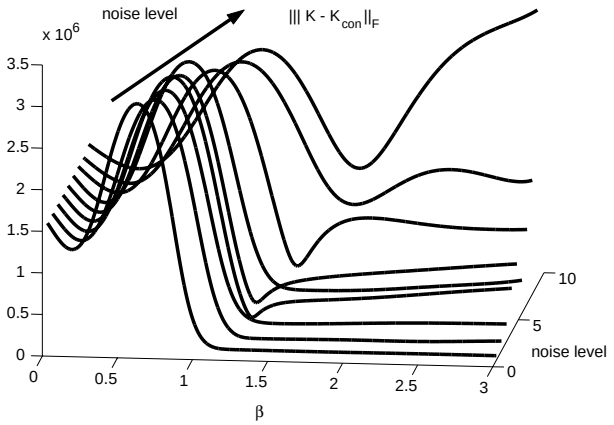

Fig. 10. The effect of measurement noise on the connectivity cost and the optimal β.

on the shape of the connectivity cost curve is shown in Fig. 10. Looking at the second local minimum, which is the global minimum, one sees the same phenomenon as in Figs. 2 and 3. With low level of noise, the connectivity cost is flat for large values of β, but it curves upwards when the noise level is increased. In the second case, three parameters, which are marked as A, B and C in Fig. 8, were used. Their deviations from the analytic model were 50%, −30%, and 0%, respectively. Using six measured modes, the parameters converged to the correct value.

7. Conclusion

The paper presents a model updating method, which brings together the reference basis and the parametric approach. The result, model updating with a posteriori connectivity assignment, maintains the advantages of the classical reference basis method and avoids some of its disadvantages. The main idea of the paper, appearing in different aspects of the algorithm, is the inter-relationships existing between the unconstrained and the constrained optimization problems. In a way, each one serves as feedback to the other; thus the algorithm results in a model which satisfies the connectivity constraints on one hand, but has a strong relationship with the unconstrained reference basis method. The basic principle that is applied, namely reducing the measurable connectivity cost as a means of reducing the unmeasurable updating error, was shown to be in agreement with the examples.

The algorithm includes iterative steps. One which is fundamental is the search for the optimal weighting matrix (the optimal parameter β). An optional step is the final iterations with fixed β. Since the amount of calculation required for each update is very small, the computational load of the overall algorithm, even with a large number of iterations, is still low.

The algorithm works with a given parameterization, but gives also indication to its adequacy. The cost of parameterization can be used for a search for the most adequate connectivity parameterization, and to rule out inadequate ones.

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Appendix A. Derivation of Eq. (25)

The optimization criterion (23) can be written as

$$\min_{\alpha_j} J = \mathrm{tr}\left[\left(\sum_{i=1}^{p}\alpha_i\mathbf{K}_i - (\mathbf{K}-\mathbf{K}_A)\right)^{\mathrm{T}}\left(\sum_{i=1}^{p}\alpha_i\mathbf{K}_i - (\mathbf{K}-\mathbf{K}_A)\right)\right]. \tag{A.1}$$

Using the linearity of the trace operator, and the symmetry of $\mathbf{K}$

$$\partial J/\partial \alpha_j = 2\,\mathrm{tr}\left[\left(\sum_{i=1}^{p} \alpha_i \mathbf{K}_j \mathbf{K}_i - \mathbf{K}_j(\mathbf{K}-\mathbf{K}_A)\right)\right]. \tag{A.2}$$

Hence, for the optimal parameters

$$\mathrm{tr}\left[\left(\sum_{i=1}^{p} \alpha_i \mathbf{K}_j \mathbf{K}_i\right)\right] = \mathrm{tr}\left[\left(\sum_{i=1}^{p} \mathbf{K}_j(\mathbf{K}-\mathbf{K}_A)\right)\right]. \tag{A.3}$$

Interchanging summation and trace, one obtains

$$\left(\sum_{i=1}^{p} \alpha_i\,\mathrm{tr}(\mathbf{K}_j\mathbf{K}_i)\right) = \left(\sum_{i=1}^{p} \mathrm{tr}(\mathbf{K}_j(\mathbf{K}-\mathbf{K}_A))\right), \quad j=1,\ldots,p. \tag{A.4}$$

Or, in a vector-matrix notation

$$\begin{bmatrix} \mathrm{tr}(\mathbf{K}_1\mathbf{K}_1) & \mathrm{tr}(\mathbf{K}_1\mathbf{K}_2) & \mathrm{tr}(\mathbf{K}_1\mathbf{K}_3) \\ \vdots & \mathrm{tr}(\mathbf{K}_2\mathbf{K}_2) & \vdots \\ \mathrm{tr}(\mathbf{K}_p\mathbf{K}_1) & \cdots & \mathrm{tr}(\mathbf{K}_p\mathbf{K}_p) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\mathbf{K}_1(\mathbf{K}-\mathbf{K}_A)) \\ \vdots \\ \mathrm{tr}(\mathbf{K}_p(\mathbf{K}-\mathbf{K}_A)) \end{bmatrix}. \tag{A.5}$$

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