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Response of a large plate–liquid system to a moving pressure step. Transient and stationary aspects

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Abstract

An unbounded system of a plate in contact with a liquid is presented. The coupling equations are set, accounting for the compressibility of the liquid and the Mindlin–Reissner plate theory. A numerical solution, founded on an explicit scheme, is validated. The resolution is applied to a strip loaded by a unit pressure step spreading uniformly along the strip axis. The simulation for every speed of loading ranging among the characteristic velocities of the coupled system points out how a steady state response can emerge from the transient one. The steady state solution is theoretically established, which agrees well with the numerical prevision. The form of the response is different for every region limited by the characteristic velocities. For a load speed greater than the speed of acoustic waves in the liquid, the pressure propagates along a straight line and the existence of a steady state response is confirmed. On the contrary, for a lower load speed, no pressure front is present and the response always keeps its transient character. The solution shows that displacements obtained in the transient part increase incessantly with time and become quickly larger than in the stationary part. Concerning the stresses, the study reveals that the amplitudes are of the same order in the two parts. The solution can be extended easily to the case of a pressure spreading with a cylindrical symmetry, which can correspond to the real conditions of detonation loading.

1. Introduction

The present work finds its place in the field of wave propagation and fluid–structure coupling. At the beginning, the problem is set to find the response of a plate in contact with a liquid when a pressure moves on the plate with a high velocity, possibly of the same order as sound velocities in the plate or in the liquid. This problem is often encountered in cases of shocks and explosions. The

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shock wave caused by a detonation can reach several thousand metres per second and presents an absolutely steep front, equating to an ideal step. To better address the problem and be able to form some general conclusion, the case is considered of a very large coupled unbounded domain so that the propagation of motion is not impeded by physical limits. Although fluid–structure coupling has been studied widely, it seems to have never been considered exactly under this assumption.

Among couplings of large plate–liquid systems, the most often encountered is that of an ice sheet on a wide surface of water. A lot of works refers to this study and a recent one of Nugroho et al. [1] provides numerous references. The usual assumption in this problem is to consider the liquid incompressible and the plate under the classical flexion theory. The liquid has a constant depth. Most often, the purpose is to find the response to various transport systems moving on the ice such as trucks or aeroplanes. The problem is well adapted to the study of stationary responses.

Another coupling concerning large plates has been considered recently by Abrahams and Wickham [2]. It deals with the study of plates in contact with a moving liquid. An impulsively started moving load hits the plates and the resultant oscillations are studied. An important theoretical work has been achieved, supported by numerical computation. In this case too, the liquid is considered incompressible and the plate under the classical flexion theory.

The last two examples considered large plate–fluid systems; in that sense, they have similarity with the present problem. Nevertheless, the response to high-velocity loading, possibly supersonic, must be obtained by accounting for compressibility of the liquid.

Under the same assumptions of the Kirchhoff theory of plates and incompressible liquid, Amabili and Kwak [3] presented a study of the vibration of circular plates on a free fluid surface. They proposed solutions to axisymmetric plates with various boundary conditions by the way of the Hankel transformation. To approach the coupled problem, it is already necessary to understand the response to a moving load of a plate alone, in the absence of coupling. In this field, the most important work has been realized in the study of beams on an elastic foundation. A lot of studies have been achieved on this topic. A recent work, by Felszeghy [4], reviews 18 specialized references extensively. The most significant results, useful in this context, are recalled.

The theoretical study of a beam loaded by a moving force or pressure step always refers to a steady state response, which would appear stationary to an observer travelling with the load. Then, the beam is imagined to be infinite in extent in front of and behind the position of loading. More generally, the beam is supposed to be resting on an elastic foundation, but the case of a completely free beam is also envisaged as the degenerated case of an infinitely soft foundation. The transient responses of beams, especially when they are semi-infinite, can be sought by means of integral transform methods. Nevertheless, even in the more simple cases, the treatment results in such an analytical complication that only an asymptotic study of the solution is practical. All the results in the expression of the whole transient response were obtain by means of a computational method, even the last by Felszeghy [5].

To study the response of a beam to a dynamical loading, the simplest model is that of the classical flexion by Euler–Bernoulli. The deficiency of this model is that it accounts, neither for inertia in longitudinal displacements due to cross-section rotations, nor for transverse shear deformation. This model can be used only for long wavelengths, much greater than the height of the beam. To make up for these defects, the Timoshenko modelling must replace it. This model is able to account for short wavelengths, of the same order as the height of the beam. For shorter wavelengths, only a three-dimensional theory can be used, which negates the simplicity of beam theories. According to these comments, well known in beam dynamics, one refers here only to the Timoshenko beam theory.

In looking for harmonic flexural waves in a beam, a dispersion equation can be established, which expresses the phase velocity versus circular frequency. If a beam, infinite in extent, is submitted to a periodic excitation, the response, after the vanishing of transient parts, is the superposition of harmonic waves. Each one travels at its own phase velocity, according to the dispersion equation. Several phase velocities can correspond to the same frequency. If a beam, infinite in extent, is submitted to a constant loading travelling at a constant velocity, the problem is somewhat different. After the vanishing of transient parts, the response can appear frozen in co-ordinates moving with the loading. Such a solution, if it exists, is called stationary or sometimes quasi-stationary. To make the understanding of the solution easier, one refers to the simplest loading, which is a travelling localized impulse (the solution to this basic loading allows to find the response to any loading by convolution).

Contrary to the previous case, in the quasi-steady state case, the loading imposes the phase velocity of the response and not its frequencies. If a quasi-stationary harmonic solution exists, its phase velocity equals the loading velocity. Thus, to analyze the possibility of a quasi-stationary response, the phase velocity must first be imposed in the dispersion equation. Only then can the possible corresponding frequencies be searched. The dispersion equation of the Timoshenko beam reveals the importance of two particular velocities, v_b and $v_s, (v_b > v_s)$, so-called characteristic, respectively the bar velocity and the modified shear velocity. For a phase velocity ranging from 0 to v_s only one frequency belongs to the interval $]0, +\infty[$. For a phase velocity greater than v_b only one frequency belongs to the interval $]\omega_c, +\infty[$, ω_c being a cut-off frequency corresponding to an infinite phase velocity. For a phase velocity belonging to the interval $]v_s, v_b[$, the dispersion equation has non-real roots. It means that a harmonic solution cannot appear; the propagating wave is changed into a near field. The demonstration of the existence of a stationary response for a Timoshenko beam to a travelling loading can be found extensively in Felszeghy [4]. To be valid, the study must exclude the neighbourhoods of the characteristic velocities for which the wavelengths of responses should be much smaller than the height of the beam.

Another significant result of the review is that it has been established that the transient solution approaches the steady state solution asymptotically, for all ranges of load speed, at least within a continuously expanding interval centred about the moving load. All these conclusions from previous studies on beams alone confirm the interest of the research of a steady state solution for coupled systems. It is hoped, by these means, to be able to describe the behaviour in the neighbourhood of the front of loading theoretically. For plates, the Mindlin–Reissner assumption corresponds to the Timoshenko theory for beams and will be chosen.

2. Statement of problem

To set the problem, consider a large plate in contact with a great quantity of liquid. The dimensions of the plate are assumed so large and the amount of liquid so huge that the coupled system seems unbounded when observed not far from the plate. The system can be imagined as the wall of a large filled tank or like an ice sheet on a wide and deep lake. The whole system is at rest when suddenly a pressure spreads on the external surface of the plate like an explosion. Using this example of loading justifies exploration of a large range of velocities—including supersonic—with respect to the sound velocities in the plate or in the liquid. At this level of presentation, it can be useful to give the range of some physical values in the topic of explosions and dynamics of plates.

Concerning explosions, the focus will be on detonation, which is deterministic, contrary to deflagration. Whether they are caused by gaseous mixtures or solid explosives like TNT, their external effects in terms of the pressure field are exactly the same, just quantified by an energetic equivalence factor. Inside an explosive cloud, for example a propane–oxygen mixture, in the ambient conditions, the overpressure step of the shock wave can reach more than 3 MPa and the velocity of the front more than 2500 m/s, if a detonation is initiated. Outside the cloud, at its boundary, a shock wave propagates, launched in the air at a supersonic velocity, remaining supersonic until the discontinuity of thermodynamics values ahead and behind the front vanishes. Only then, and after the decay due to a three-dimensional (3-D) or 2-D expansion, the perturbation in air becomes acoustic. For a strong detonation, the shock front velocity remains supersonic for a long distance. If an aerial detonation is considered, at a given height from the ground, a spherical shock wave propagates from the centre of explosion, supersonic at its beginning. When this 3-D shock wave hits the ground, the pressure on the ground propagates concentrically with a decreasing velocity [6]. At the exact time of contact of the sphere with the plane, the spreading speed of the pressure on the plane is theoretically infinite. Then, it decreases, like the overpressure step.

Concerning plates, the speed of sound in a constitutive material is generally well known. For usual metallic materials such as steel or aluminium, the speed is among the highest, more than 5000 m/s. For polymeric materials, the speed of sound can be much less: 1700 m/s for pvc and even less for some others. For ice, which has been mentioned earlier in this context, an intermediate value of 2600 m/s is acceptable.

Concerning liquids, for this paper only the most common speed of sound in water is used, i.e., 1500 m/s.

All these physical values of speed being recalled, it seems clear that all arrangements of them are possible. Especially, if one considers, for the loading, the intersection of a 3-D spherical shock wave with a plane, the spreading speed of the pressure on the plane can exceed, at least for a moment, any possible value of sound velocity for a given material. To conclude this physical aspect of loading by detonations, it is emphasized that, in a practical range, few detonations are able to conserve constant overpressure step and velocity, particularly due to geometrical expansion. Nevertheless, this is possible in two cases: in a 1-D guide, like a tunnel, or inside a homogeneous explosive gaseous cloud.

A previous (recent) work by Renard and Taazount [7] presented results obtained for the dynamic response of the plate alone, without any coupling. This work contained details, which could aid the comprehension of the present paper. In its conclusion, it underlined that the response of a plate to a pressure spreading cylindrically around a centre point, like that of a detonation, is not far from the response of a strip loaded by a pressure travelling along its axis. For this reason, this work will be limited to the study of loads moving axially along strips and using only one space variable.

Precisely, the response of the coupled system is supposed independent of the y co-ordinate, measured on the transverse direction of the strip. For the plate, it is described only in terms of the x co-ordinate measured on its longitudinal axis. To describe the liquid, the z co-ordinate is introduced. The studied strip is considered infinitely long and free, only required to be in permanent contact with the liquid. It is assumed to be at rest at time $t = 0$ and subject to the external pressure loading $p(x, t)$ for $t > 0$.

2.1. Plate governing equations

Fig. 1 presents the geometry and co-ordinates of a plate and liquid coupled system. The plate has a mass density ρ and a thickness h. Its Young's modulus is E, Poisson ratio v, shear modulus G, and shear correction factor κ.

Two displacements are necessary to describe the motion of the strip: $w(x, t)$, the deflection or displacement of the neutral axis, and $\Psi(x, t)$, the angular rotation of the cross-section. The latter function must be distinguished from the slope of the neutral axis, according to Mindlin–Reissner assumption.

The stresses induced by the bending can be deduced from w and Ψ. Let σ be the flexural stress on the external surface and τ the average shear stress in the cross-section.

To simplify the analysis of the problem, the following non-dimensional variables are introduced:

$$r_0 = \sqrt{I/A} = h/\sqrt{12}, \tag{1}$$

$$X = x/r_0, \quad W = w/r_0, \tag{2}$$

where A represents the cross-sectional area and I its moment of inertia. Using the velocity of longitudinal propagation in a plate

$$v_p = \sqrt{E/(\rho(1 - v^2))}, \tag{3}$$

and the modified shear wave velocity

$$v_s = \sqrt{\kappa G/\rho}, \quad \text{with } \kappa = 0.86 \text{ (Reismann [8])}, \tag{4}$$

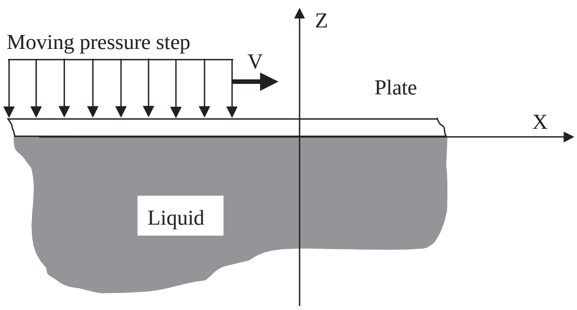


Fig. 1. Geometry of the coupled plate and liquid.

By setting

$$\theta = v_s/v_p, \quad T = tv_p/r_0, \quad P = p/(\rho V_p^2\sqrt{12}), \quad \Sigma = \sigma/(\rho V_p^2\sqrt{12}), \tag{5}$$

the non-dimensional equations of the movement are written in the well-known form

$$\partial^2 W/\partial T^2 = \theta^2(\partial^2 W/\partial X^2 - \partial\Psi/\partial X) + P(X,T), \tag{6}$$

$$\partial^2\Psi/\partial T^2 = \partial^2\Psi/\partial X^2 + \theta^2(\partial W/\partial X - \Psi), \tag{7}$$

with the initial conditions of rest

$$W(X,0) = 0, \quad \psi(X,0) = 0, \quad \partial W(X,0)/\partial T = 0, \quad \partial\Psi(X,0)/\partial T = 0. \tag{8}$$

The non-dimensional forms Σ and Γ correspond to the dimensional forms σ and τ.

$$\Sigma = -\tfrac{1}{2}\partial\Psi/\partial X, \quad \Gamma = (\theta^2/\sqrt{12})(\partial W/\partial X - \Psi). \tag{9}$$

In the study, the variables will be taken in their non-dimensional form, denoted in capital letters.

2.2. Liquid-governing equations

The liquid is supposed inviscid. It is submitted to fast loading and must be supposed compressible. Using the potential φ, one writes

$$\partial^2\varphi/\partial t^2 = v_1^2\nabla^2\varphi, \tag{10}$$

where v_1 represents the velocity of acoustic waves.

$v_1^2 = B/\rho_1$, B being the bulk compressibility modulus of the liquid and ρ_1 its mass density. The pressure and speeds are deduced by

$$p = \rho_1\partial\varphi/\partial t, \quad \partial u/\partial t = -\partial\varphi/\partial x, \quad \partial w/\partial t = -\partial\varphi/\partial z, \tag{11}$$

where u and w are the displacements in the liquid along, respectively, in the x and z co-ordinate directions. To obtain a non-dimensional form compatible with those chosen for the plate, new non-dimensional variables are introduced as

$$\Phi = \varphi/v_p r_0, \quad Z = z/r_0, \quad \delta = v_1/v_p, \quad \mu = \rho_1/\rho\sqrt{12}. \tag{12}$$

Under these conditions, and by considering that every function is independent from y, the new non-dimensional equations can be written, also using the plate characteristics as

$$\partial^2\Phi/\partial T^2 = \delta^2(\partial^2\Phi/\partial X^2 + \partial^2\Phi/\partial Z^2), \tag{13}$$

$$\partial U/\partial T = -\partial\Phi/\partial X, \quad \partial W/\partial T = -\partial\Phi/\partial Z, \quad P = \mu\partial\Phi/\partial T. \tag{14}$$

One assumes the liquid at rest at $T = 0$, then the derivatives of Φ must be null at this time.

2.3. Equations of coupling

In the assumed absence of cavitation, the continuity of forces and normal displacement at the interface needs to be ensured to account for coupling between the plate and the liquid. These conditions are summarized by the two equations valid on the interface at any time

$$P_{int} = \mu(\partial\Phi/\partial T)_{(Z=0)}, \quad (\partial W/\partial T)_{(\text{Plate})} = -(\partial\Phi/\partial Z)_{(Z=0)}, \tag{15}$$

where P_{int} is the internal pressure under the plate, and P_{ext} the external loading on the plate. According to these conditions, the coupled equations can be written as

$$\partial^2 W/\partial T^2 = \theta^2(\partial^2 W/\partial X^2 - \partial\Psi/\partial X) + \mu(\partial\Phi/\partial T)_{(Z=0)} + P_{ext}(X,T), \tag{16}$$

$$\partial^2\Psi/\partial T^2 = \partial^2\Psi/\partial X^2 + \theta^2(\partial W/\partial X - \Psi), \quad (\partial W/\partial T) = -(\partial\Phi/\partial Z)_{(Z=0)},$$
$$\partial^2\Phi/\partial T^2 = \delta^2(\partial^2\Phi/\partial X^2 + \partial^2\Phi/\partial Z^2). \tag{17−19}$$

The first three equations are valid for the plate and on the boundary of the liquid while the fourth is valid in the fluid domain and on its boundary.

3. Numerical solution and simulation

The previous system of coupled equations could take the form of an evolving system and be able to receive a solution by an explicit scheme of integration in time if all terms including it could be isolated.

3.1. An explicit scheme for time integration

To solve propagating equations, that is to say hyperbolic partial-derivative systems, an explicit scheme for time integration can be used, while the Courant–Freidrichs–Levy (CFL) stability condition is fulfilled. The central finite difference method is among the simplest and very efficient ways of resolution. It has been used with success in similar problems concerning the dynamics of plates alone [7]. In the case of coupling, the CFL condition must be verified for every domain. If the second order central finite-difference method is applied to the plate and to the liquid, it is justified that the same order of precision be used to write the interface condition. Care must be taken also to use only centred differences on the interface. This is possible by introducing virtual points beyond the boundary of the liquid. Then, neither loss in precision nor instability will happen.

To set the interface condition, it is necessary to use Eq. (19) while accounting for Eq. (18). Setting the development at the second order of the potential Φ, in the neighbourhood of the interface, one writes for every X and every T, and for an arbitrary small (negative) value of dZ

$$\Phi(\mathrm{d}Z) = \Phi(0) + \mathrm{d}Z(\partial\Phi/\partial Z)_{(Z=0)} + (\mathrm{d}Z^2/2)(\partial^2\Phi/\partial Z^2)_{(Z=0)}. \tag{20}$$

Introducing Eq. (18),

$$\Phi(\mathrm{d}Z) = \Phi(0) - \mathrm{d}Z(\partial W/\partial T) + (\mathrm{d}Z^2/2)(\partial^2\Phi/\partial Z^2)_{(Z=0)}, \tag{21}$$

so on the interface, Eq. (19) becomes

$$\partial^2\Phi/\partial T^2 = \delta^2(\partial^2\Phi/\partial X^2 + (2/\mathrm{d}Z^2)(\Phi(\mathrm{d}Z) - \Phi(0) + \mathrm{d}Z\partial W/\partial T)). \tag{22}$$

Now it is possible to have the required form with all time-dependent terms isolated. For the plate and on the interface

$$\frac{\partial^2 W}{\partial T^2} - \mu\left(\frac{\partial \Phi}{\partial T}\right)_{(Z=0)} = \theta^2\left(\frac{\partial^2 W}{\partial X^2} - \frac{\partial \Psi}{\partial X}\right) + P_{\mathrm{ext}}(X,T) = A, \tag{23}$$

$$\partial^2 \Psi/\partial T^2 = \partial^2 \Psi/\partial X^2 + \theta^2(\partial W/\partial X - \Psi) = B, \tag{24}$$

$$\frac{\partial^2 \Phi}{\partial T^2} - \frac{2\delta^2}{\mathrm{d}Z}\frac{\partial W}{\partial T} = \delta^2\left(\frac{\partial^2 \Phi}{\partial X^2} + \frac{2}{\mathrm{d}Z^2}(\Phi(\mathrm{d}Z) - \Phi(0))\right) = C, \tag{25}$$

and in the liquid

$$\partial^2 \Phi/\partial T^2 = \delta^2\left(\partial^2 \Phi/\partial X^2 + \partial^2 \Phi/\partial Z^2\right) = D. \tag{26}$$

In order to solve these equations numerically, it is necessary to use discrete values for all the functions W, Ψ and Φ. A regular mesh of sizes $\Delta X, \Delta Z, \Delta T$, can be chosen and the function evaluated at every node. The differential operators on functions can be replaced by centred finite differences on discrete values, according to a well-known technique. For each time T, the second members A, B, C and D of Eqs. (23)–(26) can be evaluated easily. For any discrete function in space at a point M_i, and called $f(T)$, one sets $f(T+\Delta T) = f_f$ (following value), $f(T) = f$ (actual value), $f(T-\Delta T) = f_p$ (previous value). According to this notation, the results of the operators are

$$\partial^2 f/\partial T^2 = (f_f - 2f + f_p)/\Delta T^2, \quad \partial f/\partial T = (f_f - f_p)/(2\Delta T). \tag{27}$$

Setting the arbitrary value $\mathrm{d}Z = \Delta Z$ in Eq. (25), Eqs. (23) and (25) can be written as

$$(\mathbf{M}_1)\begin{pmatrix} W_f \\ \Phi_f \end{pmatrix} = (\mathbf{M}_2)\begin{pmatrix} W \\ \Phi \end{pmatrix} + (\mathbf{M}_3)\begin{pmatrix} W_p \\ \Phi_p \end{pmatrix} + \begin{pmatrix} A \\ C \end{pmatrix}, \tag{28}$$

where $(\mathbf{M}_1), (\mathbf{M}_2)$ and $(\mathbf{M}_3)$ are (2×2) matrices of fixed numbers.

Multiplying by $(\mathbf{M}_1)^{-1}$ and rearranging, one obtains, a_i and b_i being constant coefficients

$$\begin{aligned} W_f &= a_1 W + a_2 W_p + a_3 \Phi + a_4 \Phi_p + a_5 A + a_6 B, \\ \Phi_f &= b_1 W + b_2 W_p + b_3 \Phi + b_4 \Phi_p + b_5 A + b_6 B. \end{aligned} \tag{29}$$

These two equations are valid for any point on the plate and on the boundary of the liquid. Eq. (24) leads directly to the following values Ψ_f, valid for any point of the plate. Eq. (26) leads directly to the following values Φ_f, valid for any point in the liquid. Under this form, the system can be solved in an explicit way. The values of the functions are calculated for any successive times separated by ΔT. The CFL stability condition will be satisfied if ΔT is chosen as the minimum of $\Delta X/(4\theta^2\Delta X^2 + 1)^{1/2}$ (valid for the plate) and $\Delta X/\delta\sqrt{2}$ (valid for the liquid). The discrete value function should be able to account for boundary conditions if the domain was not infinite. The initial conditions are also easily introduced. They are particularly simple if the system is at rest at the beginning of loading.

3.2. First simulations and their analysis

The simulation of responses of plates coupled with liquid can be made in such a large field of possibilities that it is necessary to choose among the most significant and demonstrative.

To keep a large generality, the responses are presented for infinite domains only—large plates or long strips and deep fluids. This choice is realistic, even for bounded domains, if the response is observed briefly, before the boundaries are reached.

Another fundamental choice is the form of the loading. For several reasons, the presentation will correspond to a pressure spreading from a central point with a constant velocity. The first reason is that a spreading pressure, with high velocity and a steep front, is realistic in detonation loading. It can be applied on a strip or a plate and can justify a 1-D modelling or a cylindrical symmetry assumption. The second reason for this choice is that the results clearly underline the presence of a steady state solution and suggest the possibility of an analytical study.

Assuming that the numerical resolution of the previous coupled equations is well managed, different results are presented. They all correspond to the 1-D response of an infinite strip submitted to a uniform unit pressure spreading at a constant non-dimensional velocity $V = v/v_p$, (v being the dimensional velocity), symmetrically around a central line of origin. For this reason of symmetry, only the X-positive part will be presented. The previous non-dimensional form of functions and variables is used. For the whole presentation, some parameters are fixed: μ, δ and θ. They correspond to a determined couple of plate material and liquid. For example, for aluminium and water, approximate values are: $\mu = 0.1, \delta = 0.28, \theta = 0.55$.

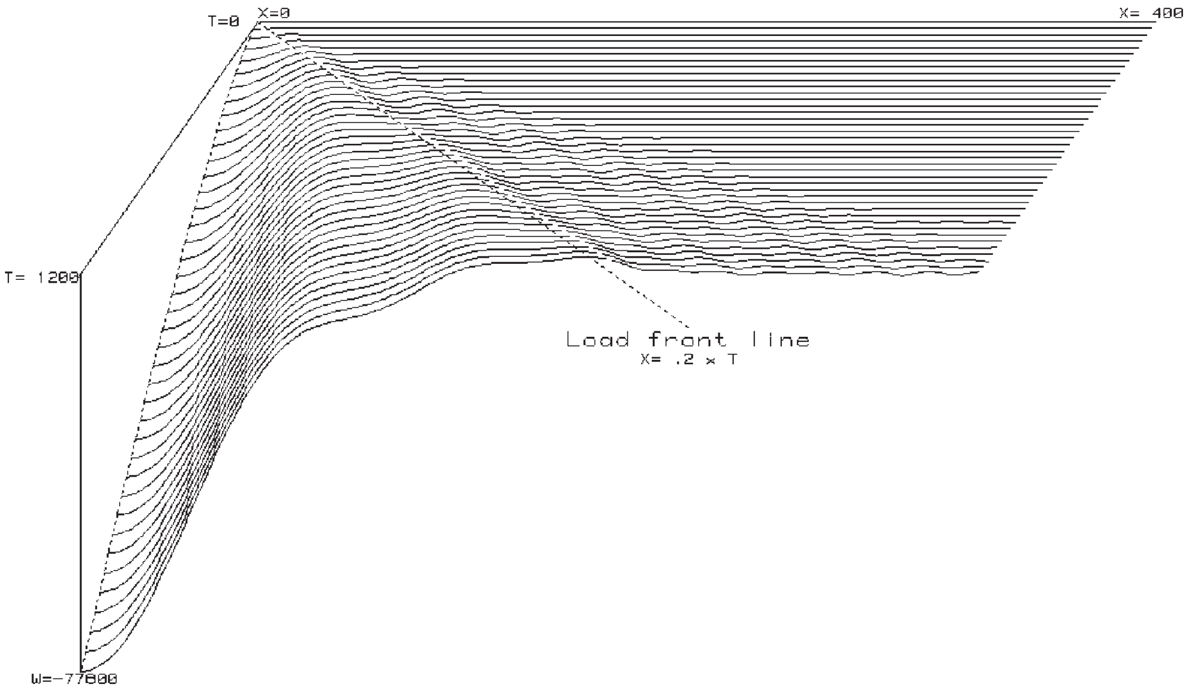
To explore many of the responses, it would be necessary to set the velocity V over a wide range. The non-dimensional characteristic velocities in the plate are $V_p = 1$ (velocity of longitudinal waves in a plate), $V_s = \theta$ (velocity of shear waves) with $V_s < V_p, V_s = v_s/v_p = \sqrt{\kappa(1-\upsilon)/2}$ and $V_s \sim 0.55$ for $\upsilon = 0.3$. For the liquid, $V_1 = \delta$, a value that can occur anywhere among V_p and V_s. Then three different cases can occur. Finally, when the velocity V of loading is chosen, it can take place anywhere among three values arranged in three different manners; to summarize, 12 different cases should be studied to explore all possibilities.

After testing every possibility, only four of them are presented in this work, showing the most significant results and being representative of all cases. For composite plates, δ may be higher than θ, but for metallic plates and water, the most common case, δ remains lower than θ. Under this latter assumption, the four cases are: case $0: V<\delta<\theta<1$; case $1: \delta<V<\theta<1$; case $2: \delta<\theta<V<1$ and case $3: \delta<\theta<1<V$.

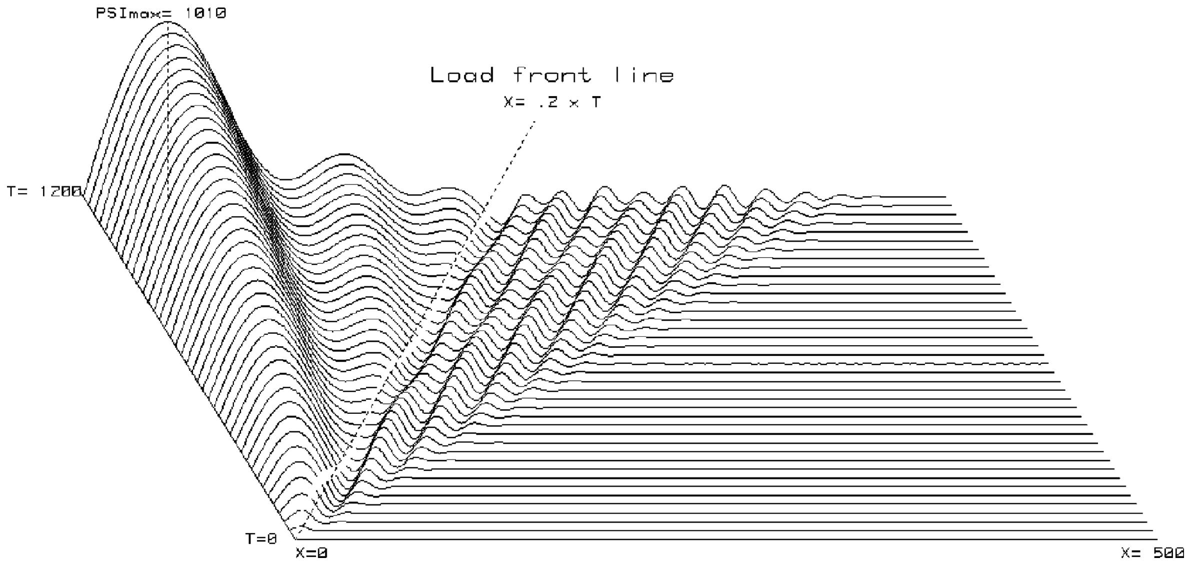
For each case, the pattern of evolution of displacements W and Ψ, and of stresses Σ and Γ is presented in Figures 2–9.

The evolution of plate motion shows clearly how a transient response takes place. Undulations appear from the centre of application of pressure, grow and progress with a non-uniform velocity. The wavelength becomes shorter and shorter with the x co-ordinate and longer and longer with time. Finally, the oscillating part decreases and links to a part of the response, which seems invariable if observed in a translation along the x co-ordinate at the velocity of the loading front (the position of the load front is marked on figures by a blank line).

In all cases, the observation of the pressure in the liquid shows two regions: one, behind the loading front tending toward the value P_{ext} of the loading pressure, and the other, ahead, tending



(a)



(b)

Fig. 2. Displacements of the plate subject to a pressure spreading at the velocity $V = 0.2$, (case 0—evolution with time): (a) deflection $W(X)$; (b) rotation $\Psi(X)$.

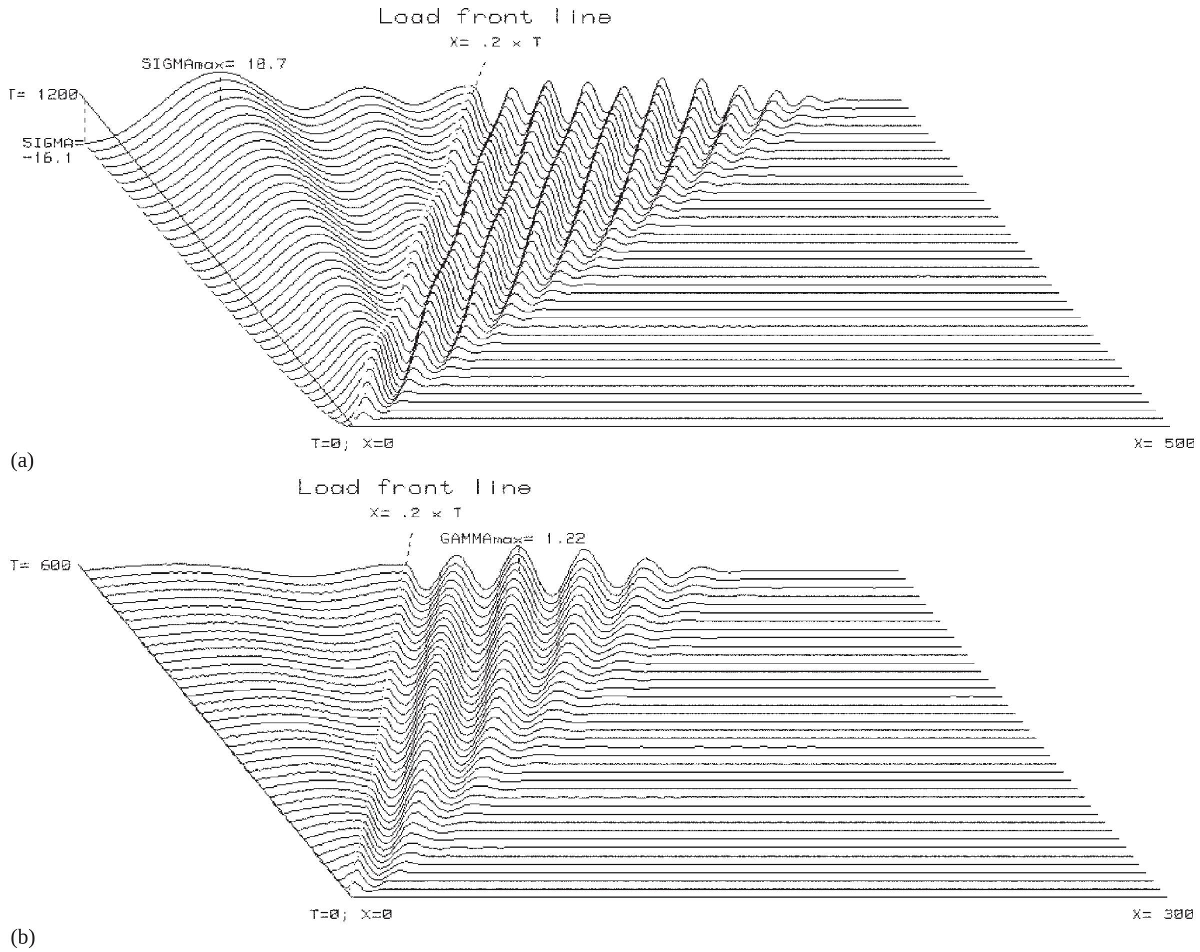
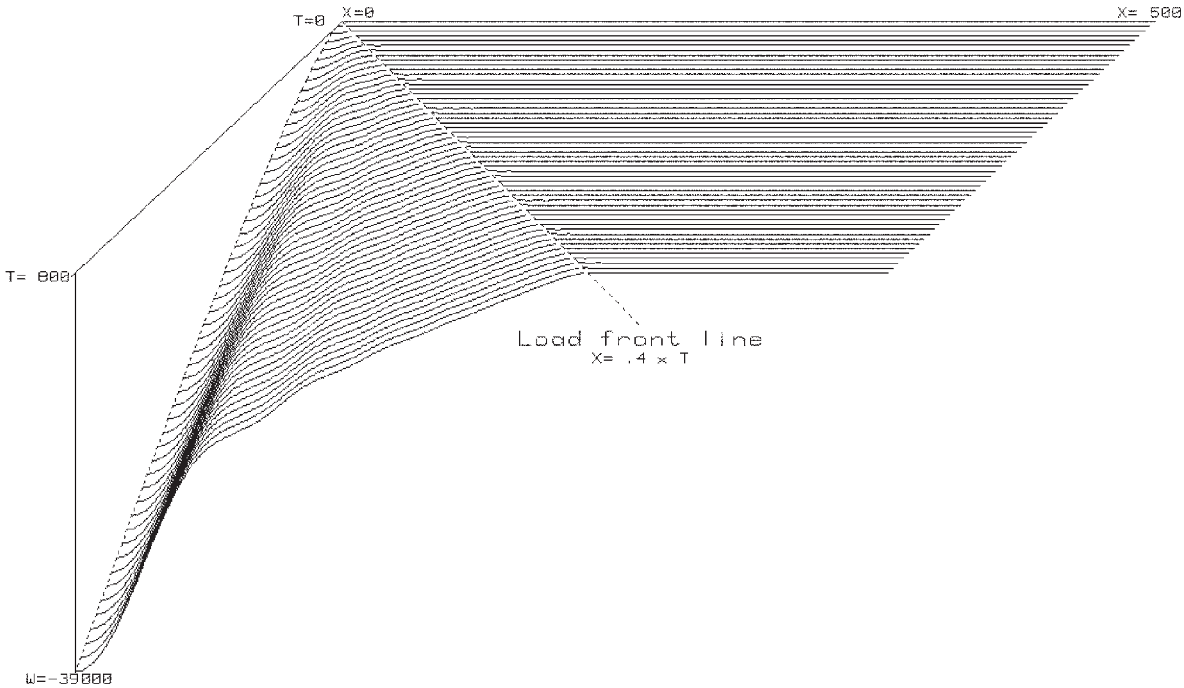


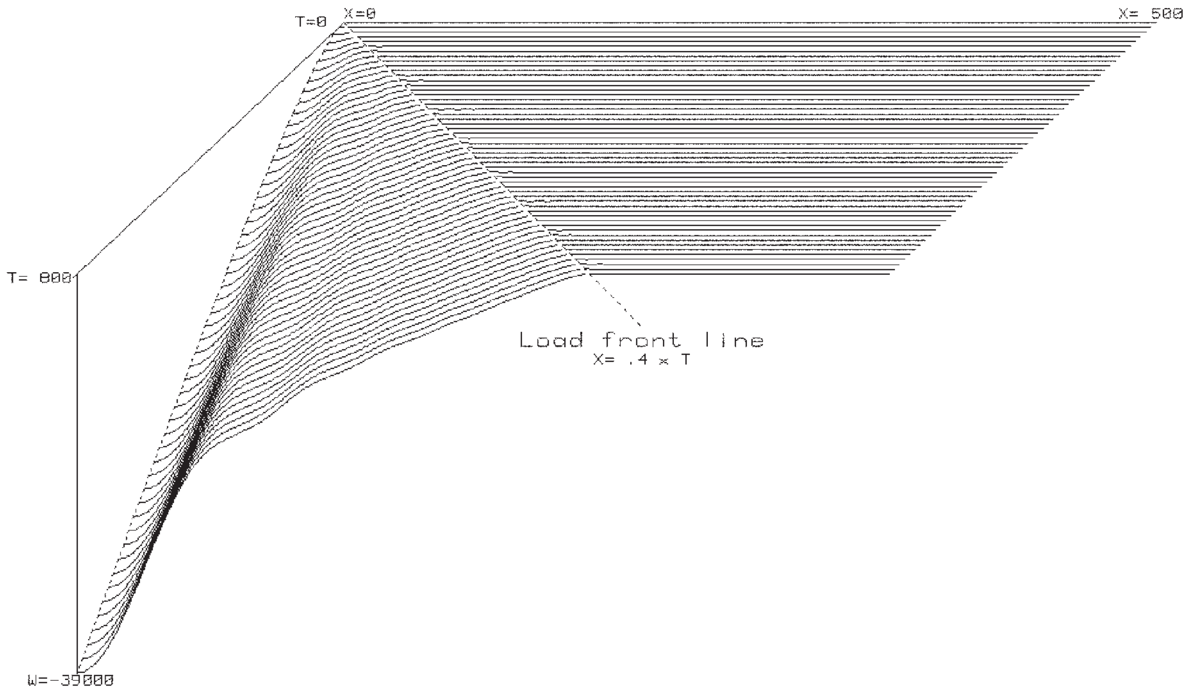
Fig. 3. Stresses in the plate subject to a pressure spreading at the velocity $V = 0.2$ (case 0—evolution with time): (a) flexural stress $\Sigma(X)$; (b) shearing stress $\Gamma(X)$.

toward zero. In the case of front load velocity greater than wave velocity in the liquid, pressure front appears clearly in the liquid. In case 0, no pressure front appears.

For any velocity of charge, the transient part of the response builds itself in a similar way. For the second part of the response, which follows the pressure front, the 3-D representation (Figs. 4–9) suggests that the solution evolves toward a stationary form. These remarks are in good agreement with the responses, already mentioned, of plates alone. If one looks precisely at the numerical results, the confirmation of this assumption is verified only in cases 1–3. On the contrary, in case 0, i.e., for $V < \delta$, the exact stationary solution is never reached. The study shows that the maximum of displacement on the front line increases incessantly (Fig. 2a). The other functions, Ψ, Σ and Γ seem nearer from a stationary response, but it is not possible to really separate a transient part from a steady state one. The existence of a steady state response is confirmed in all cases when $V > \delta$, but some differences are worth noting.

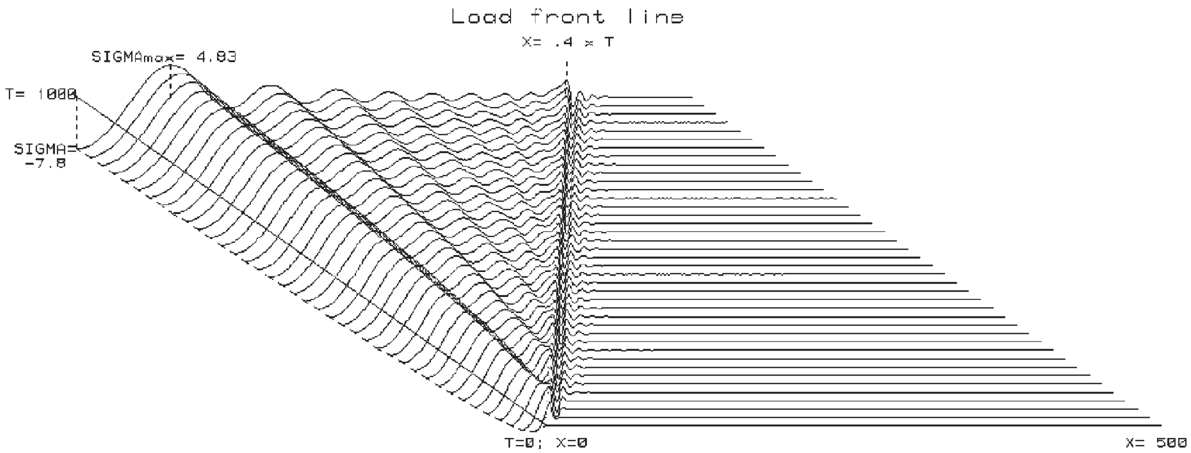


(a)

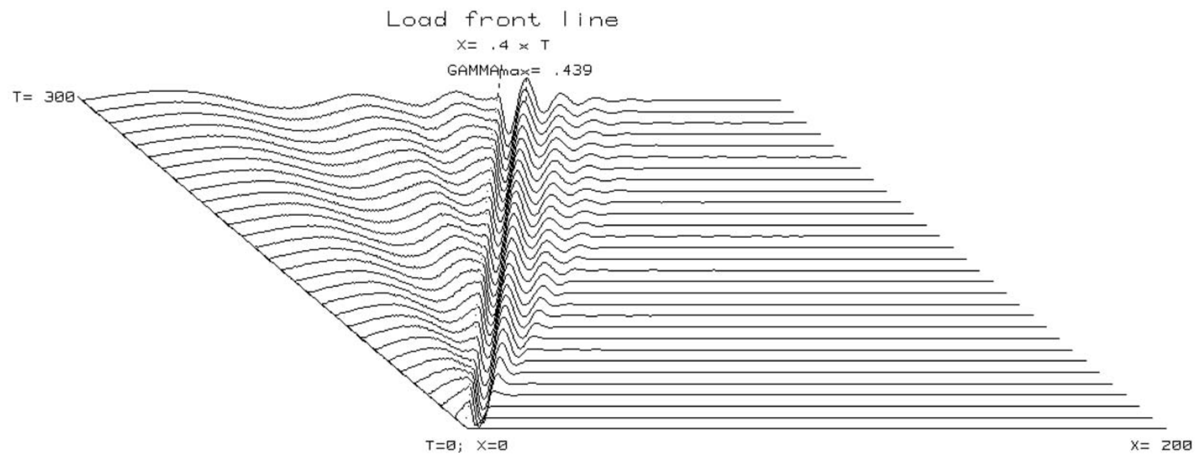


(b)

Fig. 4. Displacements of the plate subject to a pressure spreading at the velocity $V = 0.4$ (case 1—evolution with time): (a) deflection $W(X)$; (b) rotation $\Psi(X)$.



(a)



(b)

Fig. 5. Stresses in the plate subject to a pressure spreading at the velocity $V = 0.4$, (case 1—evolution with time): (a) flexural stress $\Sigma(X)$; (b) shearing stress. $\Gamma(X)$.

In case 1 (Figs. 4 and 5), the stationary response shows oscillations, only on the right of the load front. They decay quickly with X. In case 2 (Figs. 6 and 7), the stationary response contains no oscillation. A surprising single peak in the representation of stresses Γ seems to travel with the load front. In case 3 (Figs. 8 and 9), slightly decreasing oscillations are visible, but only behind the load front. No perturbation is present ahead of the front, which agrees well with the supersonic velocity of the charge.

If the response of the coupled system is calculated with the condition in $X = 0$ other than the symmetry around the central line of origin presented earlier, it is possible to verify that the transient response is strongly dependent on the boundary conditions at the origin. The condition of symmetry means than the component of speed is null in the liquid for $X = 0$. If another condition was imagined, another transient response would take place. On the contrary, the stationary part travelling with the loading always remains the same, only depending on the

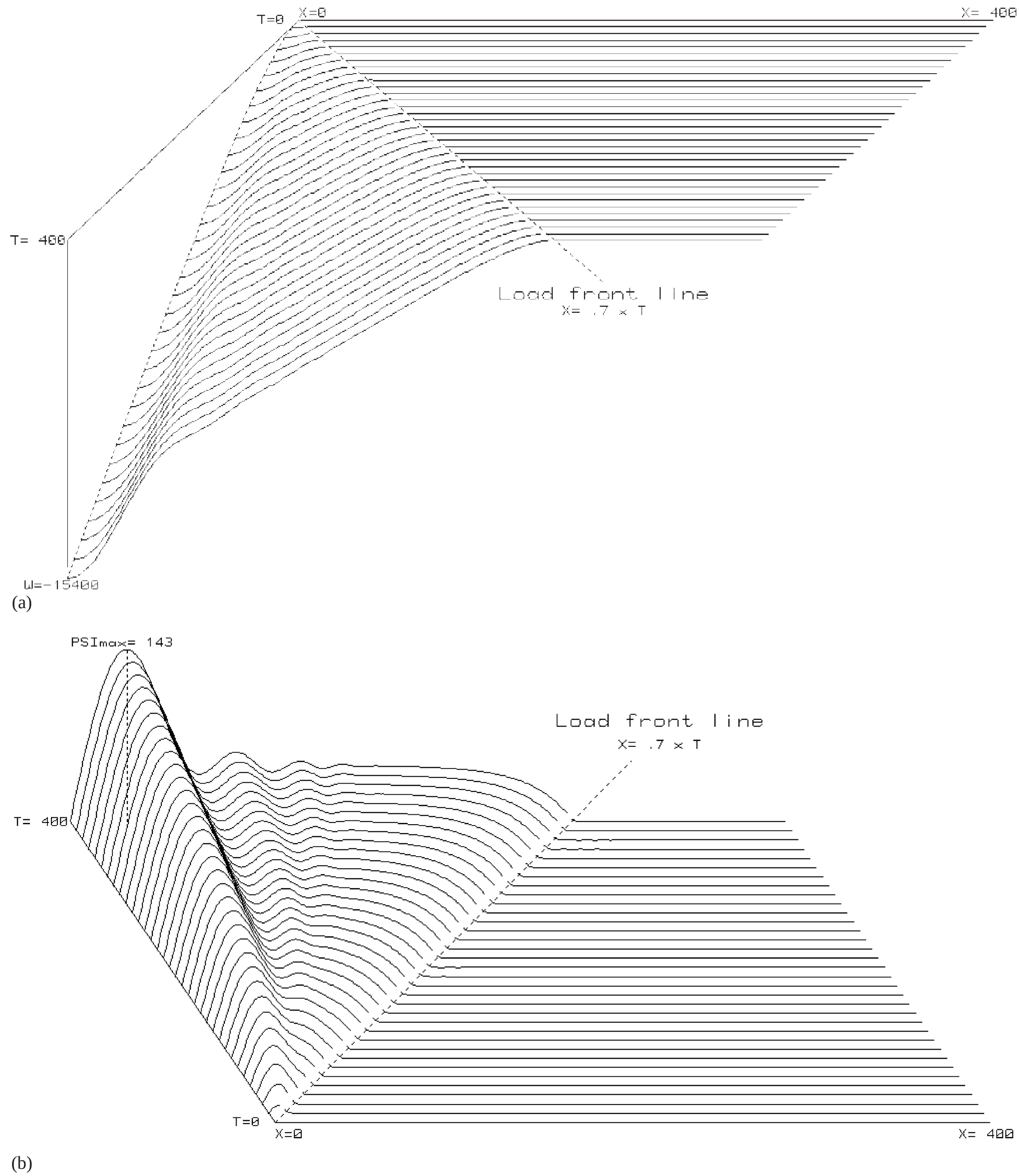


Fig. 6. Displacements of the plate subject to a pressure spreading at the velocity $V = 0.7$, (case 2—evolution with time): (a) deflection $W(X)$; (b) rotation $\Psi(X)$.

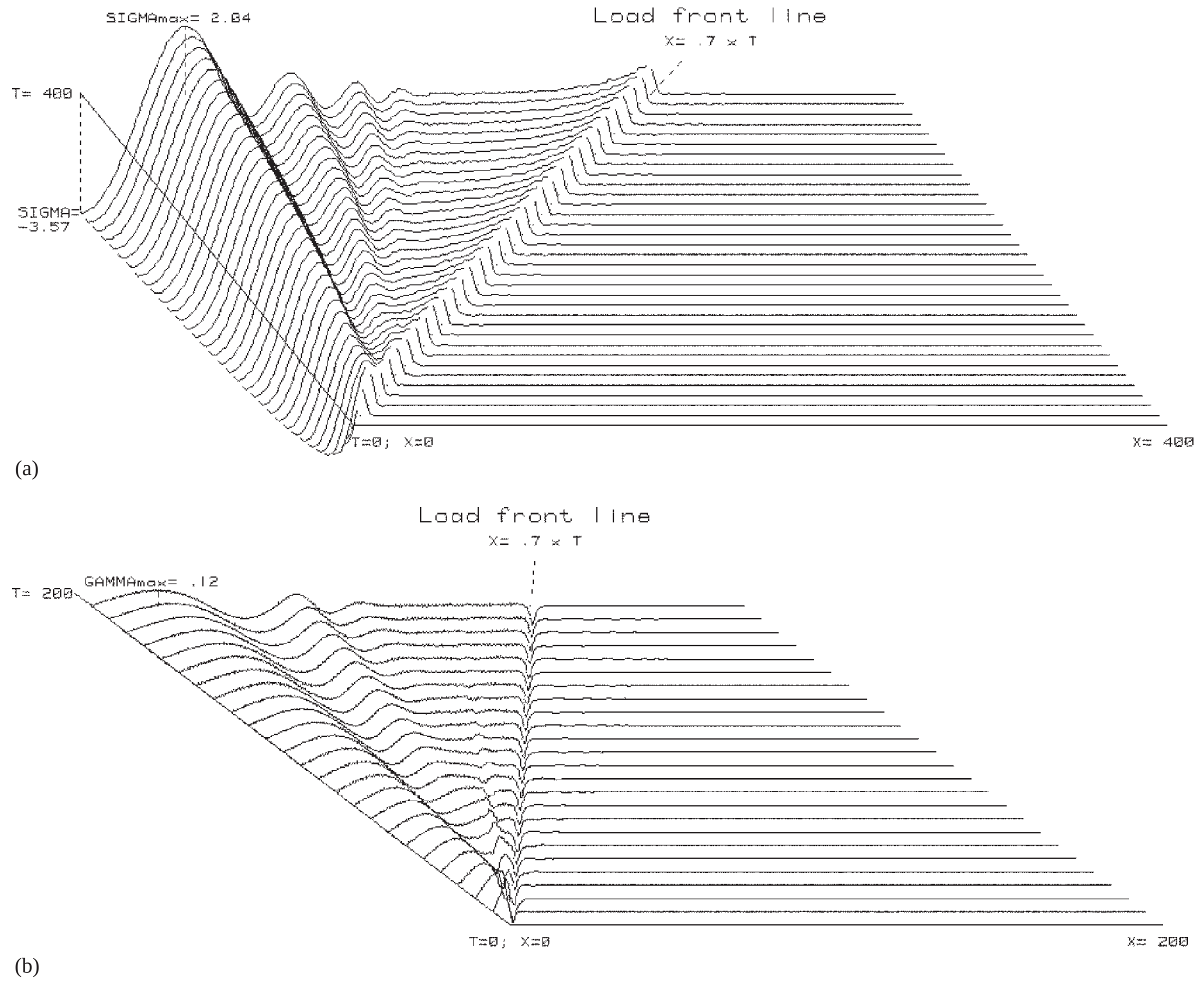
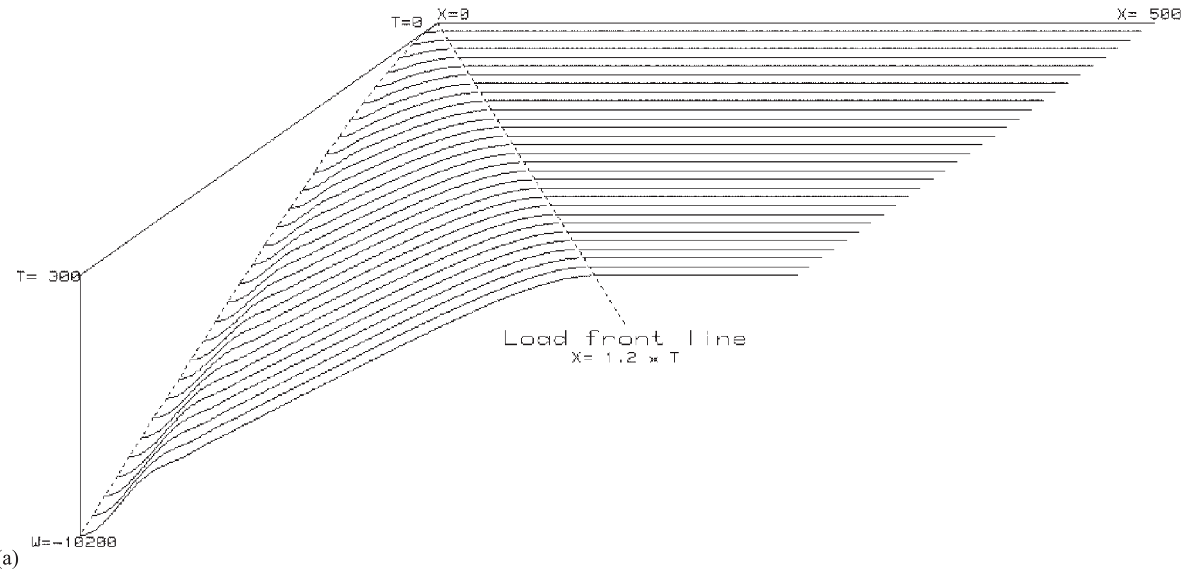


Fig. 7. Stresses in the plate subject to a pressure spreading at the velocity $V = 0.7$, (case 2—evolution with time): (a) flexural stress $\Sigma(X)$; (b) shearing stress $\Gamma(X)$.

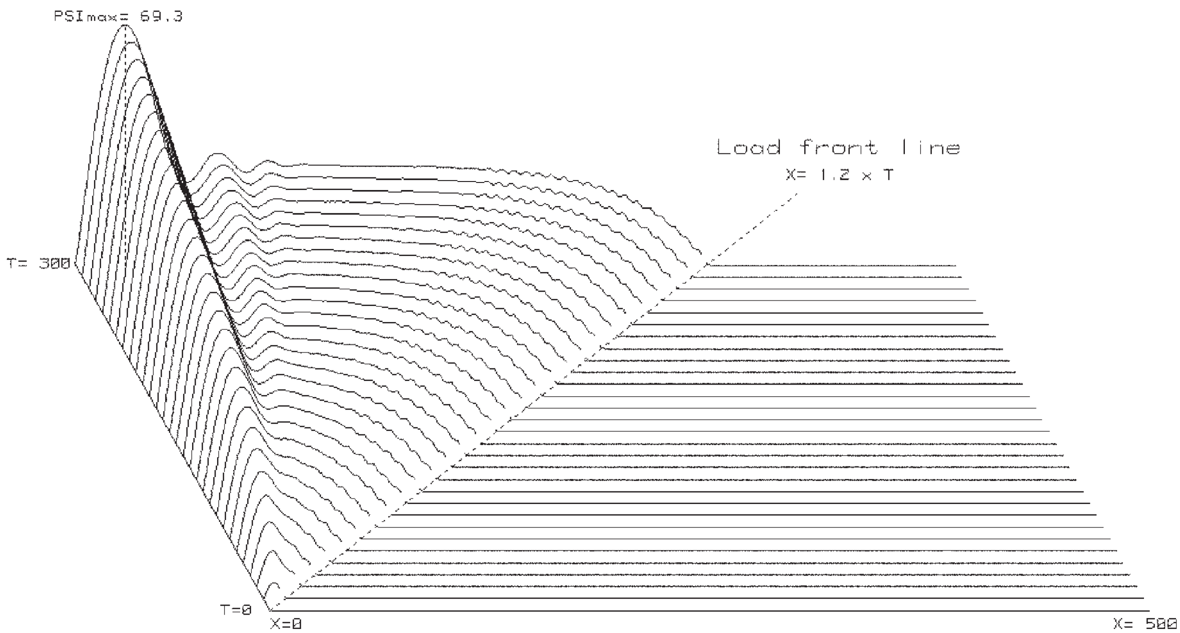
velocity of the pressure front. For displacements, the larger values are obtained in the transient response and grow with time. These significant values of displacements do not generally correspond to great deformations because the wavelengths are rather long. For the stresses, the values obtained in the stationary response are significant and give interest to the study of the steady state response.

4. The steady state solution

A steady state solution exists if the motion of the coupled system appears frozen to an observer, which moves at the load speed. It requires that a loading with a constant profile should be moving with a constant velocity on an infinite long strip.



(a)



(b)

Fig. 8. Displacements of the plate subject to a pressure spreading at the velocity $V = 1.2$, (case 3—evolution with time): (a) deflection $W(X)$; (b) rotation $\Psi(X)$.

The formulation of a steady state solution needs some abstract analysis, especially to express the displacement values at the infinite ends of the plate. These do not vanish necessarily for $X = -\infty$. Let V be the non-dimensional loading velocity ($V = v/v_p$); any function describing the

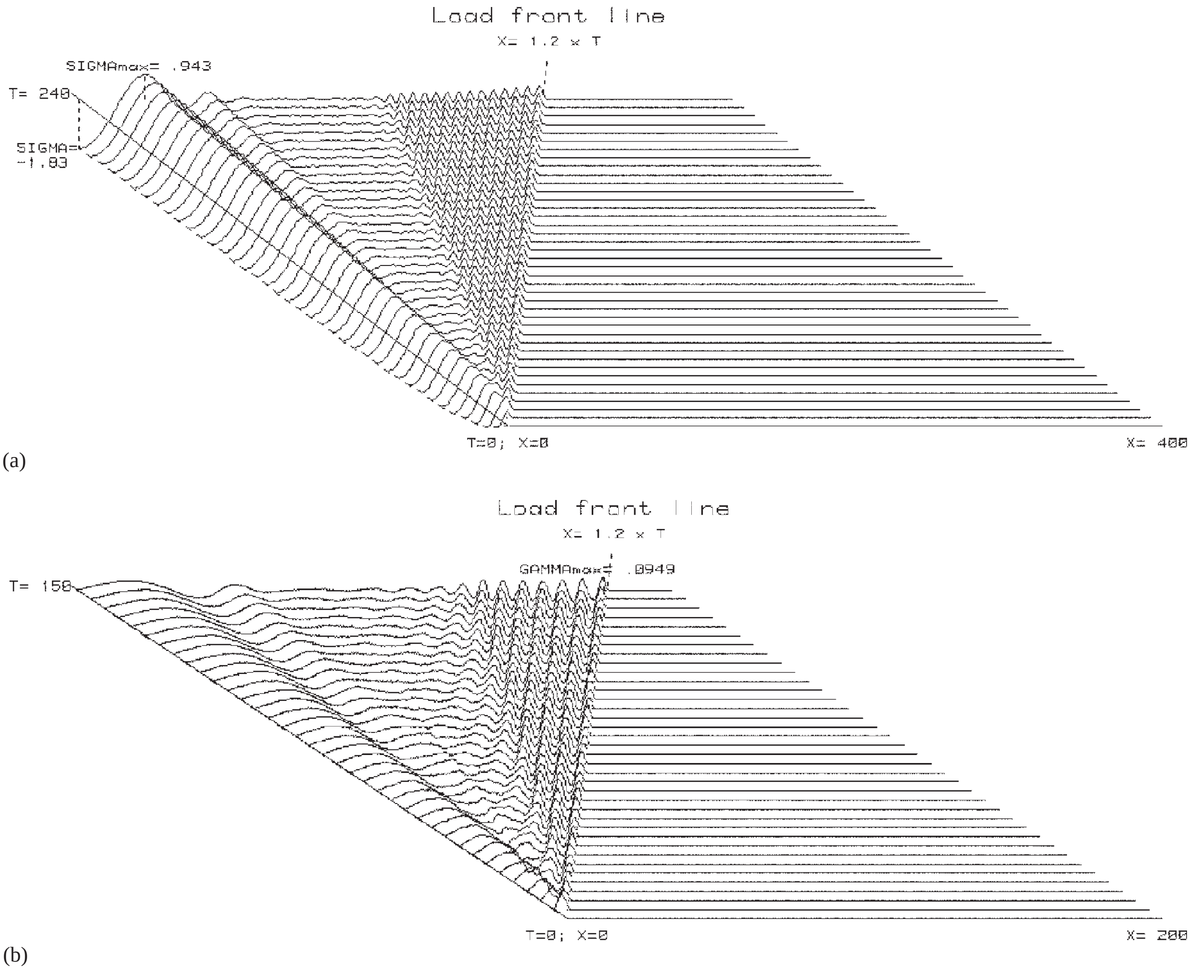


Fig. 9. Stresses in the plate subject to a pressure spreading at the velocity $V = 1.2$, (case 3—evolution with time): (a) flexural stress $\Sigma(X)$; (b) shearing stress $\Gamma(X)$.

motion of the system can be expressed according to the Y position of an observer moving with the load, where $Y = X - VT$, then,

$$W(X,T) = \hat{W}(Y), \quad \Psi(X,T) = \hat{\Psi}(Y), \quad \Phi(X,Z,T) = \hat{\Phi}(Y,Z),$$
$$\text{under the condition that } P(X,T) = \hat{P}(Y). \tag{30}$$

By convenience, the sign $\hat{}$ will be omitted. Replacing partial derivatives by ordinary ones, $\partial/\partial X = \mathrm{d}/\mathrm{d}Y, \partial/\partial T = -V\mathrm{d}/\mathrm{d}Y$, the system of coupled equations can be written as

$$V^2 = \frac{\mathrm{d}^2 W}{\mathrm{d}Y^2} = \theta^2\left(\frac{\mathrm{d}^2 W}{\mathrm{d}Y^2} - \frac{\mathrm{d}\Psi}{\mathrm{d}Y}\right) - \mu V\left(\frac{\partial \Phi}{\partial Y}\right)_{(Z=0)} + P_{ext}(Y), \tag{31}$$

$$V^2\,\mathrm{d}^2\Psi/\mathrm{d}Y^2 = \mathrm{d}^2\Psi/\mathrm{d}Y^2 + \theta^2(\mathrm{d}W/\mathrm{d}Y - \Psi), \tag{32}$$

$$V\,\mathrm{d}W/\mathrm{d}Y = (\partial\Phi/\partial Z)_{(Z=0)}, \qquad V^2\partial^2\Phi/\partial Y^2 = \delta^2(\partial^2\Phi/\partial Y^2 + \partial^2\Phi/\partial Z^2). \tag{33, 34}$$

If any free wave travels at the velocity V, the solution of homogeneous equations will take the form

$$W(Y) = W_0\mathrm{e}^{\lambda Y}, \qquad \Psi(Y) = \Psi_0\mathrm{e}^{\lambda Y}; \qquad \Phi(Y,Z) = \Phi_0(Z)\mathrm{e}^{\lambda Y}. \tag{35, 36}$$

Applying derivatives, one obtains

$$W_0\lambda^2(V^2-\theta^2) + \Psi_0\lambda\theta^2 + \lambda\mu V\Phi_0(0) = 0, \qquad W_0\lambda\theta^2 + \Psi_0(\lambda^2 - \theta^2 - \lambda^2V^2) = 0 \tag{37, 38}$$

$$(\mathrm{d}\Phi_0/\mathrm{d}Z)_{(Z=0)} = \lambda V W_0, \qquad \mathrm{d}^2\Phi_0(Z)/\mathrm{d}Z^2 - ((V^2-\delta^2)/\delta^2)\lambda^2\Phi_0(Z) = 0. \tag{39, 40}$$

$$\text{Setting } \Omega = ((V^2-\delta^2)/\delta^2)^{1/2} \text{ and } \Omega_0 = |(V^2-\delta^2)/\delta^2|^{1/2}, \tag{41}$$

$$\text{if } V>\delta, \Omega = \Omega_0, \text{real; if } V<\delta, \Omega = \mathrm{i}\Omega_0, \text{imaginary} \tag{42}$$

Eq. (40) requires that $\Phi_0(Z)$ take the form of two independent solutions, $A_-\mathrm{e}^{\lambda\Omega Z}$ and $A_+\mathrm{e}^{\lambda\Omega Z}$

Suppose the first form

$$\Phi_0(Z) = A_-\mathrm{e}^{\lambda\Omega Z}, \tag{43}$$

$$(\mathrm{d}\Phi_0/\mathrm{d}Z)_{(Z=0)} = -A_-\lambda\Omega. \tag{44}$$

By comparison with Eq. (39), A_- is deduced as

$$A_- = -(V/\Omega)W_0, \text{ and then } \quad \Phi_0(0) = -(V/\Omega)W_0. \tag{45}$$

Reintroducing $\Phi_0(0)$ into Eq. (37), the resultant system is

$$W_0(\lambda^2(V^2-\theta^2) - \lambda\mu V^2/\Omega) + \Psi_0\lambda\theta^2 = 0, \qquad W_0\lambda\theta^2 + \Psi_0(\lambda^2-\theta^2-\lambda^2V^2) = 0. \tag{46}$$

The compatibility of this system gives the characteristic equation

$$\lambda(\lambda^3(V^2-\theta^2)(1-V^2) - \lambda^2(1-V^2)\mu V^2/\Omega - \lambda\theta^2V^2 + \mu\theta^2V^2/\Omega) = 0. \tag{47}$$

The solutions are

$$\lambda_0 = 0, \lambda_1, \lambda_2, \lambda_3. \tag{48}$$

If the other choice had been made ($\Phi_0(Z) = A_+\mathrm{e}^{+\lambda\Omega Z}$), the sign of Ω in Eq. (41) would have changed and the values found as

$$\lambda_0' = 0, \lambda_1' = -\lambda_1, \lambda_2' = -\lambda_2, \lambda_3' = -\lambda_3. \tag{49}$$

To obtain explicit solutions, the real or imaginary Ω cases must be studied separately.

4.1. Case $V>\delta$: the front load velocity is greater than the wave propagation velocity in the liquid

According to Eq. (42), $\Omega = \Omega_0$, real. Eq. (47) has real coefficients. At least one of its roots is real. Non-real roots must be conjugate. The free waves, solutions of the homogeneous equations, take the form

$$W = a_0 + \sum_{j=1}^{j=3} a_j e^{\lambda_j Y} + \sum_{j=1}^{j=3} a'_j e^{\lambda'_j Y}, \qquad \Psi = b_0 + \sum_{j=1}^{j=3} b_j e^{\lambda_j Y} + \sum_{j=1}^{j=3} b'_j e^{\lambda'_j Y}, \tag{50, 51}$$

$$\Phi = c_0 + \sum_{j=1}^{j=3} c_j e^{\lambda_j (Y - \Omega_0 Z)} + \sum_{j=1}^{j=3} c'_j e^{\lambda'_j (Y + \Omega_0 Z)}, \tag{52}$$

a_0, b_0, c_0 being real constants and $a_j, b_j, c_j, a'_j, b'_j, c'_j$, complex ones.

Introducing compatibility using Eq. (46) one obtains

$$b_0 = 0, \quad b_j = a_j \frac{\lambda_j \theta^2}{\lambda_j^2 (V^2 - 1) + \theta^2}, \quad b'_j = a'_j \frac{\lambda'_j \theta^2}{\lambda_j'^2 (V^2 - 1) + \theta^2}, \; j = 1, \ldots, 3. \tag{53}$$

Introducing the compatibility using Eq. (33)

$$c_j = -a_j V / \Omega_0, \quad c'_j = -a'_j V / \Omega_0, \quad j = 1, \ldots, 3. \tag{54}$$

The velocity of loading of the plate being supersonic in respect to the liquid, no disturbance can occur in the liquid earlier than in the plate, for any value of the Y co-ordinate. Especially, if one consider a line $Y + \Omega_0$, no pressure potential can be present, for large positive values of Y, behind or ahead of this line. For this reason, coefficients c'_j are always zero, like b'_j and a'_j.

To obtain the forced response, a particular solution of the complete system must be added to the previous ones. Rewriting the external loading $P_{ext}(Y) = -P_0, \mathrm{H}(-Y)$ where P_0 is the intensity of the loading pressure and $H(\cdot)$ the Heaviside step function, one verifies easily that the following expressions are convenient for a particular solution:

for $Y<0$, *for* $Y \geqslant 0$

$$W_p = (P_0 \Omega_0 / \mu V^2) Y, \quad W_p = 0, \tag{55}$$

$$\Psi_p = P_0 \Omega_0 / \mu V^2, \quad \Psi_p = 0, \tag{56}$$

for $Y - \Omega_0 Z < 0$, *for* $Y - \Omega_0 Z \geqslant 0$

$$\Phi_p = -(P_0 / \mu V)(Y - \Omega_0 Z), \quad \Phi_p = 0. \tag{57}$$

The particular solutions W_p and Ψ_p are discontinuous in $Y = 0$ and Φ_p is discontinuous on the line $Y - \Omega_0 Z$. The free waves must be assumed different for each side of a discontinuity. Writing the expression $e^{\lambda_j Y} = e^{(\alpha_j + \beta_j) Y}$, it is clear that this kind of solution exists only for $Y<0$ if $\alpha_j > 0$ and only for $Y > 0$ if $\alpha_j < 0$.

By using this remark and introducing arbitrarily the multiplying factor P_0 into the free wave equation, the final form of the complete solution can be written

for $Y<0$

$$W = -P_0 \left(a_0 + \sum_{\alpha_j > 0} a_j e^{\lambda_j Y} - \frac{\Omega_0}{\mu V^2} Y \right), \tag{58}$$

$$\Psi = -P_0\left(b_0 + \sum_{\alpha_j>0} \frac{\lambda_j\theta^2}{\lambda_j^2(V^2-1)+\theta^2} a_j e^{\lambda_j Y} - \frac{\Omega_0}{\mu V^2}\right), \tag{59}$$

$$\Phi = -P_0\left(c_0 - \frac{V}{\Omega_0}\sum_{\alpha_j>0} a_j e^{\lambda_j(Y-\Omega_0 Z)} + \frac{1}{\mu V}(Y-\Omega_0 Z)\right), \tag{60}$$

for $Y \geqslant 0$

$$W = -P_0\left(a_0' + \sum_{\alpha_j<0} a_j e^{\lambda_j Y}\right), \qquad \Psi = -P_0\left(\sum_{\alpha_j<0} \frac{\lambda_j\theta_2}{\lambda_j^2(V^2-1)+\theta^2} a_j e^{\lambda_j Y}\right), \tag{61, 62}$$

$$\Phi = -P_0\left(c_0' - \frac{V}{\Omega_0}\sum_{\alpha_j<0} a_j e^{\lambda_j(Y-\Omega_0 Z)}\right). \tag{63}$$

c_0' can be taken to zero by convenience without consequences. a_0' is null because W tends toward zero for very large positive values of Y, the plate being originally at rest.

Five parameters have yet to be determined: a_0, a_1, a_2, a_3 and c_0. For this purpose, the continuity of five functions is assumed: $W, \Psi, \mathrm{d}W/\mathrm{d}Y, \mathrm{d}\Psi/\mathrm{d}Y$ in $Y = 0$ and Φ along the line $Y = \Omega_0 Z$. Five conditions are obtained,

$$a_0 + \sum_{\alpha_j>0} a_j = \sum_{\alpha_j<0} a_j, \qquad \sum_{\alpha_j>0} \frac{\lambda_j}{\lambda_j^2(V^2-1)+\theta^2} a_j = \sum_{\alpha_j<0} \frac{\lambda_j}{\lambda_j^2(V^2-1)+\theta^2} a_j \tag{64, 65}$$

$$\sum_{\alpha_j>0} \lambda_j a_j + \frac{P_0\Omega_0}{\mu V^2} = \sum_{\alpha_j<0} \lambda_j a_j, \quad \sum_{\alpha_j>0} \frac{\lambda_j^2}{\lambda_j^2(V^2-1)+\theta^2} a_j = \sum_{\alpha_j<0} \frac{\lambda_j^2}{\lambda_j^2(V^2-1)+\theta^2} a_j, \tag{66, 67}$$

$$c_0 - \frac{V}{\Omega_0}\sum_{\alpha_j>0} a_j = -\frac{V}{\Omega_0}\sum_{\alpha_j<0} a_j. \tag{68}$$

The comparison of the last equation with the first one gives the value $c_0 = -(V/\Omega_0)a_0$, and only the first four equations have to be considered. They can be written more simply by setting $\mathrm{sgn}(\alpha_j) = s_j$:

$$a_0 + \sum_{j=1}^{j=3} s_j a_j = 0, \qquad \sum_{j=1}^{j=3} \frac{\lambda_j}{\lambda_j^2(V^2-1)+\theta^2} s_j a_j = 0, \tag{69, 70}$$

$$\sum_{j=1}^{j=3} \lambda_j s_j a_j = \frac{\Omega_0}{\mu V^2}, \quad \sum_{j=1}^{j=3} \frac{\lambda_j^2}{\lambda_j^2(V^2-1)+\theta^2} s_j a_j = 0. \tag{71, 72}$$

Solving the previous linear system, the four values a_j are obtained and c_0 can be deduced. Then the steady state solution is completed. All desired functions are described by Eqs. (58)–(63).

To complete the solution, the values of stresses by Eq. (9) and pressure by Eq. (14) can be added:

for $Y<0$, *for* $Y \geqslant 0$

$$\Sigma = -P_0\left(-\frac{1}{2}\sum_{\alpha_j>0}\frac{\lambda_j^2\theta^2}{\lambda_j^2(V^2-1)+\theta^2}a_j \mathrm{e}^{\lambda_j Y}\right), \Sigma = -P_0\left(-\frac{1}{2}\sum_{\alpha_j<0}\frac{\lambda_j^2\theta^2}{\lambda_j^2(V^2-1)+\theta^2}a_j \mathrm{e}^{\lambda_j Y}\right), \quad (73)$$

$$\Gamma = -P_0\left(\frac{\theta^2}{\sqrt{12}}\sum_{\alpha_j>0}\frac{\lambda_j^3(V^2-1)}{\lambda_j^2(V^2-1)+\theta^2}a_j \mathrm{e}^{\lambda_j Y}\right), \Gamma = -P_0\left(\frac{\theta^2}{\sqrt{12}}\sum_{\alpha_j<0}\frac{\lambda_j^3(V^2-1)}{\lambda_j^2(V^2-1)+\theta^2}a_j \mathrm{e}^{\lambda_j Y}\right), \quad (74)$$

for $Y-\Omega_0 Z<0$, *for* $Y-\Omega_0 Z \geqslant 0$,

$$P = -P_0\left(\frac{\mu V^2}{\Omega_0}\sum_{\alpha_j>0}\lambda_j a_j \mathrm{e}^{\lambda_j(Y-\Omega_0 Z)} - 1\right), P = -P_0\left(\frac{\mu V^2}{\Omega_0}\sum_{\alpha_j<0}\lambda_j a_j \mathrm{e}^{\lambda_j(Y-\Omega_0 Z)}\right). \quad (75)$$

The latter equation shows that a pressure front exists in the liquid. Its slope corresponds exactly to the trace left by a supersonic motion in an acoustic medium.

To specify the possible steady state responses, it is necessary to look closely at the values of roots λ_j. Looking to Eq. (47), it can be observed that the velocities $V=1, V=\theta$ and $V=\delta$ appear as characteristic, able to change the nature of solutions.

Fig. 10 shows the values λ_j functions of V. The sign of their real part allows prediction if the solution vanishes for one of the sides $Y<0$ or $Y>0$. The figure summarizes the kind of solution for every region of loading speed. To illustrate the result, the (Figs. 11–14) show the shape of the steady state responses in any region, according to $V>\delta$.

The responses visible in these figures agree well with the stationary part of responses presented in Figs. 4–9. The existence of undulations behind or ahead of the load front is now clearly explained. A quantitative comparison between the theoretical stationary responses and the computed ones confirms that they correspond exactly, for frequencies as well as for amplitudes. The rear part of the theoretical response and especially its expression when Y tends toward $-\infty$ corresponds well to those predicted by the calculation.

Fig. 15 illustrates the pressure in the liquid for a given value $V=0.4$. It appears clearly that the pressure propagates in the liquid along a straight front line. The pressure profile remains the same and its peak does not decrease with the distance from the plate. This surprising result is due to the theoretical inviscid character of the fluid and to the plane modelling, without possible expansion in the transverse direction. For other choices of loading velocity, higher than δ, the pressure profile can be described by Eq. (75).

4.2. Case $V<\delta$: the front load velocity is lower than the wave propagation velocity in the liquid ($\Omega = \mathrm{i}\Omega_0$, imaginary)

In this case, even coefficients of Eq. (47) are imaginary. At least one of its roots is imaginary. Other not purely imaginary roots have the same imaginary parts and real parts opposite.

To clarify: for $\Omega = -\mathrm{i}\Omega_0$, $\lambda_1 = \mathrm{i}\gamma_1, \lambda_2 = \mathrm{i}\gamma_2, \lambda_3 = \mathrm{i}\gamma_3$, or $\lambda_1 = \mathrm{i}\gamma_1, \lambda_2 = \alpha_2 + \mathrm{i}\beta_2, \lambda_3 = -\alpha_2 + \mathrm{i}\beta_2$; for $\Omega = +\mathrm{i}\Omega_0$, $\lambda_j' = -\lambda_j$, $j = 1, \ldots, 3$.

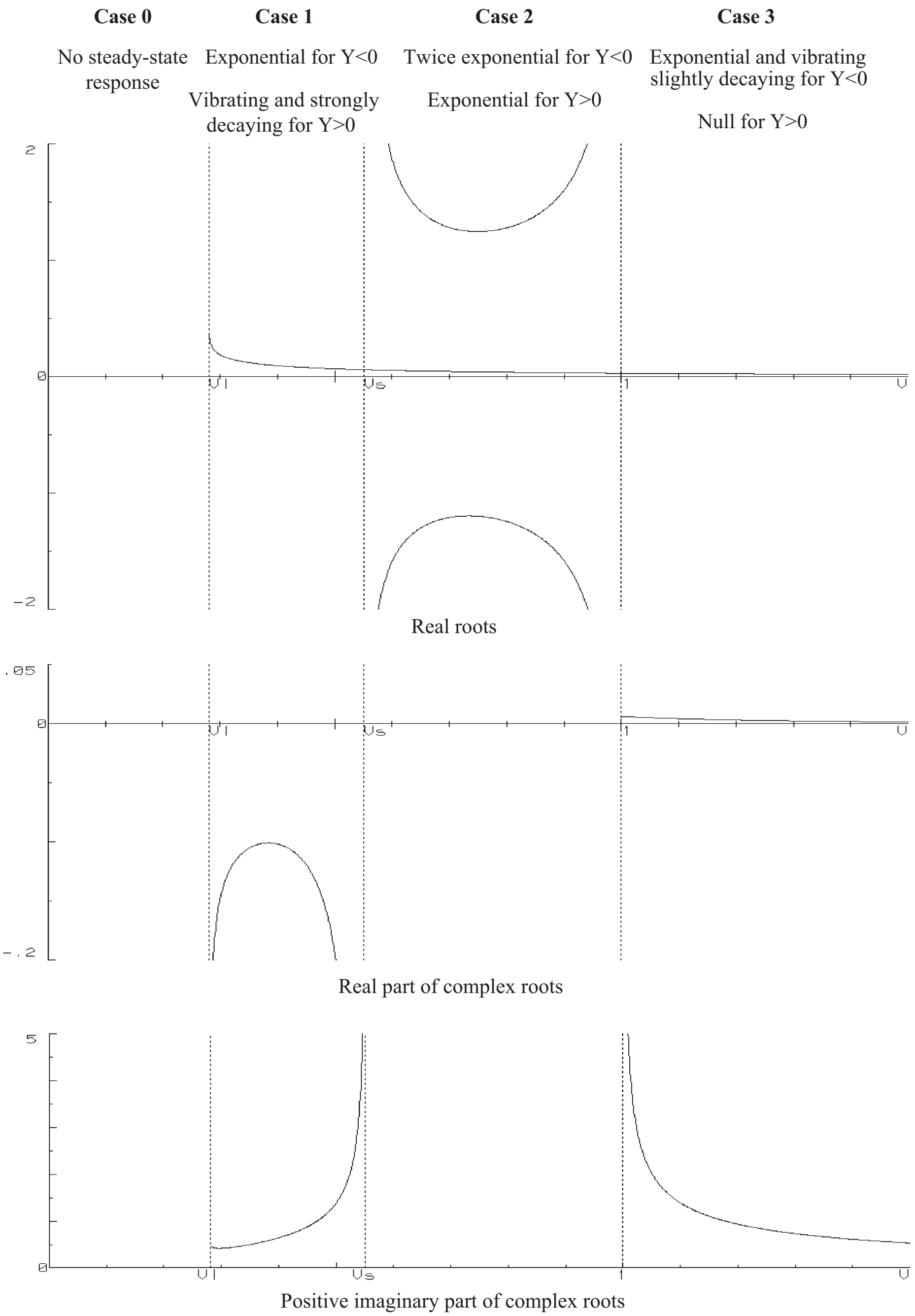


Fig. 10. Wave number and response form of a coupled plate according to the load front velocity.

To obtain the real responses W, Ψ and Φ, conjugate values must be associated. The calculation of a possible steady state response can be looked for but needs new developments. The fine observation of the numerical transient solution reveals that it evolves continuously in the

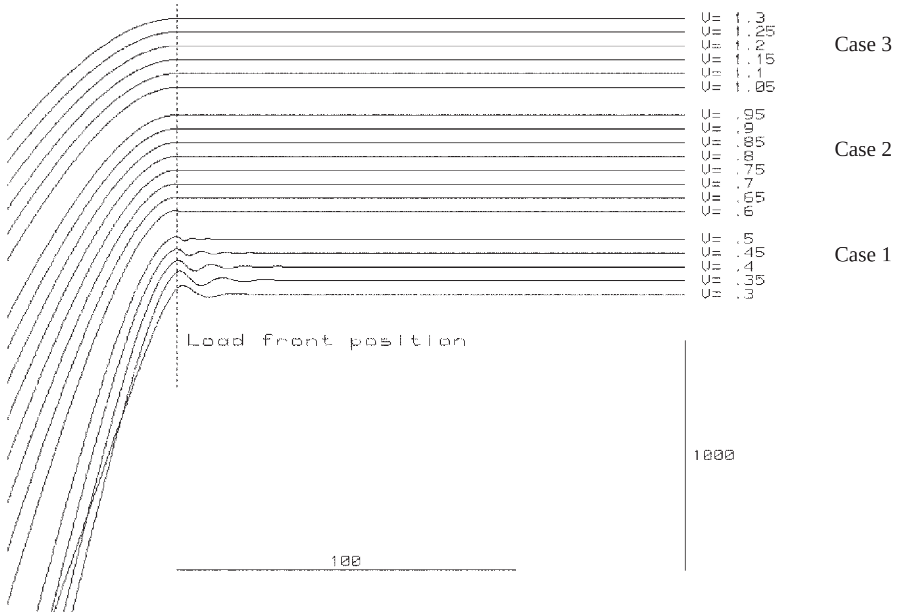


Fig. 11. Steady state deflection $W(Y)$ of a coupled plate for different cases of pressure front velocity.

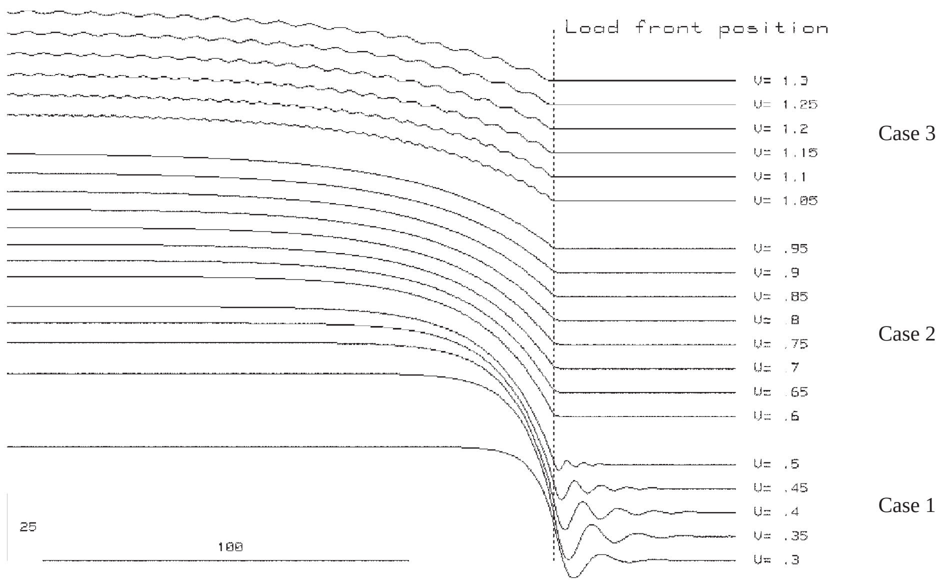


Fig. 12. Steady state rotation $\Psi(Y)$ of a coupled plate for different cases of pressure front velocity.

neighbourhood of the load front and never tends toward a frozen shape. This conclusion agrees well also with a physically reasonable solution. The velocity of the loading being subsonic in respect to the liquid, no pressure front exists. So, in an unbounded loaded liquid, no equilibrium

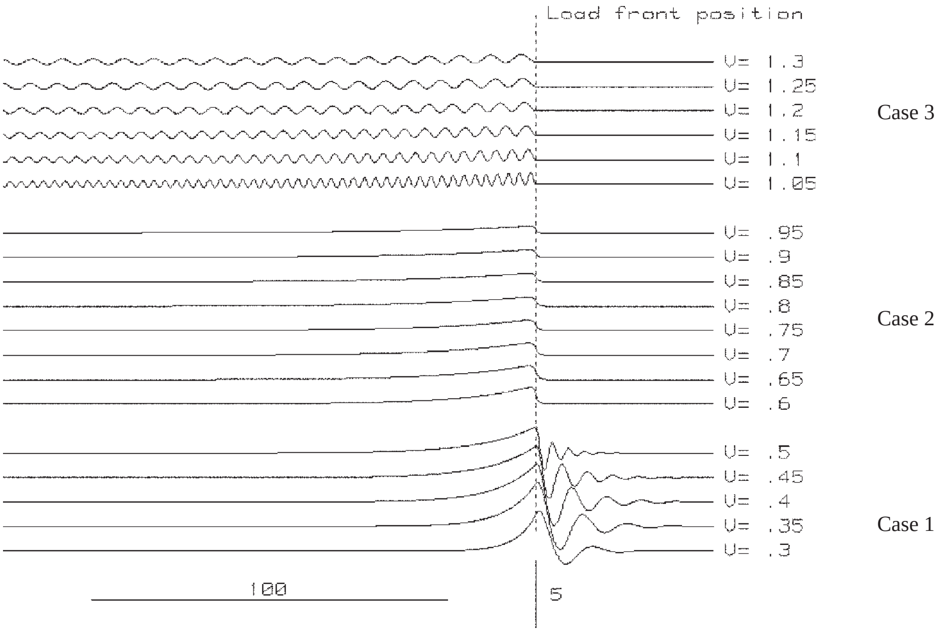


Fig. 13. Steady state flexural stress $\Sigma(Y)$ in a coupled plate for different cases of pressure front velocity.

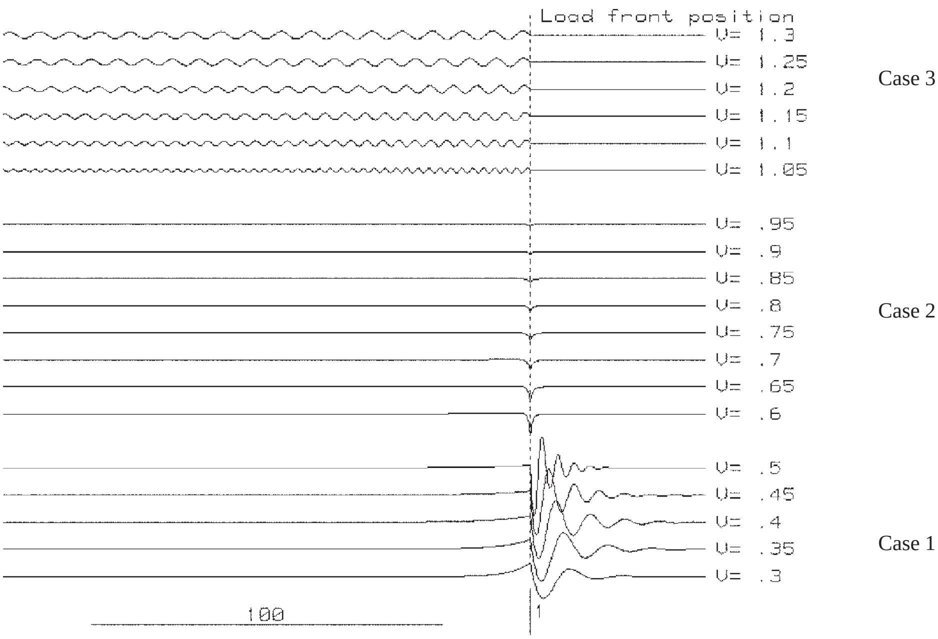


Fig. 14. Steady state shearing stress $\Gamma(Y)$ in a coupled plate for different cases of pressure front velocity.

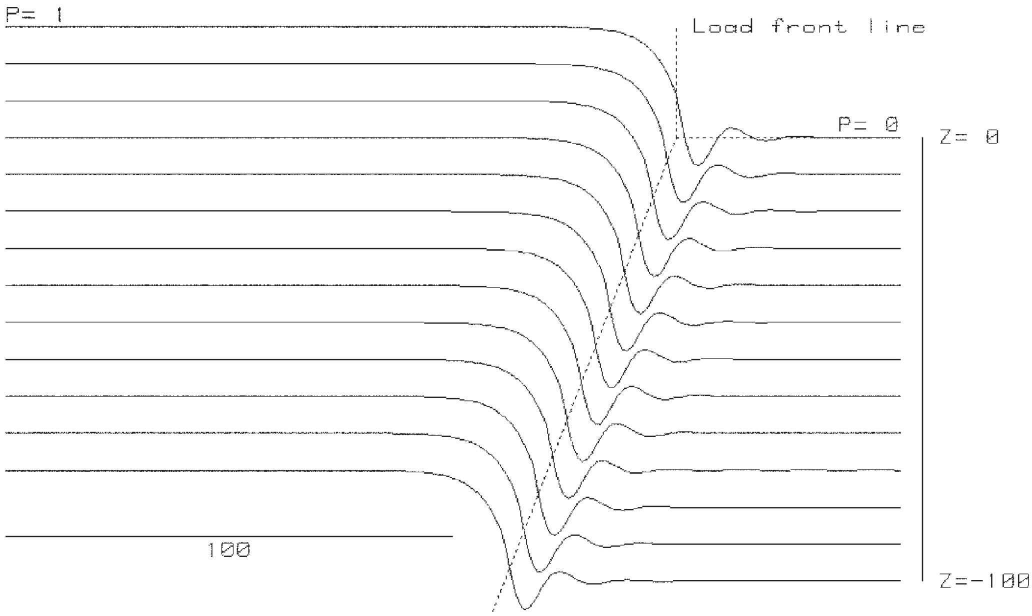


Fig. 15. Steady stress pressure $P(Y,Z)$ in the coupled liquid for load front velocity $V = 0.4$.

state can take place and the response has necessarily a transient character. Nevertheless, stresses visible in Fig. 3 seem close to a stationary form. It suggests that the response, in the neighbourhood of the loading front, could contain a stationary part although it could not be completely stationary.

5. Conclusion

The study of the coupled system made of an unbounded plate in contact with unbounded liquid shows that it is possible to predict the response to a pressure step travelling on the surface of the plate. A steady state analytical solution is found for waves travelling with the load front for all velocities of charge higher than the velocity of acoustic waves in the liquid. A numerical resolution is also proposed and validated, which is able to account for boundary conditions and for any form of loading. This computational method is applied to find the response of a plate to a pressure step spreading on a plate. The numerical solution shows how a transient response takes place and then evolves to a stationary response corresponding to the theoretical one.

The non-dimensional expression of variables enables the application of the results to any real cases of coupled liquid and plates. The chosen form of the loading, a spreading pressure, is well adapted to predict the responses to shock waves, such as those found in detonations. For these kinds of loading, a pressure step travels at very high velocities ranging among the characteristic velocities of a plate or of a liquid.

The condition of an unbounded domain is not really very common, but for a small enough time, the beginning of the response is the same, at least before the motion reaches the limits of the domain.

For more complicated loading or boundary conditions, the computational method is very simple to apply and leads to results worthy of confidence. Nevertheless, for supersonic loading, unchanged in translation, the analytical stationary solution is always adequate to describe the response in the neighbourhood of the front of loading. The computational method is a useful complement to obtain the whole transient response.

Previous work by Renard and Taazount [7], has shown that the response of a plate to a pressure spreading with a cylindrical symmetry around a central point was very close to the response of a strip to a travelling pressure travelling along its axis. The results compared the transient and steady state responses and concerned displacements and stresses. It would be interesting to confirm this result for plates coupled with liquid. The present work prepares a possible comparison.

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