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Comparison between the discrete and finite element methods for modelling an agricultural spray boom—Part 1: Theoretical derivation

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Abstract

The paper develops a systematic procedure for modelling linear flexible multibody structures of which the flexible parts are a composition of beams. The theory of mechanics of solids is fit into a general expression of virtual work, linking rigid-body motions with flexible deformations of the different bodies. A comparison is made with the finite element method for approximating the behaviour of the flexible bodies. It turns out that the discrete element method boils down to a particular selection of shape functions of which the mass matrix is inconsistent with the flexibility matrix. Furthermore, contrary to the finite element method, only point forces can be applied.

1. Introduction

Modelling techniques can be classified as black box and white box methods. A white box model is obtained by applying physical laws combined in one or a set of mathematical equations. In black box modelling, a mathematical relationship between the input and the output is established, regardless of the underlying physics. The main advantage of white box modelling is that it provides physical insight into the behaviour of the mechanism, which is essential for virtual prototyping. In this way virtual prototyping can be performed. Controllers can be designed and validated on the model or a mechatronic approach can be followed in which the mechanism and the controller are jointly optimized. Concerning robust controller design, white box modelling has the additional advantage that it is easier to model the uncertainty structure, because all

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parameters have a physical meaning. The uncertainty structure reflects bounds on the variation of physical parameters, bounds on non-linearities or bounds on neglected dynamics. Especially in μ-synthesis [1–4] and in quantitative feedback theory (QFT) [5–7] this uncertainty structure is exploited.

Once it has been decided to perform white box modelling, the first concern is to find a suitable technique to derive a linear model. In this paper methods are investigated for linear modelling, based on the basic laws of mechanics. A theoretical technique is established by a systematic and consistent procedure, minimizing the amount of intuition required during derivation of a model. The resulting model should contain a maximum of information and accuracy and has a minimum complexity. Therefore, several methods based on Newton's laws need to be investigated.

The objective of this paper is to elaborate a linear technique for modelling flexible multibody structures of which the flexible bodies are a composition of beam elements. An example of such type of systems is an agricultural spray boom, which was studied in previous work [8] by analyzing the vibrational characteristics of a flexible beam. About the modelling and control of flexible beams, an extensive amount of literature is available. A rudimentary way of modelling flexible beams is to consider them as lumped masses interconnected with springs. However, this approach needs a lot of insight into the fundamental properties of the structure and the modelling procedure becomes very intuitive when the lumped masses are mixed with rigid bodies. Considering one flexible body with the shape of a beam, a model can be derived by solving the Euler–Bernouilli equation [9]. If the moment of inertia and the contribution of shear to bending are not negligible, approximate solutions for the Timoshenko beam equation must be found [10]. Several approximation methods to solve these partial differential equations are proposed in the literature [10,11]. However, in practice, considering flexible multibody beam-like structures like agricultural spray booms, flexible robot arms or cranes, etc. as one simple flexible beam with no rigid-body modes, is too conceptual and too far from reality. In addition a lot of intuition is required to arrive at a certain model.

Ramon [8] observed that agricultural spray booms are basically linear flexible multibody systems for which the theory and methods of dynamics of flexible multibodies need to be employed. Therefore, he developed a systematic procedure to combine flexible and rigid-body motions of multibody systems in one linear model. The flexibilities are approximated by finite elements. In this paper, a procedure is outlined to derive a linear model for flexible multibody systems of which the flexible elements are a composition of beams. Flexibilities are handled by the theory of mechanics of solids. Several ways for mass allocation are studied and a comparison is made with the finite element method. As the procedure developed by Ramon [8,12] is a finite element based method for modelling linear flexible multibody systems, the procedure described in this paper can be considered as a discrete element approach for modelling linear flexible multibody systems.

2. Kinematic description of a point in space

In this section, systems having an open kinematic chain or tree structure are considered. This enables one to describe every point in space in a unique way. To derive the equations of motion systematically and consistently a good kinematic description of a point of the structure is very important. The description of a point in space is based on Shabana [13]. The kinematic description of a point involves an appropriately selected co-ordinate system and a systematic procedure to locate a point in space. Using the proposed co-ordinate system, it is shown that the position vector of a point P_{ij} is determined as a vector sum of three vectors i.e., a vector following the body when it moves as an undeformable structure, a vector pointing to point P_{ij} when the body is in undeformed state and a vector indicating the change in position of P_{ij} due to flexible deformations of the body.

A first subsection discusses the selection of the co-ordinate systems and the characterization of a position vector of a point P_{ij} as a sum of three vectors. In the next subsection, it is explained how to determine these three vectors. Generally, the position of a point is a non-linear function of the co-ordinates. The linearization process is discussed in the last subsection.

2.1. Co-ordinate system and location of a point in space

To facilitate the kinematic description of a flexible multibody system, the co-ordinate system and the procedure to locate a point in space must be selected in an appropriate way. The same kinematic description as in Ref. [13] is used. This kinematic description, depicted in Fig. 1, allows a transparent discrimination between the motion of the body as an undeformable structure and flexible deformations of the body itself. In the next section, it is shown that this turns out to be an advantage for obtaining the starting equation from which the equations of motion are derived.

To observe any point of the multibody system with respect to the same reference, a fixed absolute co-ordinate system $({}^0x, {}^0y, {}^0z)$ with origin o_0 is selected. Note that in Fig. 1 vectors with

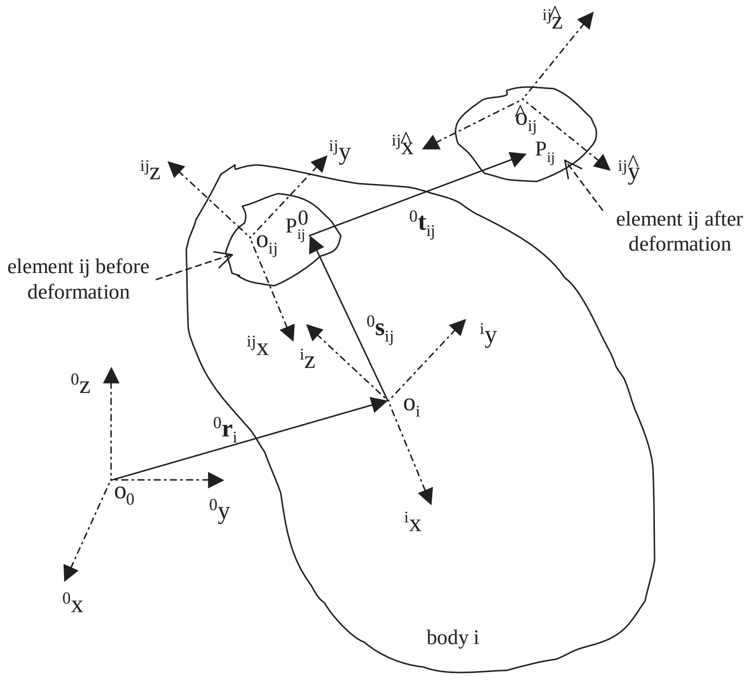

Fig. 1. Description of the position of a point in body i before and after deformation.

the left superscript 0 are interpreted in the absolute reference frame. Every body i is allocated a floating or body reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$ with origin o_i, following the rigid-body motions of body i. The rigid-body motions are those motions of a body i where all particles of the body perform the same displacements. Consequently, in case the body is flexible, the origin o_i is not connected to a physical point of the body. When body i is deforming, frame $({}^{i}x, {}^{i}y, {}^{i}z)$ is 'floating' in the body, which explains its name. A position vector ${}^{0}\mathbf{r}_i$ locates the absolute position of the origin o_i with regard to $({}^{0}x, {}^{0}y, {}^{0}z)$. By definition ${}^{0}\mathbf{r}_i$ is independent of the flexible co-ordinates of the body itself but it can change due to flexible deformations of other bodies and due to elastic deformations of the joints connecting the bodies. Because of the floating nature of the body reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$, flexible deformations of body i can be described easily with respect to this frame. From the body reference frame, every point of the body in undeformed state, P_{ij}^{0} is determined by a position vector $\mathbf{s}_{ij}$ pointing from o_i to P_{ij}^{0} (Fig. 1). The meaning of the extra right subscript j will become clear during the remainder of the text. Note that the left superscript 0 of vector $\mathbf{s}_{ij}$ is missing, which indicates that the vector is interpreted in the co-ordinate system $({}^{i}x, {}^{i}y, {}^{i}z)$ in which it is defined. As the floating reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$ follows the rigid-body motions, $\mathbf{s}_{ij}$ is fixed.

Graul [14] puts the origin of the body reference frame always in the centre of mass of the body as some body properties such as for example moments of inertia become simpler. Again the origin of the body reference frame is not connected to a physical point, since during deformation of the body, the centre of mass changes its position in the body. As a result, ${}^{0}\mathbf{r}_i$ becomes dependent on flexible co-ordinates of the body itself. With this selection of co-ordinate system, it is impossible to get a clear separation between motions of the body as an undeformable structure and flexible deformations.

Since the floating reference frame exclusively follows rigid-body motions of the body, flexible deformations are fully described within this reference frame. This is performed by a vector ${}^{i}\mathbf{t}_{ij}$, reflecting the displacement of a point P_{ij}, from the position before the deformation P_{ij}^{0} to the final position after deformation P_{ij} (Fig. 1). Strictly speaking, with the vectors ${}^{0}\mathbf{r}_i$, ${}^{0}\mathbf{s}_{ij}$ and ${}^{0}\mathbf{t}_{ij}$, the absolute reference frame and the floating reference frames, the position of a point of a flexible multibody system is perfectly defined. However, in some cases, a floating reference frame is not very suited to describe ${}^{i}\mathbf{t}_{ij}$, especially when complex flexible structures are involved, which are difficult to approximate by for example the Rayleigh–Ritz, the assumed modes or the Galerkin method [11]. In case the flexible behaviour is calculated with other approximation techniques, able to tackle complex geometries, as for example the finite element method, the body is split into a finite number of elements requiring local co-ordinate systems, of which the co-ordinates are only defined within the element. These local frames are called intermediate element reference frames $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ and their origin o_{ij} is rigidly attached to the origin of the floating reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$. The orientation of the co-ordinate system $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ is assumed to be fixed to the floating reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$. This implies that both frames $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ and $({}^{i}x, {}^{i}y, {}^{i}z)$ translate and rotate together. Their orientation is not necessarily parallel but remains constant. Note that the name element not necessarily refers to a finite element, but it is just used as a general naming. The term intermediate refers to the fact that the frames $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ are not rigidly attached to a point of the element. The co-ordinate systems of which the origin is rigidly attached to a point on the element are called element reference frames $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$. Normally, the co-ordinate axes of $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$ are selected in such a manner that they are initially parallel to the axes of the intermediate element reference frame $({}^{ij}x, {}^{ij}y, {}^{ij}z)$. The element reference frames $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$ are only used when other bodies are connected to the element in order to express their rotations and displacements, caused by the flexible deformations of the element.

2.2. Determination of vectors ${}^0\mathbf{r}_i$, ${}^0\mathbf{s}_{ij}$ and ${}^0\mathbf{t}_{ij}$

From the above, it is clear that the position vector ${}^0\mathbf{p}_{ij}$ of a point P_{ij} can be written as

$$ {}^0\mathbf{p}_{ij} = {}^0\mathbf{r}_i + {}^0\mathbf{s}_{ij} + {}^0\mathbf{t}_{ij}. \tag{1}$$

A procedure to determine ${}^0\mathbf{r}_i$, ${}^0\mathbf{s}_{ij}$ and ${}^0\mathbf{t}_{ij}$, will be outlined. This procedure is closely related to the geometrical structure of the multibody system. In this section and the next section, flexible multibody systems, having a tree-structure i.e., containing no closed kinematic chains, are investigated. In tree structured systems, there is only one unique way to go from one body to another, by following the joints and the bodies of the system. For such systems and with Eq. (1) it is possible to describe every point in an unambiguous way. Later on in Part 2 of the paper [15], it is shown how the procedure can be extended to systems with closed kinematic chains. If every point is written as a function of independent Lagrangian co-ordinates and flexible co-ordinates, defined via $\mathbf{t}_{ij}$, no constraint equations are involved in the equations of motion, reducing redundant degrees of freedom of the system. A rotation around a cylindrical joint is an example of a Lagrangian co-ordinate i.e., the relative position of two bodies can be described as a function of the rotation angle of the hinge around the hinge axis. The combination of Lagrangian and flexible co-ordinates are often called hybrid or generalized co-ordinates. In this paper, Lagrangian co-ordinates are sometimes referred to as rigid-body co-ordinates.

Once the geometrical structure of the system is known and the decision is made as to what kind of co-ordinates to use, the vectors of Eq. (1) can be calculated. In a first step, vector ${}^0\mathbf{r}_i$, pointing from the origin of the absolute reference frame to the origin of the floating reference frame is determined. This is done based on the location of a body $(i-1)$, connected through a joint k with body i. This body is named body $(i-1)$ because when starting at the inertial body, attached to the absolute reference frame, and following the bodies and the joints, this body is the last body that is met before encountering body i in the kinematic chain. From body $(i-1)$, it is supposed that the location of every point of the body is known i.e. ${}^0\mathbf{r}_{i-1}$, ${}^0\mathbf{s}_{i-1j}$ and ${}^0\mathbf{t}_{i-1j}$ as well as the orientation of the floating, the intermediate element and element reference frames. According to Fig. 2, ${}^0\mathbf{r}_i$ is determined as the vector sum of

$$ {}^0\mathbf{r}_i = {}^0\mathbf{r}_{i-1} + {}^0\mathbf{s}_{i-1a} + {}^0\mathbf{t}_{i-1a} + {}^0\mathbf{z}_{i-1i} - {}^0\mathbf{s}_{ib}. \tag{2}$$

Points a and b are the connection points between joint k and, respectively, body $(i-1)$ and i. The origins $\hat{o}_{i-1a}$ and $\hat{o}_{ib}$ of the element reference frames $({}^{i-1a}\hat{x}, {}^{i-1a}\hat{y}, {}^{i-1a}\hat{z})$, $({}^{ib}\hat{x}, {}^{ib}\hat{y}, {}^{ib}\hat{z})$ are rigidly connected to the elements in point a respectively b. Although the joint of Fig. 2 is, as a matter of an example, a cylindrical joint, allowing one rotational and one translational degree of freedom, expression (2) is completely general, as is indicated in the remainder of the text. Note that no flexibilities ${}^0\mathbf{t}_{ib}$ of body i are involved in Eq. (2) since ${}^0\mathbf{r}_i$ follows only the rigid-body motions of body i, as explained in the previous subsection. In a second step, vectors ${}^0\mathbf{s}_{ib}$ and ${}^0\mathbf{z}_{i-1i}$ need to be determined.

Vector ${}^0\mathbf{s}_{ib}$ follows the rigid-body motions of body i and can consequently be written as

$$ {}^0\mathbf{s}_{ib} = {}^0\mathbf{A}_i\mathbf{s}_{ib} \tag{3}$$

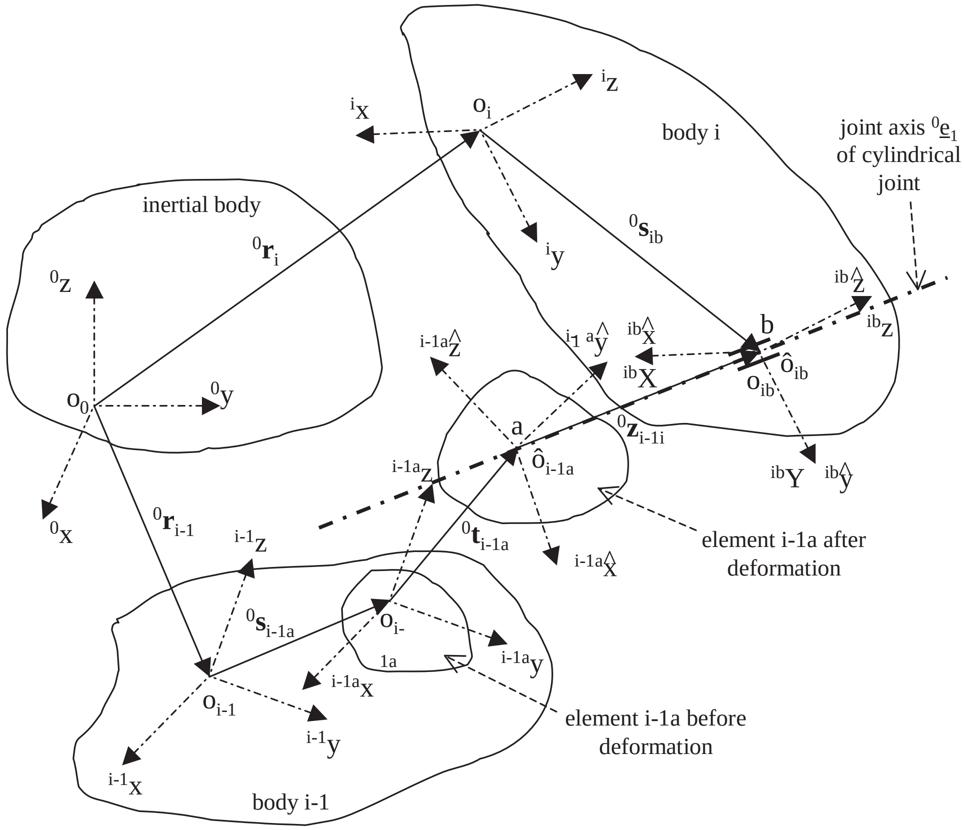

Fig. 2. Determination of $^0\mathbf{r}_i$ from the location of a body $(i-1)$ preceding body i in the kinematic chain.

in which $\mathbf{s}_{ib}$ is the vector interpreted in the floating reference frame $({}^ix, {}^iy, {}^iz)$ from the origin o_i of this frame to point b when body i is considered in undeformed state. Once frame $({}^ix, {}^iy, {}^iz)$ is allocated to body i, $\mathbf{s}_{ib}$ is known. Matrix $^0\mathbf{A}_i$ is a co-ordinate transformation matrix for transforming the co-ordinates in the floating reference frame $({}^ix, {}^iy, {}^iz)$ to the absolute reference frame $({}^0x, {}^0y, {}^0z)$. The left superscript of the transformation matrix refers to which reference frame the co-ordinates, defined in the frame indicated by the right subscript, are transformed. Body i can only change orientation with respect to body $(i-1)$, due to rotation of joint k and due to changes in orientation of the element to which point a is attached. Therefore, transformation matrix $^0\mathbf{A}_i$ can be written as a multiplication of the following matrices:

$$^0\mathbf{A}_i = {}^0\mathbf{A}_{i-1}\,{}^{i-1}\mathbf{A}_{i-1a}\,{}^{i-1a}\mathbf{A}_{\widehat{i-1a}}\,{}^{\widehat{i-1a}}\mathbf{A}_{\widehat{ib}}\,{}^{ib}\mathbf{A}_i. \tag{4}$$

The hat (^) on the subscript and superscript of the transformation matrix $\mathbf{A}$ refers to element reference frames. The transformation matrix from element reference frame $({}^{ib}\hat{x}, {}^{ib}\hat{y}, {}^{ib}\hat{z})$ to the intermediate element reference frame $({}^{ib}x, {}^{ib}y, {}^{ib}z)$ is a unity matrix because body i is in undeformed state and is omitted. Transformation matrices from a floating reference frame to an intermediate element reference frame or vice versa, such as $^{i-1}\mathbf{A}_{i-1a}$ and $^{ib}\mathbf{A}_i$, are constant matrices and can be determined once the floating reference frames and the intermediate element reference frames are placed in the body. Matrix ${}^{i-1a}\mathbf{A}_{\widehat{i-1a}}$ is only a function of the flexible co-ordinates of element $(i-1a)$ because it is the transformation matrix between the intermediate element reference frame $({}^{i-1a}x, {}^{i-1a}y, {}^{i-1a}z)$ and the element reference frame $({}^{i-1a}\hat{x}, {}^{i-1a}\hat{y}, {}^{i-1a}\hat{z})$ and can be determined. The change in orientation between the element reference frames $({}^{i-1a}\hat{x}, {}^{i-1a}\hat{y}, {}^{i-1a}\hat{z})$ and $({}^{ib}\hat{x}, {}^{ib}\hat{y}, {}^{ib}\hat{z})$ is due to rotations allowed by joint k. Consequently, ${}^{\widehat{i-1a}}\mathbf{A}_{\widehat{ib}}$ is only dependent on the rotational Lagrangian co-ordinates of joint k and can be derived immediately.

The translation of body i with respect to body $(i-1)$ due to the translational degrees of freedom of the joint k is represented by vector ${}^{0}\mathbf{z}_{i-1i}$. Because the location of body $(i-1)$ is completely known, ${}^{0}\mathbf{z}_{i-1i}$ is in general determined as

$$ {}^{0}\mathbf{z}_{i-1i} = {}^{0}\mathbf{A}_{i-1}{}^{i-1}\mathbf{A}_{i-1a}{}^{i-1a}\mathbf{A}_{\widehat{i-1a}}(\mathbf{z}^{0}_{i-1i} + q_{k1}\mathbf{e}_1 + q_{k2}\mathbf{e}_2 + q_{k3}\mathbf{e}_3). \tag{5} $$

Vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the unit vectors indicating the directions, defined in element reference frame $({}^{i-1a}\hat{x}, {}^{i-1a}\hat{y}, {}^{i-1a}\hat{z})$, attached to point a, along which the joint allows translation. Lagrangian co-ordinates q_{k1}, q_{k2}, q_{k3} represent the amount of translation performed by joint k. The representation of Eq. (5) is a general formulation. In case of a cylindrical joint as in Fig. 2, only one direction $\mathbf{e}_1$ and one co-ordinate q_{k1} is required. The distance between a and b when the joint is in rest position is determined by $\mathbf{z}^{0}_{i-1i}$.

With Eqs. (3)–(5), supposing that the location and orientation of every point of body $(i-1)$ is known, Eq. (2) can be determined. To derive vector ${}^{0}\mathbf{r}_i$ an iterative procedure is needed and the procedure is started at the bodies linked with a joint to the inertial body, attached to the absolute reference frame $({}^{0}x, {}^{0}y, {}^{0}z)$. In the inertial body all position vectors are known. Vector ${}^{0}\mathbf{r}_0$ is a (3×1) vector of zeros, ${}^{0}\mathbf{s}_{0a}$ is a constant vector known from the geometry of the inertial body and ${}^{0}\mathbf{t}_{\widehat{0a}}$ is function of the flexible co-ordinates of the inertial body. ${}^{0}\mathbf{A}_0$, is the identity matrix, ${}^{0}\mathbf{A}_{0a}$ is a constant matrix depending on the geometry of the inertial body and ${}^{0a}\mathbf{A}_{\widehat{0a}}$ is function of the flexible co-ordinates of the inertial body. Once the vectors $\mathbf{r}_i$ of the adjacent bodies of the inertial frame are determined, the succeeding vectors $\mathbf{r}_i$ in the kinematic chain are appointed by Eqs. (2)–(5). This procedure is repeated until all the origins of the floating reference frames are described.

After the calculation of ${}^{0}\mathbf{r}_i$ in Eq. (1), ${}^{0}\mathbf{s}_{ij}$ and ${}^{0}\mathbf{t}_{ij}$ still need to be derived. The determination of ${}^{0}\mathbf{s}_{ij}$ has already been explained through ${}^{0}\mathbf{s}_{ib}$ and is performed with Eqs. (3) and (4). With respect to the intermediate element reference frame $({}^{ij}x, {}^{ij}y, {}^{ij}z)$, $\mathbf{t}_{ij}$ is only a function of the flexible deformation and is consequently known, but needs to be transformed to the absolute reference frame:

$$ {}^{0}\mathbf{t}_{ij} = {}^{0}\mathbf{A}_{ij}\mathbf{t}_{ij} = {}^{0}\mathbf{A}_{i}{}^{i}\mathbf{A}_{ij}\mathbf{t}_{ij}. \tag{6} $$

The co-ordinate transformation matrix ${}^{0}\mathbf{A}_i$ is calculated in Eq. (4). Matrix ${}^{i}\mathbf{A}_{ij}$ is a constant transformation matrix and is known once the position and the orientation of the floating and intermediate element reference frames are known.

2.3. Linearization procedure

In the previous subsection, it has been shown that every position vector ${}^{0}\mathbf{p}_{ij}$ is built up sequentially. Therefore, linearization is performed during the construction of ${}^{0}\mathbf{p}_{ij}$. Summations do not cause non-linearities. Consequently, only expressions containing multiplications need to be investigated. In Eq. (3) $\mathbf{s}_{ib}$ is constant, such that ${}^{0}\mathbf{A}_i$ is left to be linearized, which is performed on Eq. (4). Only ${}^{0}\mathbf{A}_{i-1}$, ${}^{i-1a}\mathbf{A}_{\widehat{i-1a}}$ and ${}^{\widehat{i-1a}}\mathbf{A}_{\widehat{ib}}$ are functions of the hybrid co-ordinates and need linearization. Matrix ${}^{0}\mathbf{A}_{i-1}$ is supposed to be already in linear form. Matrices ${}^{i-1a}\mathbf{A}_{\widehat{i-1a}}$ and ${}^{\widehat{i-1a}}\mathbf{A}_{\widehat{ib}}$ are functions that are linearized around a nominal working point by a Taylor series expansion. After multiplication of the transformation matrices in Eq. (4), all products between co-ordinates are neglected.

Concerning the linearization of Eq. (5), transformation matrix ${}^{0}\mathbf{A}_{i-1}$ is supposed to be already linearized and ${}^{i-1}\mathbf{A}_{i-1a}$ is a constant matrix. Again in the multiplication, all products between co-ordinates are neglected.

For Eq. (6) ${}^{i}\mathbf{A}_{ij}$ is a constant matrix and ${}^{0}\mathbf{A}_i$ is already linearized. Vector $\mathbf{t}_{ij}$ is a constant vector, such that Eq. (6) is in linear form.

3. General equation for deriving the equations of motion for flexible multibody systems

In the previous section it was explained how rigid-body motions of the whole system and the flexible behaviour of the individual bodies are combined kinematically. The dynamic behaviour of the flexibilities of the different bodies and of the rigid-body motions of the system needs to be coupled. For both separately, well-documented methodologies are available. A procedure, leading to a general equation, must be found, to combine consistently the dynamics caused by body flexible deformations and the dynamics from rigid-body motions of the individual bodies.

In this paper, a virtual work formulation as in Refs. [8,12,16] is employed. This equation is based on methodologies for modelling rigid multibody systems [17,18], in which the principle of virtual work and the principle of d'Alembert are combined together with the theory of structural analysis [19]:

$$\sum_{i=1}^{n}\sum_{j=1}^{mb_i}\int_{V_{ij}}\rho_{ij}\delta^{0}\mathbf{p}_{ij}^{\mathrm{T}0}\ddot{\mathbf{p}}_{ij}\,\mathrm{d}V_{ij} = -\sum_{i=1}^{n}\sum_{j=1}^{mb_i}\int_{V_{ij}}\delta\boldsymbol{\varepsilon}_{ij}^{\mathrm{T}}\boldsymbol{\sigma}_{ij}\,\mathrm{d}V_{ij} + \sum_{i=1}^{n}\delta Ws_i + \sum_{i=1}^{n}\delta Wb_i. \tag{7}$$

Eq. (7) is proved in Ref. [16] in which n is the total number of bodies, mb_i the number of elements in body i, V_{ij} the volume of element j in body i and ρ_{ij} the mass density of element j in body i. By moving some terms to the other side of the equality sign, Eq. (7) can be understood intuitively. From an energetic point of view and with the principle of d'Alembert, it can be said that the virtual work of the inertia forces, the virtual work of the surface forces δWs_i and the body forces δWb_i, result in an increase in internal energy. The internal energy in this case is the energy created by stress $\boldsymbol{\sigma}_{ij}$ and strain $\boldsymbol{\varepsilon}_{ij}$ in the different elements [20]. The stresses $\boldsymbol{\sigma}_{ij}$ and strains $\boldsymbol{\varepsilon}_{ij}$ must be interpreted in the intermediate element reference frames. This can be understood intuitively. By the selection of the co-ordinate systems, flexibilities are described in the intermediate reference frame. Because the stress $\boldsymbol{\sigma}_{ij}$ and strain $\boldsymbol{\varepsilon}_{ij}$ are related to deformations, they must be interpreted in the intermediate reference frames. Surface forces, embedded in δWs_i, include all forces, external to the body i and not necessarily external to the system, acting on the surface of the body i, exclusive constraint forces. Constraint forces reduce the number of degrees of freedom of the system and are generally acting in joints in the directions in which no motion is allowed. The body forces, covered by δWb_i, are for example gravitational and magnetic forces.

4. Equations of motion

With Eq. (7) and the kinematic description of a point of the flexible multibody system in Eq. (1), the equations of motion can be derived. However, it has not yet been explained how to determine $\mathbf{t}_{ij}$, the displacement of a point due to flexible deformations and how to calculate σ_{ij} and ϵ_{ij} . These last two problems are related to the question of how flexibilities are handled. This section proposes two methods to treat the flexibilities of the bodies. The first, elaborated by Ramon [8,12], is based on finite elements. The second one, the discrete element approximation, is developed in this text and makes use of results of the mechanics of solids. Once it is decided how to deal with the flexibilities, the equations of motions are derived.

The main objective of this section is to arrive at the linearized equations of motion, written in matrix notation of the following form [8,12]:

$$\mathbf{M}_s \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{u}} \end{bmatrix} + \mathbf{C}_s \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{u}} \end{bmatrix} + \mathbf{K}_s \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} = \mathbf{V}_s \mathbf{v} + \mathbf{W}_s \mathbf{w} \tag{8}$$

in which $\mathbf{M}_s$, $\mathbf{C}_s$, $\mathbf{K}_s$, are, respectively, the mass, damping and stiffness matrix of the system. Co-ordinates associated with rigid-body motions are represented by $\mathbf{q}$, and flexible co-ordinates by $\mathbf{u}$. On the right hand side of Eq. (8), distinction is made between forces $\mathbf{v}$ coming from commands of a controller, and forces $\mathbf{w}$ resulting from disturbances on the system. Matrices $\mathbf{V}_s$ and $\mathbf{W}_s$ are the force input distribution matrices corresponding to $\mathbf{v}$ and $\mathbf{w}$.

4.1. Finite element approximation

In the finite element method, the body is discretized into a finite number of elements, which are connected through nodes. Nodes are certain physical points in and at the vertices of the elements. With these nodes, the state of the element i.e. the displacement caused by flexible deformations is supposed to be known. From the position of the nodes, the other points of the element are calculated through interpolation [21]. Because the interpolation involves an approximation of the shape of the element after deformation, the interpolation functions are called shape functions. Every node has a certain number of degrees of freedom, represented by nodal co-ordinates. For a certain element ij, nodal co-ordinates are collected in the vector $\hat{\mathbf{u}}_{ij}$, defined in the intermediate element reference frame $({}^{ij}x, {}^{ij}y, {}^{ij}z)$. In matrix notation, the displacement of an arbitrary point caused by flexible deformations, described within $({}^{ij}x, {}^{ij}y, {}^{ij}z)$, can be written as

$$\mathbf{t}_{ij} = \mathbf{\Psi}_{ij}^{\mathrm{T}} \hat{\mathbf{u}}_{ij}, \tag{9}$$

where $\mathbf{\Psi}_{ij}$ is a matrix containing the interpolation functions, which are function of the Cartesian co-ordinates of $({}^{ij}x, {}^{ij}y, {}^{ij}z)$. By substituting the position of the considered point within $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ when body i is in its undeformed state, $\mathbf{\Psi}_{ij}$ is determined and with known $\hat{\mathbf{u}}_{ij}$, can be calculated [8]. For coherence reasons, the nodal co-ordinates are not defined in the intermediate element reference frame $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ but in the floating reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$. Therefore $\mathbf{t}_{ij}$

turns into [8]:

$$\mathbf{t}_{ij} = \mathbf{\Psi}_{ij}^{\mathrm{T}} \mathbf{T}_{ij} \mathbf{u}_{ij} \tag{10}$$

in which $\mathbf{u}_{ij}$ are the co-ordinates described in $({}^{i}x, {}^{i}y, {}^{i}z)$ and $\mathbf{T}_{ij}$ the transformation matrix from $\mathbf{u}_{ij}$ to $\hat{\mathbf{u}}_{ij}$. Note that with $\mathbf{u}_{ij}$, containing a finite number of nodal co-ordinates, the displacement due to flexible deformations of element ij is completely characterized. Consequently, the finite element method approximates the behaviour of a flexible body by a finite number of degrees of freedom.

Once $\mathbf{t}_{ij}$, the displacement of a point P_{ij}, resulting from flexible deformation is known, $\boldsymbol{\varepsilon}_{ij}$ can immediately be calculated. In the derivation of Eq. (7), these displacements are assumed to be small. The connection between $\mathbf{t}_{ij}$ and $\boldsymbol{\varepsilon}_{ij}$ is given by

$$\boldsymbol{\varepsilon}_{ij} = \Delta_{ij} \mathbf{t}_{ij}, \tag{11}$$

where Δ_{ij} is an operator defined by

$$\begin{bmatrix} \frac{\partial}{\partial x_{ij}} & 0 & 0 \\ 0 & \frac{\partial}{\partial y_{ij}} & 0 \\ 0 & 0 & \frac{\partial}{\partial z_{ij}} \\ \frac{\partial}{\partial y_{ij}} & \frac{\partial}{\partial x_{ij}} & 0 \\ \frac{\partial}{\partial z_{ij}} & 0 & \frac{\partial}{\partial x_{ij}} \\ 0 & \frac{\partial}{\partial z_{ij}} & \frac{\partial}{\partial y_{ij}} \end{bmatrix}. \tag{12}$$

In the literature, several relations between $\boldsymbol{\varepsilon}_{ij}$ and $\boldsymbol{\sigma}_{ij}$ are considered. For Hookean materials, this relationship is linear [20,22]:

$$\boldsymbol{\sigma}_{ij} = \kappa_{ij} \boldsymbol{\varepsilon}_{ij}. \tag{13}$$

A Kelvin–Voight model [23,24], describing the behaviour of visco-elastic materials, adds an additional viscous term, proportional to the time derivative of $\boldsymbol{\varepsilon}_{ij}$:

$$\boldsymbol{\sigma}_{ij} = \kappa_{ij} \boldsymbol{\varepsilon}_{ij} + \chi_{ij} \dot{\boldsymbol{\varepsilon}}_{ij}. \tag{14}$$

Note that in case Eq. (13) is used, the material of the bodies is supposed to have no damping.

At this stage, all the ingredients are available to derive the equations of motions with the aid of Eq. (7). Ramon [8,12], worked out Eq. (7) symbolically for the finite element approximation method and obtained the linearized Eq. (8). The determination of the matrices in Eq. (8) is explained in Refs. [8,12].

4.2. Discrete element approximation

The finite element method is generally applicable for handling structural flexibilities but is probably not purposeful enough to end up with the most compact model. Flexible spray booms are a composition of elastic beams. Concentrating on other techniques for modelling beam structures will presumably yield a model with a smaller complexity but with the same accuracy. Equivalent to the finite element method, the objective of this subsection is to provide all necessary information to be able to arrive at the linear expression (8).

In finite element analysis, an approximation of the shape of the body after deformation is considered as a starting point for deriving the equations of motion. In order to obtain a good prediction of the modes of interest, the number of elements, in which the beam is divided, must be sufficiently large. Another approach is to split the flexible structure into small rigid masses, interconnected with springs and dampers. This idea of lumping the mass of a flexible structure is employed in many textbooks and papers [25,26]. Sometimes, in a finite element analysis, the mass matrix is derived by lumping the mass of the flexible bodies of the structure in undeformable masses. The resulting mass matrix is then called inconsistent or lumped mass matrix [27,28]. Contrary to finite element analysis, through the concept of approximating the flexible bodies as a lumped system with a finite number of degrees of freedom, the modes of interest of a flexible beam can be modelled by a small number of undeformable masses, interconnected by springs and dampers. For example Kino et al. [29] and Torfs et al., [30] lump a beam in a number of masses equal to the number of modes. The models predict very well the behaviour of the flexible beam and are well suited for controller design. However, the division of the beam into masses and the determination of the spring and damper constants is rather intuitive and empirical. A method should be found to perform this more systematically. Seto et al. [31] approximate the first five modes of a bridge tower by five lumped masses interconnected by springs. The allocation of masses to the five lumps and the determination of the spring constants are derived from the modal vector components. A drawback of this method is that a finite element analysis must be performed in order to know the mode shapes.

The same idea of lumping the mass of a flexible beam is used in Stodola's and Holzer-Myklestad's method for calculating the eigenfrequencies or critical speeds of shafts [27,28,32], but the approach is more analytical and systematic than the methods just discussed. The mass of the continuous shaft is lumped and concentrated in a limited number of points, named mass allocation points. Instead of interconnecting the masses with springs of which the stiffness coefficient must be determined in an empirical way, all masses are connected to the same beam spring, having the same stiffness properties of the original shaft. The accelerations at the mass allocation points induce inertia forces, which cause bending or torsion. The displacement or the twist angle of the shaft can be calculated, using the formulas of mechanics of solids. Note that the mass of the shaft is lumped, but the stiffness properties are not.

Vernon [33] follows a similar train of thought in the so-called lumped parameter approximation method, illustrated in Fig. 3. By splitting the flexible beam in a number of small undeformable beams, interconnected through a massless beam and with equal stiffness properties as the original flexible beam, the modes of the beam are calculated through a mass and stiffness matrix. Again, the massless beam plays the role of a beam spring. The mass matrix is derived from the geometrical shape of the small undeformable beam and the stiffness matrix is calculated, based on the theory of mechanics of solids.

Przemieniecki [20] employs a generalization of the concept of Vernon. As in the finite element method, he divides a flexible structure into elements, named discrete elements, which are interconnected by nodes. A global stiffness matrix is derived, based on the assembly of the

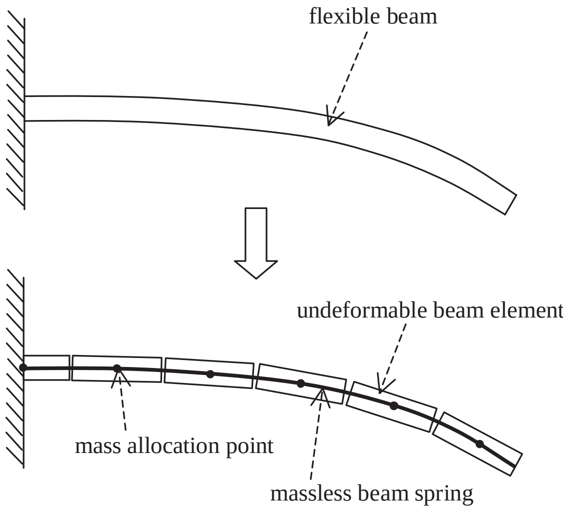

Fig. 3. Lumped parameter approximation method.

stiffness properties of the individual elements. For beam structures, the stiffness properties of an element i.e. the relation between the displacements at the nodes of an element and the loads applied to realize these displacements, are derived by means of the theory of the mechanics of solids. A mass matrix is calculated by allocating mass to the nodes. This allocation is performed by assuming a certain displacement function of the element and is actually comparable to the finite element method. The methodology of Przemieniecki [20] concentrates on one flexible body but is not limited to beam structures. It also includes structures consisting of plate and tetrahedron elements. For static problems, the procedure to determine the stiffness matrix is sometimes called the stiffness method.

In this paper Przemieniecki's method [20], also called the discrete element method, is extended to linear flexible multibody systems. The discrete element idea is fit into the virtual work expression (7) for flexible multibody systems and linearizations are performed in order to obtain the form of Eq. (8).

4.2.1. Lumping of the bodies and co-ordinate systems

In order to start the derivation of Eq. (7), co-ordinate systems must be allocated to the multibody system, as explained in Section 2.1. In the first step, each flexible body i has to be divided into mb_i elements. In the case of the discrete element method a choice has to be made whether this division is performed according to the mass or flexibility properties of the structure. In the finite element method, this choice should not be made because division in elements is solely performed on the geometry or shape of the system. The problem of dividing the structure into elements according to mass or flexibility properties for the discrete element method is highlighted in Fig. 3.

Inspection of Fig. 3 reveals that mass and stiffness are depicted differently. The mass properties are represented by a finite number of undeformable beam elements whereas the stiffness properties are comprised in one single massless beam spring. Consequently, the definition of the elements can be different when mass or stiffness properties are concerned. For example, when the stiffness properties are studied, an element can coincide with a beam or for the mass properties, a lumped mass can be taken. By this, the number of elements of body i depends on whether mass or stiffness properties of the beam are involved. However in order to be consistent with Eq. (7), the elements should be identical regardless of the fact that mass or stiffness properties of the beam are investigated.

In Ref. [20] an element is denoted as the piece between two mass allocation points. Again the interpretation of the element is dependent on whether the mass or stiffness properties of the beam are concerned. When the stiffness properties are considered the continuous beam spring between the two mass allocation points is meant. In the case of mass properties, the lumps of mass between the mass allocation points must be considered. Nevertheless, the definition of element is consistent because the location of the mass allocation points is independent of which properties, stiffness or mass are being investigated. However, this definition is not always taken here. For example, to draw a parallel between the determination of the mass matrix in the discrete and the finite element method, it is easier to let coincide the elements with the lumped masses.

For convenience, the ${}^{ij}x$ and ${}^{ij}\hat{x}$ axes point along the longitudinal axis of the beam and ${}^{ij}y$, ${}^{ij}\hat{y}$, ${}^{ij}z$ and ${}^{ij}\hat{z}$ along the principal axes of the cross-section in which the frame is attached, when the beam is in undeformed state. The principal axes of a cross-section are those axes where the cross moments of inertia equal zero [34]. The principal axes are properties of the cross-section and can be calculated for a beam in undeformed state, because it is supposed that during bending, the cross-sections only perform a displacement and a rotation and do not undergo any deformation. The element reference frame (${}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z}$), follows the orientation and the displacement of the cross-section.

4.2.2. Calculation of the first term of the right hand side of Eq. (7)

The calculation of the first term of the right hand side of Eq. (7) i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{mb_i}\int_{V_{ij}}\delta\boldsymbol{\varepsilon}_{ij}^{\mathrm{T}}\boldsymbol{\sigma}_{ij}\,\mathrm{d}V_{ij} \tag{15}$$

involves the calculation of the contributions of the flexibilities of each individual body to the global stiffness matrix $\mathbf{K}_s$ (Eq. (8)) of the system. It will be shown how the theory of mechanics of solids can be applied to derive the contributions of the flexibilities of the individual bodies to the stiffness matrix i.e., how the discrete element methodology for beam structures, which is developed for one single body, can be used for multibody systems.

For linear elastic isotropic homogeneous materials [34] the formulas of mechanics of solids apply and the strain energy equals the complementary energy [35]. Therefore, as proved in Ref. [16], Eq. (15) can be written as a function of the quantities of the cross-section of the beam

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{mb_i}\int_{L_{ij}}\left(\frac{\delta N_{ij}N_{ij}}{E_{ij}A_{ij}}+\frac{\delta M_{yij}M_{yij}}{E_{ij}I_{yij}}+\frac{\delta M_{zij}M_{zij}}{E_{ij}I_{zij}}+\frac{\kappa_{yij}\delta S_{yij}S_{yij}}{G_{ij}A_{ij}}+\frac{\kappa_{zij}\delta S_{zij}S_{zij}}{G_{ij}A_{ij}}+\frac{\delta M_{tij}M_{tij}}{G_{ij}J_{ij}}\right)\mathrm{d}l_{ij}\right\} \tag{16}$$

in which L_{ij} is the total length and $\mathrm{d}l_{ij}$ an infinitesimal piece of the beam element ij. The first three terms in Eq. (16) express the virtual work of the beam due to extension or contraction of the cross-section and can be caused by a tension force N_{ij} or a bending moment M_{yij} or M_{zij} around one of the principle axes. The following two terms are the contributions of shear forces S_{yij} or S_{zij}, also directed along the principal axes and acting on the cross-section. The last term represents the virtual work due to torsion M_{tij} of the cross-section. All the quantities N_{ij}, M_{yij}, M_{zij}, S_{yij}, S_{zij}, and M_{tij} are interpreted in the intermediate element reference frame $({}^{ij}x, {}^{ij}y, {}^{ij}z)$. Note that N_{ij}, M_{yij}, M_{zij}, S_{yij}, S_{zij} and M_{tij} are actually scalar functions of which the interpretation is unambiguous once the principal axes of the cross-section are known. The other symbols are material properties or properties of the cross-section i.e. E_{ij}, Young's modulus of element j in body i, G_{ij}, the shear modulus of element j in body i, A_{ij}, the cross-section of element j in body i, I_{yij} (I_{zij}), the bending moment of the cross-section around the $y(z)$ axis of the intermediate element reference frame of element j in body i, J_{ij}, the torsion moment of inertia of element j in body i. Expression (16) is also valid for curved beams, as long as the radius of curvature is not smaller than two times the height or the width of the cross-section [22,34].

The theory of elasticity proves that the effects of various loads may be superimposed, provided that the strains and the rotations are sufficiently small and the elasticity is linear [33,35]. Keeping this in mind, the quantities N_{ij}, M_{yij}, M_{zij}, S_{yij}, S_{zij}, and M_{tij} are linear functions of the external loads (forces and couples) applied to body i and not only of the forces working on element ij. As the location and orientation of $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ remains constant with respect to the floating reference frame $({}^{i}x, {}^{i}y, {}^{i}z)$, N_{ij}, M_{yij}, M_{zij}, S_{yij}, S_{zij}, and M_{tij} can be interpreted as parts of the global N_i, M_{yi}, M_{zi}, S_{yi}, S_{zi}, and M_{ti} functions of body i for element ij. Therefore, without loss of generality, N_{ij}, M_{yij}, M_{zij}, S_{yij}, S_{zij}, and M_{tij} can be seen as linear functions of a load vector $\mathbf{ld}_{ik}$, in which $\mathbf{ld}_{ik}$ is the kth external load acting on body i, interpreted in $({}^{i}x, {}^{i}y, {}^{i}z)$. Consequently, the partial derivatives, $\partial N_{ij}/\partial \mathbf{ld}_{ik}$, $\partial M_{yij}/\partial \mathbf{ld}_{ik}$, $\partial M_{zij}/\partial \mathbf{ld}_{ik}$, $\partial S_{yij}/\partial \mathbf{ld}_{ik}$, $\partial S_{zij}/\partial \mathbf{ld}_{ik}$ and $\partial M_{tij}/\partial \mathbf{ld}_{ik}$ are constant with respect to $\mathbf{ld}_{ik}$. Because one single load $\mathbf{ld}_{ik}$ can be decomposed along ${}^{i}x$, ${}^{i}y$, ${}^{i}z$, it is represented by a vector.

Referring to the basic concept of the method, which is represented in Fig. 3, it is important to note that $\mathbf{ld}_{ik}$ represents all types of external forces, including inertia forces from the masses in the mass allocation points. The above derivations are valid when the system is in static equilibrium. According to the principle of d'Alembert, acceleration of mass can be interpreted as inertia forces. By this interpretation, the structure can be considered in static equilibrium such that the above-mentioned derivations can be applied. By using the operator

$$\Delta_{ld_i} = \left[\frac{\partial}{\partial \mathbf{ld}_{i1}} \quad \cdots \quad \frac{\partial}{\partial \mathbf{ld}_{ik}} \quad \cdots \quad \frac{\partial}{\partial \mathbf{ld}_{in_{ld}}} \right]^{\mathrm{T}} \tag{17}$$

in which sub-subscript n_{ld} is the total number of loads acting on body i, and by collecting all loads $\mathbf{ld}_{ik}$ acting on body i in the vector $\mathbf{Ld}_i$ Eq. (16) can be rewritten in

$$\sum_{i=1}^{n} \Bigg\{ \delta \mathbf{Ld}_i^{\mathrm{T}} \sum_{j=1}^{mb_i} \int_{L_{ij}} \Bigg(\frac{\Delta_{ld_i} N_{ij} N_{ij}}{E_{ij} A_{ij}} + \frac{\Delta_{ld_i} M_{yij} M_{yij}}{E_{ij} I_{yij}} + \frac{\Delta_{ld_i} M_{zij} M_{zij}}{E_{ij} I_{zij}} + \frac{\kappa_{yij} \Delta_{ld_i} S_{yij} S_{yij}}{G_{ij} A_{ij}} + \frac{\kappa_{zij} \Delta_{ld_i} S_{zij} S_{zij}}{G_{ij} A_{ij}} + \frac{\Delta_{ld_i} M_{tij} M_{tij}}{G_{ij} J_{ij}} \Bigg) \mathrm{d}l_{ij} \Bigg\}. \tag{18}$$

According to Castigliano's theorem [34,35], every sum:

$$\mathbf{u}_{ik} = \sum_{j=1}^{mb_i} \int_{L_{ij}} \left(\frac{N_{ij}}{E_{ij}A_{ij}} \frac{\partial N_{ij}}{\partial \mathbf{ld}_{ik}} + \frac{M_{yij}}{E_{ij}I_{yij}} \frac{\partial M_{yij}}{\partial \mathbf{ld}_{ik}} + \frac{M_{zij}}{E_{ij}I_{zij}} \frac{\partial M_{zij}}{\partial \mathbf{ld}_{ik}} + \frac{\kappa_{yij}S_{yij}}{G_{ij}A_{ij}} \frac{\partial S_{yij}}{\partial \mathbf{ld}_{ik}} + \frac{\kappa_{zij}S_{zij}}{G_{ij}A_{ij}} \frac{\partial S_{zij}}{\partial \mathbf{ld}_{ik}} + \frac{M_{tij}}{G_{ij}J_{ij}} \frac{\partial M_{tij}}{\partial \mathbf{ld}_{ik}} \right) \mathrm{d}l_{ij} \tag{19}$$

represents the displacement or rotation $\mathbf{u}_{ik}$ of the point in which the load $\mathbf{ld}_{ik}$ is acting. The points of which the displacements and rotations are determined by the vectors $\mathbf{u}_{ik}$ are called nodal co-ordintates or nodal points by analogy with the finite element method. They must be interpreted in the floating reference frame because $\mathbf{ld}_{ik}$ is also considered in this frame $({}^i x, {}^i y, {}^i z)$. Note that with Eq. (18), the co-ordinates $\mathbf{u}_{ik}$ can only be calculated on the point of action of the loads. Consequently, contrary to the finite element method, the selection of the nodal points and the elements is not completely free. This implies that points where loads are acting must be nodal points. The mass allocation is still free. Furthermore, only concentrated loads may be applied to the beams of body i, otherwise the integral expression in Eq. (19) loses its interpretation of a flexible co-ordinate. By the finite element method, distributed loads are divided among the nodal co-ordinates by the shape functions. Actually, the same method can be applied here. By collecting all vectors $\mathbf{u}_{ik}$ of body i in a vector $\mathbf{u}_i$, Eq. (18) reduces to

$$\sum_{i=1}^{n} \{\delta \mathbf{Ld}_i^{\mathrm{T}} \mathbf{u}_i\}. \tag{20}$$

Eq. (20) indicates how the stiffness method or the discrete element method [20] for beam elements, can be applied to derive the stiffness matrix for a flexible multibody system. For one single body currently called i, without rigid-body degrees of freedom, these methods provide a relation between the loads applied to the body i and the nodal co-ordinates $\mathbf{u}_i$ i.e.,

$$\mathbf{Ld}_i = \mathbf{K}_i \mathbf{u}_i. \tag{21}$$

The construction of matrix $\mathbf{K}_i$ is explained in Ref. [16] and consists of assembling matrices of a fixed form $\mathbf{K}_{ij}$, depending on the length of the element ij, some quantities of the cross-section and material properties of the element. With Eqs. (20) and (21), the contribution of the flexibilities of the bodies, collected in matrix $\mathbf{K}$, to the stiffness matrix $\mathbf{K}_s$ can be written as

$$\mathbf{K} = \mathrm{diag}[(\mathbf{K}_i)_i]. \tag{22}$$

This approach is also used in the finite element method where it is applied to each body to obtain stiffness matrices $\mathbf{K}_i$, which are collected in a global $\mathbf{K}$-matrix. Also in finite element analysis, matrix $\mathbf{K}_i$ is assembled by some fixed matrices $\mathbf{K}_{ij}$. The shape of $\mathbf{K}_{ij}$ depends on the type of shape function selected. It turns out that the matrix $\mathbf{K}_{ij}$ for the discrete element method, is in some cases exactly the same for the finite element method. This happens, for example, in 2-D problems when Hermitian interpolation functions are used for beam elements [16]. This is also shown in Ref. [15].

For the derivation of the stiffness matrix, in case of the discrete element method, use has been made of the theory of mechanics of solids, assuming linear elastic materials according to Eq. (13). This implies that by the discrete element method, only material damping can be introduced in an artificial way, for example modal damping [36]. For the finite element method, damping can be introduced by using the Kelvin–Voight model of Eq. (14).

4.2.3. Determination of the mass matrix

The notion of mass allocation point is broadened to nodal point. From now on, mass allocation points also include points in which loads are acting and to which no mass is allocated. In order to keep the mass matrix $\mathbf{M}_s$ regular, it is correct to allocate mass to every mass allocation point. This mass matrix $\mathbf{M}_s$ is calculated by equating the left hand side of expression (7). As can be seen from Eq. (7), the position of every point ${}^0\mathbf{p}_{ij}$ needs to be known, which can be calculated by Eq. (1). The only unknown is the displacement vector ${}^0\mathbf{t}_{ij}$, representing the flexible deformations of body i. Fortunately, this vector can easily be determined, as every flexible body is lumped in undeformable masses, which are rigidly connected to the mass allocation points. Consequently according to Fig. 4, within the element reference frame $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$, connected to the mass allocation point, having nodal co-ordinates $\mathbf{u}_{ik}$, every point P_{ij} of the lumped mass is characterized by a constant position vector $\widehat{{}^{ij}\mathbf{t}_{ij}}$ in the undeformable mass with respect to $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$. As the lumped mass is undeformable ${}^{ij}\mathbf{t}_{ij}$ in $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ and $\widehat{{}^{ij}\mathbf{t}_{ij}}$ in $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$ are equal. Vector ${}^0\mathbf{t}_{ij}$ can be expressed as

$$ {}^0\mathbf{t}_{ij} = {}^0\mathbf{A}_i {}^i\mathbf{A}_{ij}\mathbf{t}_{ij} \tag{23} $$

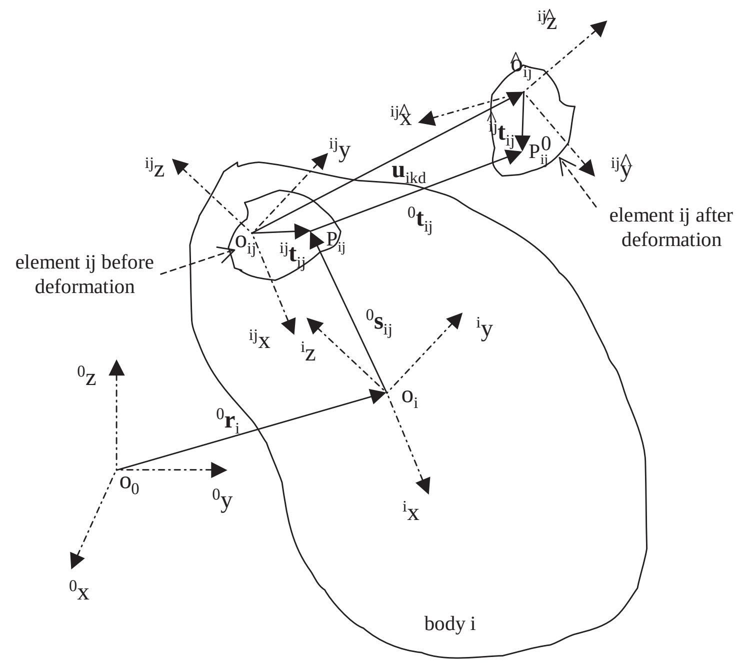

Fig. 4. Description of the position of a point in body i, before and after deformation in case of the discrete element method.

with:

$$\mathbf{t}_{ij} = -{}^{ij}\mathbf{t}_{ij} + {}^{ij}\mathbf{A}_i\mathbf{u}_{ikd} + {}^{ij}\mathbf{A}_{\widehat{ij}}\,{}^{\widehat{ij}}\mathbf{t}_{ij} \tag{24}$$

in which $\mathbf{u}_{ikd}$ are the nodal co-ordinates of $\mathbf{u}_{ik}$, expressing displacements. For convenience, the elements coincide with the lumped masses. Note that according to Eq. (19), $\mathbf{u}_{ik}$ must be interpreted in the floating reference frame. The vector $\mathbf{u}_{ikd}$ is ordered consistently such that the first entry is the displacement along the ${}^{ij}x$-axis, the second entry the displacement along the ${}^{ij}y$-axis and the last entry the displacement along the ${}^{ij}z$-axis of the intermediate element reference frame.

As already noted, the transformation matrix ${}^{ij}\mathbf{A}_i$ is a constant matrix. In the case when the rotations, due to flexible deformations, are small, the transformation matrix ${}^{ij}\mathbf{A}_{\widehat{ij}}$ can be linearized. The matrix ${}^{ij}\mathbf{A}_{\widehat{ij}}$ is a function of the rotational co-ordinates of ${}^{ij}\mathbf{u}_{ik}$, as it describes the difference in orientation between the intermediate element and the element reference frame, caused by the flexible deformations.

Left superscripts ij are added to ${}^{ij}\mathbf{u}_{ik}$ to indicate that the vector must be interpreted in the intermediate element reference frame (${}^{ij}x, {}^{ij}y, {}^{ij}z$) and not in the floating reference frame (${}^{i}x, {}^{i}y, {}^{i}z$). Suppose that the three rotational co-ordinates of $\mathbf{u}_{ik}$ are grouped in $\mathbf{u}_{ik\theta}$ and ordered in the same consistent way as $\mathbf{u}_{ikd}$ but now with respect to rotations around the ${}^{ij}x$-, ${}^{ij}y$-, ${}^{ij}z$-axis of the intermediate element reference frame, then ${}^{ij}\mathbf{A}_{\widehat{ij}}\,{}^{\widehat{ij}}\mathbf{t}_{ij}$ can be written in linearized form as

$${}^{ij}\mathbf{A}_{\widehat{ij}}\,{}^{\widehat{ij}}\mathbf{t}_{ij} = (\mathbf{I}_3 + {}^{ij}\tilde{\mathbf{u}}_{ik\theta})\,{}^{\widehat{ij}}\mathbf{t}_{ij}, \tag{25}$$

in which $\tilde{}$ is the tilde operator, defined as

$$\tilde{\mathbf{s}} = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}, \tag{26}$$

turning a vector into a matrix. The state in which the body i is undeformed is selected as linearization point, implying all u_{ik} equal to zero. Substitution of Eq. (25) into Eq. (24) and taking into account the following equality, which is a standard result of the tilde operator calculus, Eq. (24) turns into

$$\mathbf{t}_{ij} = {}^{ij}\mathbf{A}_i\mathbf{u}_{ikd} + {}^{\widehat{ij}}\tilde{\mathbf{t}}_{ij}^{\mathrm{T}}\,{}^{ij}\mathbf{A}_i\mathbf{u}_{ik\theta}. \tag{27}$$

Note that the transformation of $\mathbf{u}_{ik\theta}$ to ${}^{ij}\mathbf{u}_{ik\theta}$ is written explicitly with the constant transformation matrix ${}^{ij}\mathbf{A}_i$ In the most general case, $\mathbf{u}_{ik}$ is a (6×1) vector, because the position and orientation of every lumped undeformable mass is characterized by three translations and three rotations. Without loss of generality, every vector $\mathbf{u}_{ik}$ can be ordered such that $\mathbf{u}_{ik} = [\mathbf{u}_{ikd}\ \ \mathbf{u}_{ik\theta}]^{\mathrm{T}}$. Therefore Eq. (27), the position $\mathbf{t}_{ij}$ of a point of a lumped mass as seen in the intermediate element reference frame (${}^{ij}x, {}^{ij}y, {}^{ij}z$), is a linear function of the co-ordinates of the mass allocation point to which the lumped mass is attached:

$$\mathbf{t}_{ik} = [I_3 \quad \widehat{{}^{ij}\tilde{\mathbf{t}}_{ij}^{\mathrm{T}}}] \begin{bmatrix} {}^{ij}\mathbf{A}_i & O_3 \\ O_3 & {}^{ij}\mathbf{A}_i \end{bmatrix} \mathbf{u}_{ik}. \tag{28}$$

Eq. (28) has exactly the same form as Eq. (10), expressing displacements for the finite element approach, in which the transformation matrix $\mathbf{T}_{ij}$ of Eq. (10) is written more explicitly as

$$\begin{bmatrix} {}^{ij}\mathbf{A}_i & \mathbf{O}_3 \\ \mathbf{O}_3 & {}^{ij}\mathbf{A}_i \end{bmatrix}.$$

However, in the finite element method, in order to know the displacement of a point $\mathbf{t}_{ij}$ of an element ij after deformation, its position must be known as a function of the Cartesian co-ordinates of $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ when the body i is in its undeformed state. These co-ordinates are filled in, in $\mathbf{\Psi}_{ij}^{\mathrm{T}}$ of Eq. (10) and $\mathbf{t}_{ij}$ is obtained. This is in contrast to expression (28), where matrix $[\mathbf{I}_3 \quad \widehat{{}^{ij}\tilde{\mathbf{t}}_{ij}^{\mathrm{T}}}]$ is a function of the Cartesian co-ordinates of the element reference frame $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$. In the discrete element method, the flexibilities are framed in the displacements and rotations of the lumped masses. The lumped masses themselves are undeformable even if the body is in distorted state. Consequently, all points of a lumped mass, connected to the element reference frame $({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z})$, are located in this frame in the same way as in the intermediate element reference frame $({}^{ij}x, {}^{ij}y, {}^{ij}z)$, when the body is in undeformed state. Therefore, just as $\mathbf{\Psi}_{ij}^{\mathrm{T}}$, $[\mathbf{I}_3 \quad \widehat{{}^{ij}\tilde{\mathbf{t}}_{ij}^{\mathrm{T}}}]$ can be interpreted as dependent of the Cartesian co-ordinates of $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ when the body is undistorted. $\mathbf{\Psi}_{ij}^{\mathrm{T}}$ and $[\mathbf{I}_3 \quad \widehat{{}^{ij}\tilde{\mathbf{t}}_{ij}^{\mathrm{T}}}]$ are completely equivalent and the latter can be interpreted as a matrix containing Eq. (10) is more general, because an arbitrary function of the Cartesian co-ordinates of $({}^{ij}x, {}^{ij}y, {}^{ij}z)$ can be used. Furthermore $\mathbf{t}_{ij}$ in Eq. (10), can be a function of more than six nodal co-ordinates (3 rotational and 3 translational). Vector $\mathbf{u}_{ij}$ covers all nodal co-ordinates of one element ij of body i, whereas $\mathbf{u}_{ik}$ denotes the nodal co-ordinates of mass allocation point k of body i. This implies that with Eq. (10), the real position of a point in an element can be better approximated. As Eq. (10) is a generalization of Eq. (28), the same procedure as for the finite element method can be used to calculate the mass matrix $\mathbf{M}_s$ [8,12,16]. Note that Eq. (28) is derived under the assumption of small rotations. This supposition is also made in the classical finite element analysis [37].

By Eqs. (28) and (10), it turns out that the discrete element method makes use of linear interpolation functions to derive the mass matrix. This is inconsistent with the stiffness matrix. In the theory of mechanics of solids, for example in bending problems, the second derivative of the displacement of the beam, is proportional to the applied moment to the structure [22]. Consequently, the stiffness matrix obtained by the discrete element method cannot be obtained from linear displacement functions.

4.2.4. Calculation of the virtual work delivered by the external forces

The external forces to the bodies are represented by the last two terms on the right hand side of Eq. (28). They include forces of springs and dampers connected to the bodies, gravity force and control and disturbance forces. For all these forces, the virtual displacements of the point of action need to be expressed as a function of the Lagrangian and nodal co-ordinates. Because every point where a force is acting is a mass allocation point, having nodal co-ordinates, the derivation is easier than in the finite element method. Vector $\mathbf{t}_{ij}$, the displacement due to flexible deformations, becomes then simply $\mathbf{u}_{ik}$. The procedure to express the displacement of a point as a function of the generalized co-ordinates is the same for both the discrete and the finite element methods and has already been discussed in previous sections of the paper. Based on that, the damping and the total stiffness matrices $\mathbf{C}_s$ and $\mathbf{K}_s$ can be completed by adding the contributions of dampers and springs to the contributions of the flexible bodies itself. Similarly, the control and disturbance input distribution matrices $\mathbf{V}_s$ and $\mathbf{W}_s$ can be derived. Note that in the case of the discrete element method only concentrated forces may be used.

5. Conclusions

A systematic procedure has been proposed to derive the linear equations of the motion of flexible multibody systems of which the flexible parts are a composed of beams. To approximate the flexibilities, the discrete element method has been used in which the flexible beams are divided into small undeformable bodies. The flexible connections between these bodies are handled by the theory of mechanics of solids. By using a general expression of virtual work and an appropriate description of a point in space, rigid-body motions and flexible deformations are integrated elegantly in a linear matrix expression, consisting of a mass, stiffness, damping and input distribution matrices. A comparison has been made between the discrete and finite element method for approximating the flexible behaviour of the bodies. It turns out that the discrete element method corresponds to a finite element approximation with an inconsistent mass and stiffness matrix. The discrete element method assumes implicitly, through the theory of mechanics of solids, that no material damping is present such that the flexible material itself has no contribution in the damping matrix. If material damping is required, it must be introduced in an artificial way by for example modal damping. Furthermore, with the discrete element method, only concentrated and not distributed forces can be applied to the flexible bodies.

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