Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 266 (2003) 193–197
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Mode locking on the non-linear notes of the steelpan

A. Achong*

Department of Physics, University of the West Indies, St. Augustine, Trinidad and Tobago, West Indies

Received 27 November 2002; accepted 5 December 2002

1. Introduction

The steelpan as a musical instrument of the percussion family, with rigid vibrators, has been shown to operate as a system of non-linear mode-localized oscillators [1–3]. Each note consists of a shallow dome-shaped shell formed on the indented face of a steel drum [1]. The first mode (the fundamental) of each note is tuned according to the musical scale with the second mode as an upper octave and the third mode as a musical twelfth. The unique tonality obtained on this instrument is supplied by the amplitude and frequency modulations produced by the non-linear quadratic and cubic interactions between the tuned modes on a note or between two sympathetic notes. To obtain acceptable tonality however, the second and third modes must never be tuned as exact harmonics of the first mode [4].

In the non-musical (exotic) applications, under continuous sinusoidal excitation the instrument has been shown to display Hopf bifurcation and the jump phenomenon [1]. Using electronic synthesis, the instrument has also shown interesting chaotic behaviour [5,6].

The purpose of this communication is to show that the vibrational state of a steelpan note, when played by striking with the stick (or mallet), remains essentially in the *acquisition mode* (in the terminology of control theory) but under the right conditions, may correspond to the oscillations of *locked modes*. The treatment considers interaction of the first two (tuned) modes and is limited to quadratic non-linearities only.

2. Modal equations

Following the analysis of Ref. [1], the governing equations for these rigid vibrators with u_n as the modal displacements are

$$\ddot{u}_n + \omega_n^2 u_n + \varepsilon\left[2\mu_n\dot{u}_n - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\alpha_{j,k,n}u_j u_k - f(t)\right] = 0. \tag{1}$$

*Tel.: +868-662-2002; fax: +868-645-7132.
E-mail address: aachong@steelpanresearch.com (A. Achong).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00035-X

Here, $\ddot{u}$ is the inertial term (the dot represents differentiation with respect to time t), $\omega_n^2 u_n$ is the structural stiffness term with as ω_n the natural frequencies of the linearized system, $\mu_n \dot{u}$ represents viscous damping while μ_n are the damping coefficients; $\sum_{j=1}^{3}\sum_{k=1}^{3} \alpha_{jkn} u_j u_k$ represents quadratic stiffness with α_{jkn} as the coupling parameters ($\alpha_{jkn} = \alpha_{kjn}$), ε is an order parameter, $f(t)$ is a short duration external impulse, which is set to zero after impact for the free system.

For the purposes of this analysis, it is sufficient to consider only the free oscillation phase for the following reasons: Generally for normal playing action on this percussion instrument, the duration of the initial impact phase is less than one period of oscillation of the fundamental (first) mode. In addition, the first mode takes a few periods of oscillation to rise to maximum amplitude while the second mode, as it receives energy from the first mode through quadratic coupling, reaches its maximum somewhat later.

For *phase locking* to occur the natural frequencies ω_1 and ω_2 must be nearly harmonically related as described by the detuning parameter σ where

$$\omega_2 = 2\omega_1 + \varepsilon\sigma. \tag{2}$$

In the present context, $\varepsilon\sigma$ $(= \Omega)$ is the initial frequency detuning. To solve the set of equations in Eq. (1), the multi-time method [7,1] is used. Assuming a set of solutions (for zeroth order in ε) of the form

$$u_{n0} = \tfrac{1}{2} a_n(t_1) \mathrm{e}^{\mathrm{i}(\omega_n t_0 + \phi_n(t_1))} + cc, \tag{3}$$

where a_n and ϕ_n are functions of the slow time t_1 $(= \varepsilon t)$ and represent the amplitude and phase of the nth Fourier component of the displacement, respectively, the (free) system equations are reduced to the following solvability equations:

$$a_1' = -\mu_1 a_1 + \frac{\alpha^*_{121}}{4\omega_1} a_1 a_2 \sin\gamma_1, \qquad a_2' = -\mu_2 a_2 - \frac{\alpha_{112}}{4\omega_2} a_1^2 \sin\gamma_1,$$

$$a_1\phi_1' = -\frac{\alpha^*_{121}}{4\omega_1} a_1 a_2 \cos\gamma_1, \qquad a_2\phi_2' = -\frac{\alpha_{112}}{4\omega_2} a_1^2 \cos\gamma_1, \tag{4a–d}$$

where $\alpha^*_{jkn} = \alpha_{jkn} + \alpha_{kjn}$, the prime denotes $\mathrm{d}/\mathrm{d}t_1$, and

$$\gamma_1 = \phi_2 - 2\phi_1 + \sigma t_1. \tag{5}$$

3. Growth and decay of modal amplitudes

From Eqs. (4a) and (4b) the following conditions are obtained:

1. Mode 1: the amplitude a_1 grows when $(\alpha^*_{121}/4\omega_1)a_2 \sin\gamma_1 > \mu_1$; which requires that $\sin\gamma_1 > 0$.
2. Mode 2: the amplitude a_2 grows when $-(\alpha_{112}/4\omega_2)(a_1^2/a_2)\sin\gamma_1 > \mu_2$; which requires that $\sin\gamma_1 < 0$.
3. The complements of (1) and (2) are the conditions for decay.

Mode coupling is clearly evident in these conditions. As $\sin\gamma_1$ swings between $+1$ and -1, energy is exchanged between the two modes and the modal amplitudes and frequencies are modulated. The modulation rates are described by the slowly varying phase angle γ_1.

4. Mode locking

From Eqs. (4c) and (4d) if one defines $e_{f1} = -(\alpha^*_{121}/4\omega_1)a_2 \cos\gamma_1$ and $e_{f2} = -(\alpha_{112}/4\omega_2)(a_1^2/a_2)\cos\gamma_1$ one can write

$$\phi_1(t_1) = -\frac{\alpha^*_{121}}{4\omega_1}\int a_2 \cos\gamma_1 \, dt_1 = \int e_{f1}\, dt_1,$$

$$\phi_2(t_1) = -\frac{\alpha_{112}}{4\omega_2}\int \frac{a_1^2}{a_2}\cos\gamma_1 \, dt_1 = \int e_{f2}\, dt_1. \quad (6a, b)$$

Eqs. (6a) and (6b) show that the actions of the non-linear shell oscillators are those of integrators generating the phase angles ϕ_n of the oscillator signals $\frac{1}{2}a_n(t_1)e^{i(\omega_n t_0+\phi_n(t_1))}$. This is precisely the role played by the voltage-controlled oscillator (VCO) in an electronic phase locked loop; see for example Ref. [8]. In the case of the steelpan however, because of the non-linear dynamics, the notes are displacement-controlled oscillators (DCOs). In the sense of servo theory, e_{f1} and e_{f2}, are "error displacements" with γ_1 as the "phase error".

The equivalent loop model is shown in Fig. 1. Here, the outputs of the analog multipliers (synthesizing the quadratic parametric excitations) interact (mix) with the normal modes producing the error signals that control the DCOs. The multiplier outputs are the parametric excitations observed on the notes of the steelpan [2,9] while the error signals correspond to the observed amplitude modulations [1]. The inclusion of the low-pass filters (LPF) completes the description of the system as a well-tuned musical instrument where the (weak) high-frequency parametric modes [9] play no significant role.

An essential difference between the operation of the present system and that of an electronic phase-locked loop is that, in the latter, only phase information (and not energy) is transferred to the VCO after comparison is made between the VCO frequency and the reference frequency. On a steelpan note, both phase information and energy are transferred among the various tuned vibrating modes.

By the definition of "lock", the system is locked or in the synchronous mode, if $\phi'_n = 0$. From the cosine in the integrands of Eqs. (6a) and (6b), this requires that ϕ_n arrive at one of the stable

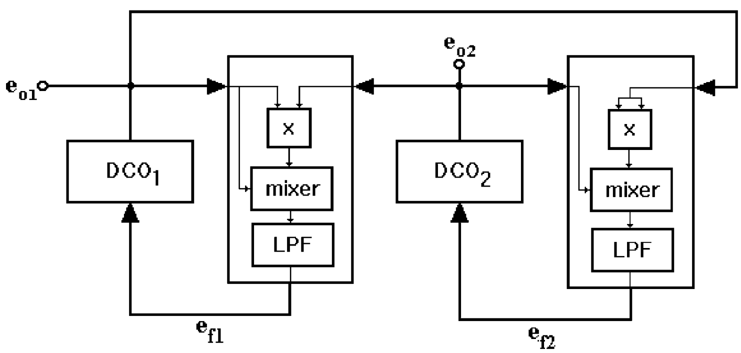

Fig. 1. Model of the coupled non-linear modes on a steelpan note. Mode 1 on the left, $e_{01} = \frac{1}{2}a_1e^{i\omega_1 t+\phi_1}$, and mode 2, $e_{02} = \frac{1}{2}a_2e^{i\omega_2 t+\phi_2}$, on the right.

nulls $\gamma_1 = (4m \pm 1)\pi/2$. In practice, the conditions for steady state are less precise and may be more realistically defined by the conditions $|\phi_n'| \leqslant \delta_\omega$ and $|\phi_n - 2m\pi| \leqslant \delta_\phi$. But this is a steady state condition that cannot be maintained over the duration of the tone driven initially by an impact. The system will therefore constantly be in the acquisition mode, with smooth passages through the synchronous mode if the conditions are right. This behaviour produces frequency modulations that exceed the frequency limit δ_ω.

If the present system is required to lock "harmonically", with $\omega_2 \approx 2\omega_1$, a clearer understanding of the operation of this system can be obtained by considering that initially, the system is not in lock but that the frequency relation is closely harmonic. The quadratic non-linearity, described by the coupling parameter α_{112}, will generate from mode 1 (with displacement u_1), both constant and second harmonic terms:

$$u_1^2 \approx (\cos\omega_1 t)^2 = \tfrac{1}{2}(1 + \cos 2\omega_1 t). \tag{7}$$

This second harmonic component will beat against the second mode at ω_2. Another component at frequency $\omega_2 - \omega_1$ generated through the quadratic coupling defined by the parameter α_{121} will beat against the first mode at ω_1. Under these conditions the error displacements will be two beat components of low frequency $\cong |\omega_2 - 2\omega_1|$. These error displacements, by generating the phase changes in Eqs. (4a) and (4b), will produce changes in the instantaneous frequencies $(\omega_1 + \phi_1', \omega_2 + \phi_2')$ of mode 1 and mode 2, respectively, and at some point these frequencies will be harmonically related and lock may result. While the error displacements may assume levels sufficient to attain lock, the system will be losing energy through material damping and acoustic radiation. The locked state can therefore only be transitional. These changes show up as amplitude and frequency modulations of the decaying musical tones of the instrument.

Mode locking with a 1:2 frequency ratio occurs when

$$\omega_2 + \phi_2' = 2(\omega_1 + \phi_1'), \tag{8}$$

which may be written

$$\omega_2 - 2\omega_1 = 2\phi_1' - \phi_2', \tag{9}$$

and finally as

$$\Omega = \left(\frac{\alpha_{112}}{4\omega_2}\frac{a_1^2}{a_2} - \frac{\alpha_{121}^*}{2\omega_1}a_2\right)\cos\gamma_1. \tag{10}$$

Since $-1 \leqslant \cos\gamma_1 \leqslant +1$, this locking condition is equivalent to

$$|\Omega| \leqslant \left|\frac{\alpha_{112}}{4\omega_2}\frac{a_1^2}{a_2} - \frac{\alpha_{121}^*}{2\omega_1}a_2\right|. \tag{11}$$

The expression on the right-hand side of Eq. (11) gives the greatest frequency separation between the second mode signal at ω_2 and the frequency-doubled signal at $2\omega_1$ in order to attain lock. This expression is therefore the *pull-in range*, which, for the present system, is dependent on the modal amplitudes (an interesting non-linear behaviour).

$$\textit{Pull-in range} = \left|\frac{\alpha_{112}}{4\omega_2}\frac{a_1^2}{a_2} - \frac{\alpha_{121}^*}{2\omega_1}a_2\right|. \tag{12}$$

Mode locking therefore requires strong coupling through the parameter α_{112} or relatively large values, the modal amplitude a_1, together with small values of α_{121} and a_2. In typical phase-locked loops, the reference oscillator is of stable and fixed frequency. In the present system the first mode acts as the reference but its frequency is modulated by $\phi_1'[= -(\alpha_{121}^*/4\omega_1)a_2\cos\gamma_1]$, which effectively reduces the pull-in range as the second term in Eq. (12) shows.

5. Conclusions

It is clear that on this musical instrument, where significant amplitude and frequency modulations are necessary for good tonality, the sounding note will remain in the acquisition mode. However, during the tuning of the instrument, a note can be placed in the state where mode 2 is at the exact second harmonic frequency of mode 1 (equivalent to making $\Omega = 0$). This produces a musically unpleasant tone, called the "pung tone" by the present author [4], in which the fundamental (mode1) after the initial rapid attack, decays rapidly in intensity, with mode 2 in the meanwhile rising more slowly then decaying to zero. There are no additional amplitude modulations and the frequencies remain stable throughout the duration of the tone.

References

[1] A. Achong, The steelpan as a system of non-linear mode-localized oscillators, Part I: theory, simulations, experiments and bifurcations, Journal of Sound and Vibration 197 (1996) 471–487.
[2] A. Achong, Non-linear analysis of compressively/thermally stressed elastic shell structures on the steelpan and the underlying theory of the tuning process, Journal of Sound and Vibration 222 (1999) 597–620.
[3] A. Achong, The theory of the steelpan, in: Proceedings of the First International Conference on the Science and Technology of the Steelpan, Vol. 1, TT Govt. Printery, Trinidad, 2000, pp. 1–16.
[4] A. Achong, Detuning and tonal structure of steelpan notes, in: Proceedings of the First International Conference on the Science and Technology of the Steelpan, Vol. 1, TT Govt. Printery, Trinidad, 2000, pp. 41–52.
[5] A. Achong, Non-linear behaviour and chaos on an electronically simulated steelpan note, in: M. Ding, W.L. Ditto, L.M. Pecora, M.L. Spano (Eds.), Proceedings of the Fifth Experimental Chaos Conference, World Scientific, Singapore, 1999, pp. 133–140.
[6] A. Achong, Pan on the road to chaos, in: Proceedings of the First International Conference on the Science and Technology of the Steelpan, Vol. 1, TT Govt. Printery, Trinidad, 2000, pp. 273–287.
[7] H. Nayfeh, D.T. Mook, Nonlinear Oscillations, McGraw-Hill, New York, 1979.
[8] F.M. Gardner, Phaselock Techniques, Wiley-Interscience, New York, 1979.
[9] A. Achong, The harmonically related parametric excitations on steelpan notes, in: Proceedings of the First International Conference on the Science and Technology of the Steelpan, Vol. 1, TT Govt. Printery, Trinidad, 2000, pp. 33–40.