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Letter to the Editor

Further investigation into eigenfrequencies of a two-part beam–mass system

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1. Introduction

In a recently published article, Kopmaz and Telli [1] have reported a detailed study on the eigenfrequencies of a two-part beam–mass system with simply supported boundary conditions. The article is perceived to be an enhancement of earlier investigations [2–5] in which only the effects of a concentrated (point) mass on the natural frequencies of a beam undergoing flexural vibration with various boundary conditions have been studied. The work of Kopmaz and Telli [1] considers the free vibration problem of a system consisting of two beam segments between which there is a rigid mass element that may have rotary inertia. The authors have outlined their theory and produced numerical results, which are illustrated in non-dimensional form, by varying significant parameters of the system. The present authors have read this new development with considerable interest. Unfortunately they have found errors both in the theory and the results presented in this work, which disqualify its authors from their claims. The object of this note is to correct the theory of Ref. [1] and present a set of amended results. Wherever possible the same notation of Ref. [1] is used, the theory is redeveloped and errors are identified. The results obtained from the present theory are compared with those obtained from the erroneous theory of Ref. [1] and the differences are highlighted.

2. Theory

Figs. 1 and 2 show the undeflected and deflected configurations of a two-part simply supported beam–mass system, respectively. The lengths of the two beam segments are a and c, and that of the rigid element which joins them is b. The total length of the system is L as shown. In the usual notation the two beam segments i ($i = 1, 2$) are described by their material and cross-sectional properties ρ_i, E_i, A_i and I_i ($i = 1, 2$) which are the density, Young's modulus, cross-sectional area

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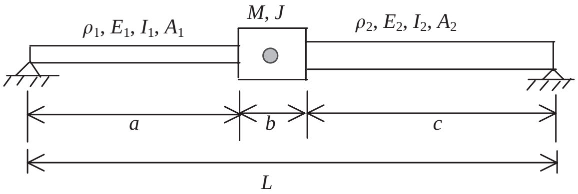

Fig. 1. A two-part beam–mass system.

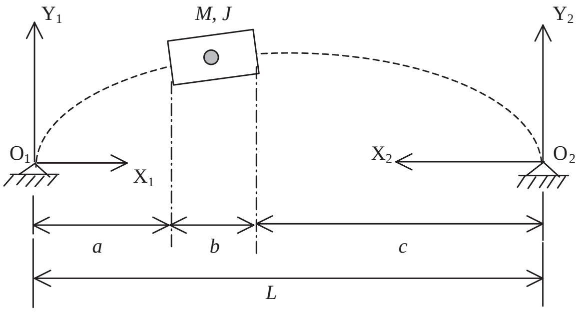

Fig. 2. Deflected shape of the elastic lines of the beam segments together with the rigid mass.

and second moment of area, respectively. The mass and mass moment of intertia of the central element are denoted by M and J, respectively. To be consistent with the notations of Ref. [1], two co-ordinate systems, namely $(O_1X_1Y_1)$ and $(O_2X_2Y_2)$ are chosen for the left-hand end and right-hand beam elements, respectively (see Fig. 2).

The governing differential equations of motion in free vibration for the two beam segments using the two co-ordinate systems are, respectively given by [6,7]

$$E_1I_1y_1'''' + \rho_1A_1\ddot{y}_1 = 0 \tag{1}$$

and

$$E_2I_2y_2'''' + \rho_2A_2\ddot{y}_2 = 0, \tag{2}$$

where a prime and an overhead dot represent differentiation with respect to x_1 or x_2 and time t, respectively.

For harmonic oscillation

$$y_1(x_1, t) = Y_1(x_1)e^{i\omega t}, \quad y_2(x_2, t) = Y_2(x_2)e^{i\omega t}. \tag{3}$$

Substituting Eq. (3) into Eqs. (1) and (2) gives

$$E_1I_1Y_1'''' - \rho_1A_1\omega^2Y_1 = 0 \tag{4}$$

and

$$E_2I_2Y_2'''' - \rho_2A_2\omega^2Y_2 = 0. \tag{5}$$

Introducing the non-dimensional lengths (variables) ξ_1 and ξ_2 where

$$\xi_1 = x_1/a, \quad \xi_2 = x_2/c. \tag{6}$$

Eqs. (4) and (5) can be reduced

$$(D_1^4 - \lambda_1^4)Y_1 = 0 \tag{7}$$

and

$$(D_2^4 - \lambda_2^4)Y_2 = 0, \tag{8}$$

where

$$\lambda_1^4 = \frac{\rho_1 A_1 \omega^2 a^4}{E_1 I_1}, \quad \lambda_2^4 = \frac{\rho_2 A_2 \omega^2 c^4}{E_2 I_2} \tag{9}$$

and

$$D_1 = \frac{\mathrm{d}}{\mathrm{d}\xi_1}, \quad D_2 = \frac{\mathrm{d}}{\mathrm{d}\xi_2}. \tag{10}$$

The solutions of the differential equatios (7) and (8) are given by

$$Y_1(\xi_1) = A_1 \cosh \lambda_1\xi_1 + B_1 \sinh \lambda_1\xi_1 + C_1 \cos \lambda_1\xi_1 + D_1 \sin \lambda_1\xi_1, \tag{11}$$

$$Y_2(\xi_2) = A_2 \cosh \lambda_2\xi_2 + B_2 \sinh \lambda_2\xi_2 + C_2 \cos \lambda_2\xi_2 + D_2 \sin \lambda_2\xi_2. \tag{12}$$

2.1. Boundary conditions

Since the two-part beam–mass system is simply supported (see Figs. 1 and 2), the boundary conditions at the left hand of beam 1 in the $(O_1X_1Y_1)$ co-ordinate system are

$$Y_1(\xi_1 = 0) = 0, \quad E_1I_1Y_1''(\xi_1 = 0) = 0 \tag{13}$$

and at the right-hand end of beam 2 in the $(O_2X_2Y_2)$ co-ordinate system are

$$Y_2(\xi_2 = 0) = 0, \quad E_2I_2Y_2''(\xi_2 = 0) = 0. \tag{14}$$

Thus

$$A_1 = C_1 = A_2 = C_2 = 0. \tag{15}$$

$Y_1(\xi_1)$ and $Y_2(\xi_2)$ can now be written as

$$Y_1(\xi_1) = B_1 \sinh \lambda_1\xi_1 + D_1 \sin \lambda_1\xi_1 \tag{16}$$

and

$$Y_2(\xi_2) = B_2 \sinh \lambda_2\xi_2 + D_2 \sin \lambda_2\xi_2. \tag{17}$$

2.2. Matching conditions for dynamic equilibrium

At the intersections at $x_1 = a$ and $x_2 = c$, the following deformation and load continuity must be satisfied (see Figs. 2 and 3).

Continuity of slope:

$$y_1'(a,t) = -y_2'(c,t). \tag{18}$$

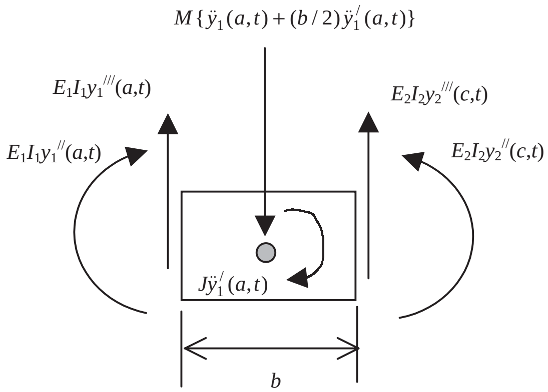

Fig. 3. Forces and moments acting on the rigid mass joining the beam segments.

Compatibility of displacements:

$$y_1(a,t) + by_1'(a,t) = y_2(c,t). \tag{19}$$

Equation of transverse motion:

$$E_1I_1y_1'''(a,t) + E_2I_2y_1'''(c,t) = M\left[\ddot{y}_1(a,t) + \frac{b}{2}\ddot{y}_1'(a,t)\right]. \tag{20}$$

Equation of rotational motion:

$$-E_1I_1y_1''(a,t) + E_2I_2y_2''(c,t) - \frac{b}{2}\left[E_1I_1y_1'''(a,t) - E_2I_2y_2'''(c,t)\right] = J\ddot{y}_1'(a,t). \tag{21}$$

The first three of the above equations are identical with Eqs. (7)–(9) of Ref. [1], but the last one differs from Eq. (10) of Ref. [1]. This is due to the omission in Ref. [1] of the contributions from the shear forces at the end of the two beam segments to the equation of the rotary motion of the central element. The free-body diagram of the rigid mass showing all the forces and moments is illustrated in Fig. 3. The omission of the shear forces in formulating the moment equation seriously compromises the model developed by Kopmaz and Telli [1] and leads to significant numerical errors as will be shown later.

Assuming harmonic oscillation as in Eq. (3), and introducing the non-dimensional length parameters ξ_1 and ξ_2 of Eq. (6), the above four equations can be written as

$$(1/a)Y_1'(1) = (-1/c)Y_2'(1), \tag{22}$$

$$Y_1(1) + (b/a)Y_1'(1) = Y_2(1), \tag{23}$$

$$(E_1I_1/a^3)Y_1'''(1) + (E_2I_2/c^3)Y_2'''(1) = -M\omega^2\left[Y_1(1) + (b/2a)Y_1'(1)\right], \tag{24}$$

$$\begin{aligned} &-(E_1I_1/a^2)Y_1''(1) + (E_2I_2/c^2)Y_2''(1) - (b/2)\left[(E_1I_1/a^3)Y_1'''(1) - (E_2I_2/c^3)Y_2'''(1)\right] \\ &\quad = -(J\omega^2/a)Y_1'(1), \end{aligned} \tag{25}$$

where a prime now denotes differentiation with respect to ξ_1 (or ξ_2) instead of x_1 (or x_2), respectively.

Substituting for $Y_1(\xi_1)$ and $Y_2(\xi_2)$ from Eqs. (16) and (17) into Eqs. (22–25) one obtains four equations in B_1, D_1, B_2 and D_2 which can be expressed in the matrix form

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} B_1 \\ D_1 \\ B_2 \\ D_2 \end{bmatrix} = 0 \tag{26}$$

or

$$\mathbf{DC} = \mathbf{0}, \tag{27}$$

where

$$d_{11} = \lambda_1 \cosh \lambda_1, \quad d_{12} = \lambda_1 \cos \lambda_1, \tag{28}$$

$$d_{13} = \frac{\eta_1}{\eta_2}\lambda_2 \cosh \lambda_2, \quad d_{14} = \frac{\eta_1}{\eta_2}\lambda_2 \cos \lambda_2, \tag{29}$$

$$d_{21} = \sinh \lambda_1 + \frac{\eta_3}{\eta_1}\lambda_1 \cosh \lambda_1, \quad d_{22} = \sin \lambda_1 + \frac{\eta_3}{\eta_1}\lambda_1 \cos \lambda_1, \tag{30}$$

$$d_{23} = -\sinh \lambda_2, \quad d_{24} = -\sin \lambda_2, \tag{31}$$

$$d_{31} = \lambda_1^3\left[\cosh \lambda_1 + \frac{\mu(\eta_1 + \phi\eta_2)}{\eta_1}\lambda_1\left\{\sinh \lambda_1 + \tfrac{1}{2}\lambda_1\frac{\eta_3}{\eta_1}\cosh \lambda_1\right\}\right], \tag{32}$$

$$d_{32} = \lambda_1^3\left[-\cos \lambda_1 + \frac{\mu(\eta_1 + \phi\eta_2)}{\eta_1}\lambda_1\left\{\sin \lambda_1 + \tfrac{1}{2}\lambda_1\frac{\eta_3}{\eta_1}\cos \lambda_1\right\}\right], \tag{33}$$

$$d_{33} = \xi\left(\frac{\eta_1}{\eta_2}\right)^3 \lambda_2^3 \cosh \lambda_2, \quad d_{34} = -\xi\left(\frac{\eta_1}{\eta_2}\right)^3 \lambda_2^3 \cos \lambda_2, \tag{34}$$

$$d_{41} = \lambda_1^2\left[-\sinh \lambda_1 + \psi\mu\frac{(\eta_1 + \phi\eta_2)\eta_3^2}{\eta_1^3}\lambda_1^3 \cosh \lambda_1 - \tfrac{1}{2}\frac{\eta_3}{\eta_1}\lambda_1 \cosh \lambda_1\right], \tag{35}$$

$$d_{42} = \lambda_1^2\left[\sin \lambda_1 + \psi\mu\frac{(\eta_1 + \phi\eta_2)\eta_3^2}{\eta_1^3}\lambda_1^3 \cos \lambda_1 + \tfrac{1}{2}\frac{\eta_3}{\eta_1}\lambda_1 \cos \lambda_1\right], \tag{36}$$

$$d_{43} = \xi\left(\frac{\eta_1}{\eta_2}\right)^2 \lambda_2^2\left[\sinh \lambda_2 + \tfrac{1}{2}\frac{\eta_3}{\eta_2}\lambda_2 \cosh \lambda_2\right], \tag{37}$$

$$d_{44} = -\xi\left(\frac{\eta_1}{\eta_2}\right)^2 \lambda_2^2\left[\sin \lambda_2 + \tfrac{1}{2}\frac{\eta_3}{\eta_2}\lambda_2 \cos \lambda_2\right] \tag{38}$$

with

$$\eta_1 = a/L, \quad \eta_2 = c/L, \quad \eta_3 = b/L = (1 - \eta_1 - \eta_2), \tag{39}$$

$$\phi = \frac{\rho_2 A_2}{\rho_1 A_1}, \quad \xi = \frac{E_2 I_2}{E_1 I_1}, \quad \mu = \frac{M}{(\rho_1 A_1 a + \rho_2 A_2 c)}, \quad \psi = \frac{J}{Mb^2}, \quad \lambda_2 = \lambda_1 \frac{\eta_2}{\eta_1} \sqrt[4]{\frac{\phi}{\xi}}. \tag{40}$$

The frequency equation for the complete system can now be obtained by equating the determinant of **D** to zero. The expression for ϕ in Eq. (24) of Ref. [1] is surely a mistake (the numerator and the denominator must be interchanged). It should be noted that in Ref. [1] the final row of **D** matrix is wrong as its Eq. (10) is faulty.

The frequency equation can thus be written as

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{vmatrix} = 0. \tag{41}$$

The elements of Δ are functions of the non-dimensional parameters λ_1, η_1, η_2, ϕ, ψ, ξ and μ only, where λ_1 alone depends on the frequency ω. (Note that λ_2 is related to λ_1, see Eq. (40).) The dimensionless natural frequencies can therefore, be expressed in terms of λ_1 alone. As in Ref. [1] most of the numerical results are obtained and presented in terms of λ_1 instead of ω.

2.3. Degenerate case

The degenerate case when the lumped mass is assumed to be concentrated at a point as shown in Fig. 4 can be investigated as a special case in which $\eta_3 = 0$, $\eta_2 = 1 - \eta_1$, $\xi = 1$, $\phi = 1$ and $\psi = 0$. For this particular case the elements of **D** are given by

$$d_{11} = \eta_1 \lambda \cosh(\eta_1 \lambda), \quad d_{12} = \eta_1 \lambda \cos(\eta_1 \lambda), \quad d_{13} = \eta_1 \lambda \cosh(\lambda - \eta_1 \lambda), \quad d_{14} = \eta_1 \lambda \cos(\lambda - \eta_1 \lambda), \tag{42}$$

$$d_{21} = \sinh(\eta_1 \lambda), \quad d_{22} = \sin(\eta_1 \lambda), \quad d_{23} = -\sinh(\lambda - \eta_1 \lambda), \quad d_{24} = -\sin(\lambda - \eta_1 \lambda), \tag{43}$$

$$d_{31} = (\eta_1 \lambda)^3 \{\cosh(\eta_1 \lambda) + \lambda \mu \sinh(\eta_1 \lambda)\}, \quad d_{32} = (\eta_1 \lambda)^3 \{-\cos(\eta_1 \lambda) + \lambda \mu \sin(\eta_1 \lambda)\}, \tag{44}$$

$$d_{33} = (\eta_1 \lambda)^3 \cosh(\lambda - \eta_1 \lambda), \quad d_{34} = -(\eta_1 \lambda)^3 \cos(\lambda - \eta_1 \lambda), \tag{45}$$

$$d_{41} = -(\eta_1 \lambda)^2 \sinh(\eta_1 \lambda), \quad d_{42} = (\eta_1 \lambda)^2 \sin(\eta_1 \lambda), \tag{46}$$

$$d_{43} = (\eta_1 \lambda)^2 \sinh(\lambda - \eta_1 \lambda), \quad d_{44} = -(\eta_1 \lambda)^2 \sin(\lambda - \eta_1 \lambda), \tag{47}$$

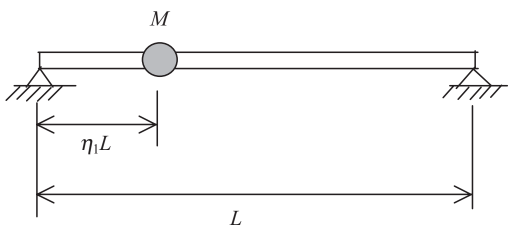

Fig. 4. A simply supported beam carrying a concentrated mass.

where

$$\lambda = \frac{\lambda_1}{\eta_1} = \frac{\lambda_2}{\eta_2}. \tag{48}$$

The determinant $\Delta = |\mathbf{D}|$ can now be expanded and the frequency equation takes the simplified form (this was assisted by the symbolic computation package REDUCE [8])

$$\begin{aligned}\Delta = {} & 2\sin\lambda\sinh\lambda + \lambda\mu\{\sin\lambda\sinh\lambda(\cosh\eta_1\lambda\sinh\eta_1\lambda - \cos\eta_1\lambda\sin\eta_1\lambda) \\ & + \cos\lambda\sinh\lambda\sin^2\eta_1\lambda - \sin\lambda\cosh\lambda\sinh^2\eta_1\lambda\} = 0.\end{aligned} \tag{49}$$

In order to be consistent with Ref. [1], when presenting results the new parameter λ $(= \lambda_1/\eta_1 = \lambda_2/\eta_2)$ has been introduced instead of λ_1 or λ_2 to non-dimensionalize the natural frequencies with respect to the total length L instead of a or b (see Fig. 1 and Eq. (9)). This particular problem has recently been investigated by Low [5] and the above frequency equation agrees completely with his Eq. (8j).

3. Numerical results

In order to compare results, the illustrative examples given in Ref. [1] have been analyzed so that the two different, but related, beam models with rigid mass were used. The first of the two models is based on a point concentrated mass as illustrated in Fig. 4. This model is relatively very simple ($\eta_3 = 0$, $\psi = 0$) and was investigated earlier by Low [5], amongst others. This is referred to below as a beam with concentrated mass model (BCMM). The second one is based on the theory of a two-part beam-mass model (TPBMM) presented in this article and in Ref. [1] where the rotational behaviour of the central (rigid) element is taken into account.

The calculations reported in Ref. [1] were repeated, so that μ and η_3 $(= b/L)$ were held constant. For $\phi = \xi = 1$ and $\psi = 0$ results were obtained for a range of η_1 values which represent the location of the rigid mass. Representative results for the first three natural frequency parameters

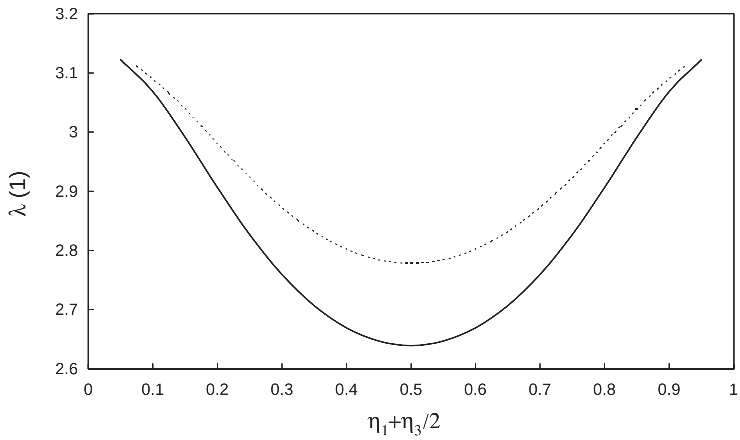

Fig. 5. Effect of the location of the rigid mass on the first natural frequency parameter, using beam with concentrated mass model (BCMM) and two-part beam–mass model (TPBMM): (—) BCMM ($\eta_3 = 0$), (- - -) TPBMM ($\eta_3 = 0.05$).

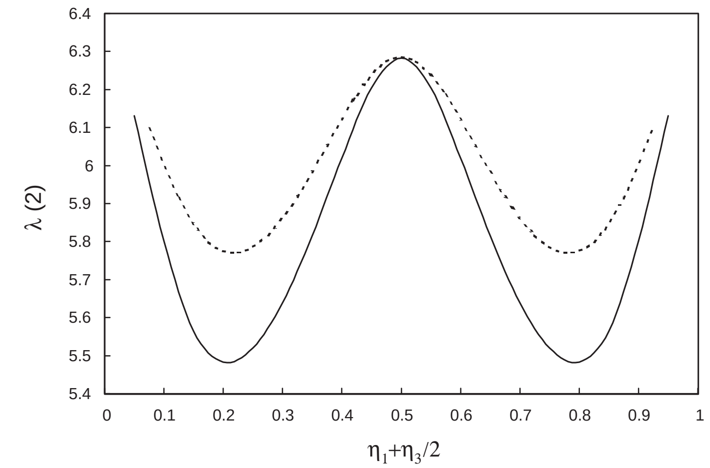

Fig. 6. Effect of the location of the rigid mass on the second natural frequency parameter, using beam with concentrated mass model (BCMM) and two-part beam–mass model (TPBMM): (—) BCMM ($\eta_3 = 0$), (- - -) TPBMM ($\eta_3 = 0.05$).

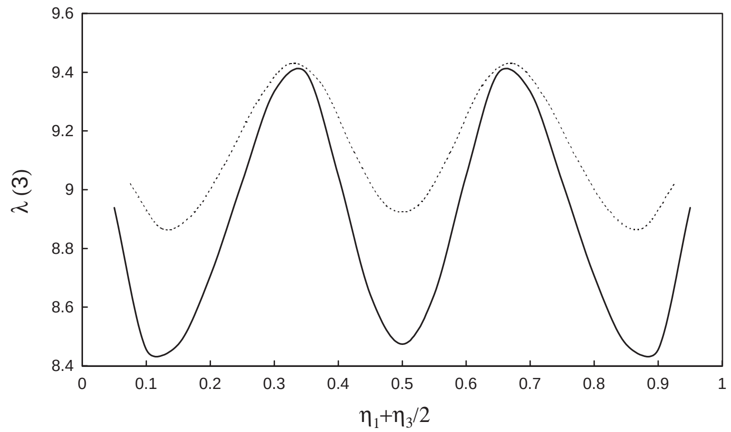

Fig. 7. Effect of the location of the rigid mass on the third natural frequency parameter, using beam with concentrated mass model (BCMM) and two-part beam–mass model (TPBMM): (—) BCMM ($\eta_3 = 0$), (- - -) TPBMM ($\eta_3 = 0.05$).

are shown in Figs. 5–7 using the two models, showing the variation of λ with the length parameter $(\eta_1 + \eta_3/2)$.

The two sets of results illustrated in each of Figs. 5–7 corresponding to $\eta_3 = 0$ for the solid line and $\eta_3 = 0.05$ for the broken line do not match with the corresponding results of Figs. 4–6 of Ref. [1]. Complete agreement for the case with concentrated point mass (shown by solid lines) is expected, but not found. Despite the error in the theory presented in Ref. [1], for small values of η_3 the results should not be as different as they appear to look. The determinantal frequency equation (34) of Ref. [1] has been programmed afresh and the results obtained do not approximate those of Ref. [1]. The BCMM results shown by solid lines in Figs. 5–7 of this article

Table 1
Natural frequencies of a simply supported beam carrying a concentrated (point) mass ($\mu = 0.5$, $\eta_3 = 0$, $\xi = 1$, $\phi = 1$ and $\psi = 0$)

η_1	$\omega_i\sqrt{\rho_1 A_1 L^4/(E_1 I_1)}$		
	$i = 1$	$i = 2$	$i = 3$
0.1	9.4152	33.654	71.498
0.2	8.4476	30.057	75.804
0.3	7.6139	31.798	87.140
0.4	7.1236	36.226	81.863
0.5	6.9660	39.478	71.816

Table 2
Natural frequencies of a two part beam mass system ($\mu = 0.5$, $\eta_3 = 0.05$, $\xi = 1$, $\phi = 1$ and $\psi = 0$)

η_1	η_2	$\omega_i\sqrt{\rho_1 A_1 L^4/(E_1 I_1)}$		
		$i = 1$	$i = 2$	$i = 3$
0.05	0.90	9.6850	37.186	81.338
0.10	0.85	9.4021	34.977	78.620
0.15	0.80	9.0588	33.626	79.619
0.20	0.75	8.7091	33.299	82.761
0.25	0.70	8.3916	33.835	86.562
0.30	0.65	8.1278	35.023	88.924
0.40	0.55	7.7937	38.256	83.290
0.45	0.50	7.7269	39.355	80.035

Table 3
Error assessment as a result of incorrect theory presented in Ref. [1] ($\mu = 0.5$, $\eta_1 = 0.5$, $\eta_2 = 0.3$, $\eta_3 = 0.2$, $\xi = 1$, $\phi = 1$ and $\psi = 0$)

i	$\omega_i\sqrt{\rho_1 A_1 L^4/(E_1 I_1)}$		% error
	Present theory	Ref. [1]	
1	10.758	11.036	2.584
2	43.389	49.978	15.19
3	100.92	117.90	16.82

were further checked using the work of Low [5] and complete agreement was found with that work. It must be concluded that the frequencies computed in Ref. [1] are faulty.

For the interest of readers who wish to check their own theory and computer program for a beam–mass system, Tables 1 and 2 provide numerical values of the non-dimensional natural frequencies for typical cases using the two models. The final set of results was obtained to assess the errors introduced by the authors of Ref. [1] due to incorrect formulation of the moment equilibrium equation which led to wrong frequency equation solution. The incorrect frequency equation of Ref. [1] was programmed in Fortran alongside the amended theory of this article. For

$\psi = 0$, $\phi = \xi = 1$, $\mu = 0.5$, $\eta_1 = 0.5$, $\eta_2 = 0.3$ and $\eta_3 = 0.2$, the results are shown in Table 3. The errors in the natural frequencies using the incorrect theory of Ref. [1] are clearly evident.

4. Conclusions

Following a recent publication and subsequent detection of errors within it, the free vibration behaviour of a two-part beam–mass system has been investigated. Numerical results show that there are serious errors in the results published. These are due to a basic error in the theory that has been identified and judiciously corrected.

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