Comparison between the discrete and finite element methods for modelling an agricultural spray boom—Part 2: Automatic procedure for transforming the equations of motion from force to displacement input and validation

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Abstract

This paper validates the discrete element method for linear flexible multibody systems, elaborated in Part 1 of the paper, of which the flexible bodies are a composition of flexible beams. An automatic procedure is developed to convert the linear equations of motion of a multibody system from force to displacement input. By this procedure, support motions and displacements of actuators between the bodies can be employed as an input to the system. Furthermore, using this procedure, the methodology explained in Part 1, which was valid for tree structured systems can be extended to systems containing closed kinematic chains. The methodology of Part 1 is applied for the discrete and finite element approximations to model the horizontal behaviour of an agricultural spray boom. As the inputs to the spray boom are known under the form of positions, the equations of motions are converted from force to position inputs. The discrete and finite element approximations are compared based on accuracy and the complexity of the resulting models.

1. Introduction

The equations of motion of a mechanism can be derived in several ways. In Part 1 of this article [1] the discrete and finite element method for modelling linear flexible multibody systems are compared theoretically. In this part of the article, both methods are applied to a model of an

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agricultural spray boom. The accuracy with respect to the complexity of the models is assessed. The models are evaluated by applying input forces and observing the response of the system by measuring accelerations or displacements. In case the input forces **v** and **w** are known or measured, the response can be calculated by solving the following differential equation:

$$\mathbf{M}_s \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{u}} \end{bmatrix} + \mathbf{C}_s \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{u}} \end{bmatrix} + \mathbf{K}_s \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} = \mathbf{V}_s \mathbf{v} + \mathbf{W}_s \mathbf{w}, \tag{1}$$

in which $\mathbf{M}_s$, $\mathbf{C}_s$, $\mathbf{K}_s$, are, respectively, the mass, damping and stiffness matrices of the system, **q** and **u** are co-ordinate vectors describing rigid and flexible body motions respectively. On the right hand side of Eq. (1), distinction is made between forces **v** from commands of a controller, and forces **w** created by disturbances in the system. Matrices $\mathbf{V}_s$ and $\mathbf{W}_s$ are the force and disturbance input distribution matrices corresponding to **v** and **w**.

However, in many applications the input excitations are not known as forces but rather as displacements, velocities or accelerations. A typical example can be found in the field of earthquake or shock engineering, in which the intensity of the earthquake is registered in an accelerogram, representing the ground acceleration time history diagram [2–6]. To minimize the effect of support motion on a certain structure, the displacement of the support, instead of forces is considered as the input [7–12]. In the study of the response of vehicles to guideway unevenness, the guideway profile is known and in case the speed of the vehicle is given, the vertical displacements of the contact point of the tires can be deduced as a function of time [3]. Another application, which is related to the latter, can be found in service load simulation [13–15] in the automotive sector. In service load simulation, procedures are developed to determine the inputs, applied to the actuators of a shaker, in order to simulate or to reproduce road measurements on a vehicle. Apart from some spindle-coupled test rigs, the shakers are position controlled instead of force controlled systems and displacements must be determined as inputs instead of forces.

In this paper a similar problem is met. The soil profile of a standard track is known from which, together with a dynamic model of the tractor or trailer, disturbances in the form of positions, velocities or accelerations to the spray boom itself can be calculated, but not the forces. In other cases, the accelerations, measured at the three-point hitch of the tractor or at the connections of the boom to the vehicle are known, instead of the forces applied by the vehicle to the boom. These accelerations are inputs to the model of the spray boom. By this, disturbances **w** and control forces **v** of Eq. (1), should be transformed to positions, velocities or accelerations.

As a consequence, before comparing the accuracy of the models from the discrete and finite element approach on the agricultural spray boom, the problem of position input in relation to Eq. (1) is solved. Initially, a literature survey is performed. Subsequently, formulas to transform Eq. (1) from force to position input are derived. These formulas are applied to models of the agricultural spray boom. The resulting displacement input models, based on a discrete and finite element approximation, are evaluated. Finally, conclusions are drawn.

2. State of the art

In the literature, techniques are available to transform the equations of motion of a mechanical system to positions, velocities or acceleration inputs, but only for a limited number of situations.

Very often, especially in earthquake engineering, it is supposed that motions of the base, to which the structure is connected, excite the system. Furthermore, the assumption is made that the base is free to move and acts like an undeformable, solid structure, such that the motion of all connection points is the same. In many textbooks [2,3,6], this situation is called ground or base motion or excitation by support motion. Most of the time, the procedure how to use the accelerations of the base as inputs, is illustrated by a simple single-degree-of-freedom system. In the more general case of a finite multiple-degree-of-freedom system, the accelerations of the base can be easily transformed to input forces. The equations of motion are derived by considering no input forces and all co-ordinates are expressed relatively to the base. As the base is free to move, the co-ordinates of the base, have no contribution in the stiffness and damping matrix. The columns of the mass matrix corresponding to the co-ordinates of the base motion are extracted from the mass matrix and serve as inertia forces exciting the system.

In continuous systems, consisting of one flexible body, described by a partial differential equation, the suppositions of base or support motion are considered too. In this case, the absolute displacement of every point can be split into a rigid-body translation of the support and the elastic body deformations [16,17]. Contrary to the elastic body deformations, the motion of the base is independent of the spatial co-ordinates and does not contribute to the stiffness of the flexible body. Therefore, the base acceleration, multiplied by the total mass of the system can be considered as an inertia force exciting the system.

An alternative method to introduce positions and velocities as inputs to the equations of motion can be found in systems, having a finite number of degrees of freedom. Sometimes, the supposition can be made that only forces, resulting from springs and dampers, connected between the structure and the supports, excite the structure. Again, this problem is handled in many textbooks about vibrations, which have already been mentioned. Most of them illustrate the principle by a simple single-degree-of-freedom system. In Ref. [16], the problem is treated more extensively. Contrary to the previous situation, in which the structure was considered as excited by the accelerations of the support or the base, the latter is not included in the derivation of the equations of motion of the system. It is not necessary that each support excecutes the same motion. Based on these assumptions, input distribution matrices driven by the positions of the supports can be derived. Since the system is only excited by forces induced by the motion of the springs and dampers between the supports and the structure, the force input distribution matrices are zero and can be omitted. In the construction of the stiffness and damping matrix, discrimination is made between spring and damping forces, resulting from the connections to the supports and the other internal spring and damping forces. The contributions of the support springs and dampers in these matrices are separated and moved to the other side of the equations. The new input distribution matrices are obtained by grouping the support positions and the support velocities in different matrices. In the case in which the deformation of the support springs or dampers are used as co-ordinates, the dampers and the springs between the structure and the support do not need to have a linear behaviour, because their deformation itself is given as an input. This is illustrated in Ref. [18], where the vibrations of an aerial vehicle subjected to earthquakes are studied. In this case, the soil under the vehicle is considered as a non-linear spring, parallel to a non-linear damper.

In the previously discussed methods, suppositions (structure is excited by a base or by springs and dampers) had to be made to transform the equations of motion to position, velocity or

acceleration inputs. A general solution to write the equations of motion as a function of motions of certain locations on the structure is found in continuous systems, described by partial differential equations. The motion of certain points, serving as inputs to the system are introduced in the equation of motion as time-varying boundary conditions [17,19]. Unfortunately, the solution of the partial differential equations becomes very difficult, especially in case of multi-support motions [20,21] and the selection of the numerical technique to solve the differential equations must be chosen very carefully. Using an appropriately selected approximation method, the partial differential equations with moving boundary conditions turn into an equation comparable to Eq. (1) in which $\mathbf{v}$ and $\mathbf{w}$ represent co-ordinates describing the motion of the boundary.

In this paper, the underlying idea for introducing displacement inputs instead of force inputs is different from the previous described methods. The causality on which the laws of Newton are based, is preserved i.e., a change in motion is caused by a force. This implies that in order to excite a system, forces are required. Therefore, the equations of motion are derived by using forces as inputs, resulting in Eq. (1). No bases, supports, dummy springs or dampers are introduced. In case of continuous structures, instead of using varying boundary conditions, the equations of motion are derived by introducing forces on the boundaries. After selecting an appropriate approximation method, the equations of motion turn into Eq. (1). Since the motions of the input or at the boundaries are known, it is possible, by proper selection of equations in Eq. (1), to calculate the forces causing these motions. Once the forces are known, the remaining unknown co-ordinates of the left hand side of Eq. (1) can be computed. It turns out that these calculations are pure algebra, which can performed automatically once the equations of motion are known.

3. Introduction of displacement inputs into the equation of motion

In order to excite a dynamical system, forces and torques are required. In the case in which the change of position of a point of the system is considered to be the excitation signal, a force must be delivered, of which it is assumed implicitly that enough power is available, to realize the displacement. Therefore, the equations of motion are derived with forces and torques as input signals, leading to Eq. (1) in case of linear systems. In this section, there is no need to distinguish between disturbance and control forces $\mathbf{v}$ and $\mathbf{w}$ but rather to discriminate between known forces and torques $\mathbf{f}_f$ and forces and torques $\mathbf{f}_p$ of which the motion of their point of action is given:

$$\mathbf{M}_s \ddot{\mathbf{q}\mathbf{u}} + \mathbf{C}_s \dot{\mathbf{q}\mathbf{u}} + \mathbf{K}_s \mathbf{q}\mathbf{u} = \mathbf{J}_s \mathbf{f}_p + \mathbf{F}_s \mathbf{f_f} \tag{2}$$

in which $\mathbf{J}_s$ and $\mathbf{F}_s$ are the input distribution matrices corresponding to $\mathbf{f}_p$ and $\mathbf{f_f}$ respectively. For ease of notation, the Lagrangian co-ordinates and the flexible co-ordinates are combined in $\mathbf{q}\mathbf{u}$. In the presented methodology, the forces $\mathbf{f}_p$, will be replaced by the displacement $\mathbf{d}_p$ at their point of action pointing in the direction of the forces $\mathbf{f}_p$:

$$\mathbf{d}_p = \mathbf{J}_s^{\mathrm{T}} \mathbf{q}\mathbf{u}. \tag{3}$$

It is easy to see that the matrix $\mathbf{J}_s$ of Eq. (3) is similar to the force input distribution matrix of the forces $\mathbf{f}_p$ in Eq. (2). Eq. (2) describes the dynamic equilibrium of the flexible multibody system. When the system is subjected to a virtual displacement, which does not violate the kinematic

constraints, the total work performed by all forces, equals zero. By using Lagrangian co-ordinates **q** and by restricting the flexible deformations to a limited number of co-ordinates **u**, the total number of co-ordinates **qu**, equals the number of the degrees of freedom of the system. For open kinematic chains, they uniquely determine the state of the system, satisfying the constraints. Because of this both sides of Eq. (2) could be premultiplied by $\delta\mathbf{qu}^{\mathrm{T}}$, the virtual change of the hybrid co-ordinates **qu**:

$$\delta\mathbf{qu}^{\mathrm{T}}\mathbf{M}_s\ddot{\mathbf{qu}} + \delta\mathbf{qu}^{\mathrm{T}}\mathbf{C}_s\dot{\mathbf{qu}} + \delta\mathbf{qu}^{\mathrm{T}}\mathbf{K}_s\mathbf{qu} = \delta\mathbf{qu}^{\mathrm{T}}\mathbf{J}_s\mathbf{f}_p + \delta\mathbf{qu}^{\mathrm{T}}\mathbf{F}_s\mathbf{f_f}. \tag{4}$$

The term $\delta\mathbf{qu}^{\mathrm{T}}\mathbf{J}_s$, equal to the variation of Eq. (3), represents the virtual displacements in the direction of the forces $\mathbf{f}_p$, which are allowed by the system. The above derivation looks straightforward, because the equations of motion of expression (2) have been derived by the principle of virtual work. However, even if Eq. (2) would have been obtained by another method, expression (3) is still valid on condition that the system is tree structured and the co-ordinates used, determine the state of the system uniquely and satisfy the constraints imposed by the system.

In non-linear multibody dynamics, expression (3) is interpreted as a constraint equation, which is incorporated by a Lagrange multiplier in Eq. (2). The solution of such problems generally leads to differential–algebraic equations (DAE), that should be solved by advanced algorithms [22]. In case of linear systems, Eq. (3) can be substituted in Eq. (2). Generally, the number of forces $\mathbf{f}_p$, equal to the number n_d of the displacements $\mathbf{d}_p$, is smaller than the number n_{qu} of the co-ordinates **qu**. As the co-ordinates **qu** uniquely determine the state of the system, $\mathbf{J}_s$ can be permuted to split it in a $n_d \times n_d$ regular upper part $\mathbf{J}_{sr}$ and a singular $(n_{qu} - n_d) \times n_d$ lower part $\mathbf{J}_{ss}$. Equivalently, the co-ordinates **qu** are permuted and split in a similar way. Inserting this in Eq. (3) results in

$$\mathbf{d}_p = \mathbf{J}_{sr}^{\mathrm{T}}\mathbf{qu}_r + \mathbf{J}_{ss}^{\mathrm{T}}\mathbf{qu}_s \tag{5}$$

in which $\mathbf{qu}_r$ can be interpreted as the n_d co-ordinates of **qu** which are directly influenced by $\mathbf{d}_p$ and $\mathbf{qu}_s$ as the $(n_{qu} - n_d)$ remaining co-ordinates of **qu**. The co-ordinates $\mathbf{qu}_r$ can then be considered as consisting of a known part, determined by the displacements $\mathbf{d}_p$ and an unknown part determined by $\mathbf{qu}_s$:

$$\mathbf{qu}_r = (\mathbf{J}_{sr}^{\mathrm{T}})^{-1}\mathbf{d}_p - (\mathbf{J}_{sr}^{\mathrm{T}})^{-1}\mathbf{J}_{ss}^{\mathrm{T}}\mathbf{qu}_s. \tag{6}$$

By performing the same permutations on Eq. (2) and by splitting the co-ordinates **qu** in $\mathbf{qu}_r$ and $\mathbf{qu}_s$, it is proven in Appendix A that Eq. (2) can be written as a function of the known displacements $\mathbf{d}_p$:

$$\begin{bmatrix} \mathbf{M}_{dd} & \mathbf{M}_{dq} \\ \mathbf{M}_{qd} & \mathbf{M}_{qq} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{d}}_p \\ \ddot{\mathbf{qu_s}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{dd} & \mathbf{C}_{dq} \\ \mathbf{C}_{qd} & \mathbf{C}_{qq} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{d}}_p \\ \dot{\mathbf{qu_s}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{dd} & \mathbf{K}_{dq} \\ \mathbf{K}_{qd} & \mathbf{K}_{qq} \end{bmatrix} \begin{bmatrix} \mathbf{d}_p \\ \mathbf{qu_s} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{F}_d \\ \mathbf{F}_q \end{bmatrix} \mathbf{f_f} + \begin{bmatrix} \mathbf{I}_{dd} \\ \mathbf{O}_{qd} \end{bmatrix} \mathbf{f}_p, \tag{7}$$

in which $\mathbf{I}_{dd}$ is an $(n_d \times n_d)$ unity matrix and $\mathbf{O}_{qd}$ an $(n_{qu} - n_d) \times n_d$ matrix of zeros. Der Kiureghian and Neuenhofer [23] postulate an equation, having a comparable form as Eq. (7), based on the derivations in the book of Clough and Penzien [6]. However, they do not consider the force inputs $\mathbf{f}_f$ and formulas to calculate the block matrices of Eq. (7) are not described. In

Ref. [6], the problem of multiple support systems is handled for finite-dimensional structures but an equation having the shape of Eq. (7) is not derived.

In the lower part of Eq. (7), $\mathbf{f}_f$ and $\mathbf{d}_p$, together with its derivatives are known such that the unknown co-ordinates $\mathbf{qu}_s$ can be calculated. In the upper part of Eq. (7), $\mathbf{d}_p$ and $\mathbf{f}_f$ are given, and as $\mathbf{qu}_s$ has already been determined, $\mathbf{f}_p$ can be computed. To describe the motion and deformation of the system, it is not necessary to know $\mathbf{f}_p$ as $\mathbf{d}_p$ is supposed to be known. Consequently, the upper part of Eq. (7) can be omitted and only the lower part of Eq. (7) suffices to calculate the dynamic behaviour of the system:

$$\mathbf{M}_{qq}\ddot{\mathbf{q}}\mathbf{u}_s + \mathbf{C}_{qq}\dot{\mathbf{q}}\mathbf{u}_s + \mathbf{K}_{qq}\mathbf{q}\mathbf{u}_s = -\mathbf{M}_{qd}\ddot{\mathbf{d}}_p - \mathbf{C}_{qd}\dot{\mathbf{d}}_p - \mathbf{K}_{qd}\mathbf{d}_p + \mathbf{F}_q\mathbf{f}_f. \tag{8}$$

Note that the number of degrees of freedom of the system is reduced by n_d, because Eq. (3) imposes n_d constraints on Eq. (2). In [6,23], $\mathbf{K}_{qd}$ is eliminated, based on static considerations. Nevertheless, both the first and the second derivative of $\mathbf{d}_p$ must be known at the same time. In many situations, only the positions are given in a sampled vector and the first and second derivative need to be calculated numerically. Appendix B proves that Eq. (8) can easily be transformed into state-space form, in which no derivatives of $\mathbf{d}_p$ are required to calculate $\mathbf{qu}_s$:

$$\begin{bmatrix} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{qq} & \mathbf{I}_{qq} \\ -\mathbf{M}_{qq}^{-1}\mathbf{K}_{qq} & -\mathbf{M}_{qq}^{-1}\mathbf{C}_{qq} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{qf} & \mathbf{A}_c \\ -\mathbf{M}_{qq}^{-1}\mathbf{F}_d & -\mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}\mathbf{A}_c - \mathbf{A}_k \end{bmatrix} \begin{bmatrix} \mathbf{f}_f \\ \mathbf{d}_p \end{bmatrix}, \tag{9}$$

$$\mathbf{qu}_s = -\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd}\mathbf{d}_p + \mathbf{x}_1. \tag{10}$$

The equation of motion, transformed into Eqs. (9) and (10) has the advantage that the behaviour of the mechanical flexible multibody system can be assessed with tools of system theory and linear algebra, which is of large value in control engineering. The states retain a physical meaning since the matrices $\mathbf{M}_{qq}$, $\mathbf{C}_{qq}$ and $\mathbf{K}_{qq}$ can still be interpreted, respectively, as mass, damping and stiffness matrices. The state vector $\mathbf{x}_1$ corresponds to positions or rotations, whereas $\mathbf{x}_2$ corresponds to velocities, as it is the derivative of $\mathbf{x}_1$.

If the accelerations or velocities of $\mathbf{qu}_s$ are required as output, the state-space equation must be augmented by an accelerometer model or by filters performing the derivatives. In the case of no forces $\mathbf{f}_f$ are present, derivatives of $\mathbf{qu}_s$ are obtained by taking the corresponding derivatives of $\mathbf{d}_p$ as an input to the system, which follows from Eq. (8).

In many situations, $\mathbf{qu}_s$ is not required as an output but rather a linear combination $\mathbf{y}$ of $\mathbf{qu}_r$ or $\mathbf{qu}_s$:

$$\mathbf{y} = \begin{bmatrix} \mathbf{L}_1^{\mathrm{T}} & \mathbf{L}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{qu}_r \\ \mathbf{qu}_s \end{bmatrix}. \tag{11}$$

If the components of $\mathbf{qu}_s$ are relative co-ordinates and the absolute position of a point needs to be known, also $\mathbf{qu}_r$ must be taken into account. Eq. (11) is transformed to known quantities by combining expressions (10) and (6):

$$\mathbf{y} = (\mathbf{L}_2^{\mathrm{T}} - \mathbf{L}_1^{\mathrm{T}}(\mathbf{J}_1^{\mathrm{T}})^{-1}\mathbf{J}_2^{\mathrm{T}})\mathbf{x}_1 + (\mathbf{L}_1^{\mathrm{T}}(\mathbf{J}_1^{\mathrm{T}})^{-1} + \mathbf{L}_1^{\mathrm{T}}(\mathbf{J}_1^{\mathrm{T}})^{-1}\mathbf{J}_2^{\mathrm{T}}\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd} - \mathbf{L}_2^{\mathrm{T}}\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd})\mathbf{d}_p. \tag{12}$$

The method presented can also be employed to derive the dynamic equations for continuous structures, described by partial differential equations, having multiple support motions as dealt by Chen et al. [20,21], Schlager et al. [19] and Meirovitch [17]. Instead of first proposing a solution in the form of a restricted linear combination of wave forms which leads ultimately to a discrete approximation of the structure, the structure is first discretized. This can for example be performed with one of the methods proposed in Ref. [1]. On the structure where support motion occurs, forces are introduced and the equations of motions are derived. By the procedure of this subsection the introduced forces exerted by the supports on the structure are converted to support motions. The application of complex numerical techniques can as such be avoided, which is a major advantage. Clough and Penzien [6] suggested a similar procedure.

The technique elaborated in this section is not restricted to support motions only. Changing the lengths of actuators and linking two bodies of the system, can also by used as inputs to the multibody system. The procedure is the same as in the case of support motions. The equations of motion are derived with the actuator force as an input. By the procedure explained in this paper, the equations of motion are converted automatically from force to position input.

In addition, methods for deriving the equations of motion of tree structured flexible multibody systems, can be extended to systems with closed kinematic chains by using the procedure presented. In such systems, certain points in the structure cannot be described in a unique way, because these points can be reached through different paths along the joints and the bodies. A joint keeps two adjacent bodies together by exerting constraint forces and torques on the bodies in the directions in which no motion is allowed. These constraint forces and torques acting from one body to the other are for coupled bodies equal in magnitude and direction but have opposite sign. Therefore, after all closed kinematic chains in a mechanism are broken by removing joints, the constraint torques and forces due to the joints, are retained and considered as external forces. Because, the manipulated structure has become a tree structure, the methodology to derive the equations of motion, developed in Ref. [1], can be used. Subsequently, the external forces and torques are transformed to position inputs. As the external forces and torques of a removed joint act on two disconnected bodies, the displacement inputs express the change of relative position between the two bodies. Consequently, if all relative positions between all disconnected bodies are set equal to zero, the kinematic chains are closed again. This is accomplished by removing the corresponding columns in expressions (9), (10) and (12).

4. Modelling the horizontal behaviour of an agricultural spray boom

In this section, the horizontal behaviour of an agricultural spray boom is modelled. As in the horizontal plane, mainly yawing and jolting are responsible for an irregular spray distribution pattern, only these motions are investigated. Fig. 1 shows the top view of the system to be modelled. The structure consists of a platform driven by two excitation actuators and on which a spray boom with horizontal suspension is mounted. The platform is able to reproduce yawing and jolting tractor motions inducing horizontal boom vibrations. The two hydraulic actuators of the shaker are fixed onto the upper part of the platform such that their pistons can reproduce the motion of the two drawbar tips of the tractor three point hitch. The double acting double-rod linear pistons of the two cylinders have a stroke of 100 mm and net area of 290 mm^2 (Rexroth,
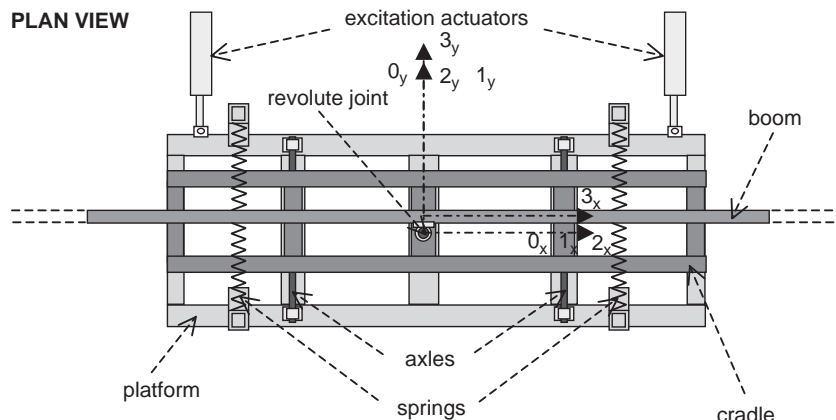
Fig. 1. Top view of a simplified agricultural spray boom.

type CG 70 E 25/16-XXZ13/01 HFUM11T). Two LVDT position sensors (Solartron DC 50) measure the position of the pistons in the cylinders. The piston positions are fed back to two PID controllers (Rexroth VT1600S3X) regulating two four-way servovalves (Rexroth, type 4 WS 2 EM6-1 X/5B1ET315Z7EM). The boom is represented by a 12 m long slender beam, having vibrational characteristics comparable to a large spray boom, with a working width ranging between 32 and 40 m. In order to isolate the boom from the tractor, it is provided with a translational degree of freedom along the driving direction of the tractor and a rotational degree of freedom around a vertical axis. The translational degree of freedom is realized by a cradle, sliding through bearings (shaft sliding collars) along two axles, connected to the platform. A revolute joint, which is composed of a vertical axle and a bearing, provides the rotational degree of freedom. Connecting springs, between the platform and the spray boom, create a passive suspension.

The flexible multibody system consists of two rigid bodies i.e., the platform and the cradle and one flexible body, the boom. The bodies are numbered in ascending order, according to their position in the kinematic tree, starting from the absolute reference frame $(^0x, ^0y, ^0z)$, which is selected at the position of the vertical axle of the revolute joint when the system is at rest. This point coincides with the centre of mass of the platform and the cradle. Floating reference frames $(^1x, ^1y, ^1z)$, $(^2x, ^2y, ^2z)$ are put at the centres of mass of, respectively, the platform and the cradle and coincide with $(^0x, ^0y, ^0z)$ when the system is in rest position. The floating reference $(^3x, ^3y, ^3z)$ of the boom is located at the centre of mass of the beam, when it is in undeformed state. All Z-axes are perpendicular to the sketch and point towards the reader.

In order to allocate the intermediate element and the element reference frames, the flexible beam must be divided into elements. In case of the finite element method, no restrictions are placed on the division of the structure into elements. The easiest way is to take elements of equal length. However, for the discrete element method, all points of the body, which are connected to other bodies and points in which forces are acting must be mass allocation points and are consequently the start or end of an element. The selection criterion in case of the discrete element method is depicted in Fig. 2. This figure shows two element sizes, but care has been taken that they
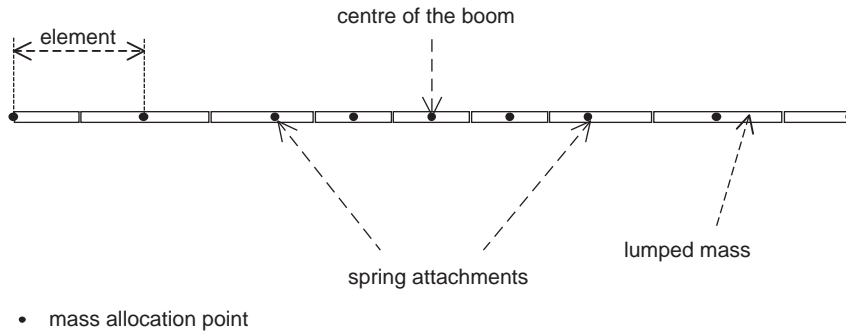
Fig. 2. Selection of the elements in case of the discrete element method.

do not differ too much. Between the end points of the boom and the spring attachments, and between the two spring attachments, the elements have the same size. The nodal points serve as mass allocation points. Mass is concentrated around the nodal points and the lumped masses cover, except for the lumped masses at the tips of the boom, two elements. In this way, four types of lumped masses, different in size, are created. The location of the intermediate element reference frame must be such that the deformation can be described easily. The co-ordinates of the most common shape functions in finite element analysis for beam structures are most of the time defined with respect to an intermediate element reference frame, which is considered at the node at the left tip of the beam element. Their orientation is parallel to the floating reference frame (^3x, ^3y, ^3z). There is no need to allocate element reference frames, as the beam itself is the last body in the kinematic tree.

To be able to describe every point of the system kinematically, in the case of the finite element method, shape functions must be selected. Because the height and the width of the beam are almost negligible with regard to its length, deflection due to shear should not be investigated. Furthermore, as only yawing and jolting motions are studied, deformations along the beam can be neglected. Therefore, Hermitian interpolation functions, defined as

$$\psi_{3j1} = 1 - 3\frac{x_{3j}^2}{L_{3j}^2} + 2\frac{x_{3j}^3}{L_{3j}^3}, \qquad \psi_{3j2} = x_{3j} - 2\frac{x_{3j}^2}{L_{3j}} + \frac{x_{3j}^3}{L_{3j}^2},$$

$$\psi_{3j3} = 3\frac{x_{3j}^2}{L_{3j}^2} - 2\frac{x_{3j}^3}{L_{3j}^3}, \qquad \psi_{3j4} = -\frac{x_{3j}^2}{L_{3j}} + \frac{x_{3j}^3}{L_{3j}^2} \tag{13}$$

are sufficient to approximate the flexible behaviour of the beam. Symbol L_{3j} represents the length of the beam element $3j$ and x_{3j} is the x-co-ordinate of the intermediate element reference frame (^{3j}x, ^{3j}y, ^{3j}z). The displacement of a point on a certain element is described as a linear combination of these functions, in which the linear interpolation coefficients are displacements and rotations of, respectively, the left and the right node. In this case every node can move in the ^3y-direction and rotate around the ^3z direction of the floating reference frame (^3x, ^3y, ^3z). To calculate the rotation of the cross-section in every point of the beam, the interpolation functions are derived once with respect to x_{3j}. To derive the equations of motion of the system the relation between the displacement due to flexible deformations and the strain is required [24], which is for small

displacements:

$$\boldsymbol{\varepsilon}_{ij} = \Delta_{ij}\mathbf{t}_{ij}, \tag{14}$$

where Δ_{ij} is an operator defined by

$$\begin{bmatrix} \dfrac{\partial}{\partial x_{ij}} & 0 & 0 \\[2mm] 0 & \dfrac{\partial}{\partial y_{ij}} & 0 \\[2mm] 0 & 0 & \dfrac{\partial}{\partial z_{ij}} \\[2mm] \dfrac{\partial}{\partial y_{ij}} & \dfrac{\partial}{\partial x_{ij}} & 0 \\[2mm] \dfrac{\partial}{\partial z_{ij}} & 0 & \dfrac{\partial}{\partial x_{ij}} \\[2mm] 0 & \dfrac{\partial}{\partial z_{ij}} & \dfrac{\partial}{\partial y_{ij}} \end{bmatrix}. \tag{15}$$

The stress strain relationship is, similarly to the discrete element method, considered as the stress strain relationship of a Hookean material:

$$\boldsymbol{\sigma}_{ij} = \kappa_{ij}\boldsymbol{\varepsilon}_{ij}. \tag{16}$$

Material damping is introduced through modal damping. Following Refs. [24–26], the equations of motion of the system, in which the flexibilities are approximated by the finite element method, are derived.

For the discrete element method, mass, stiffness, damping and input distribution matrices are derived. As indicated in Part 1 of this article [1] the contribution of the flexible bodies to the stiffness matrix is performed by assembling precalculated matrices, which are dependent on the material, properties of the cross-section and the length of the beam. These elementary blocks are obtained by calculating the elastic curve of each beam element. As no shear force is involved and only concentrated forces can be applied [1], the elastic curve of a beam element is a third order polynomial [27]. Consequently, the basic building blocks for calculating the contribution of the flexible bodies to the global stiffness matrix are exactly the same as the ones for the finite element approximation whenever Hermitian interpolation functions are used.

As proved in Ref. [1], construction of the mass matrix for the discrete element method, is the same as for the finite element method by using linear interpolation functions. This produces for the discrete element method, in terms of the finite element method, an inconsistent mass matrix. To assess the impact from this inconsistency, a consistent mass matrix is calculated for the discrete element method by using Hermitian interpolation functions. In this case the only difference between the equations of motion, obtained for the discrete and finite element approximation, is the difference in size of the elements selected for both methods.

Table 1
Eigenfrequencies of the horizontal behaviour of an agricultural spray boom for the three approximation methods

Measured (Hz)	Finite element method (6 elements) (Hz)	Discrete element method (Hermitian mass allocation, 6 elements) (Hz)	Discrete element method (geometrical mass allocation 20 elements (Hz)
0.34	0.2450	0.2442	0.2436
0.59	0.5869	0.5866	0.5814
0.88	0.9352	0.9326	0.9305
2.8	2.7874	2.7984	2.9316
4.2	4.0393	4.0501	3.9881

Similarly to the finite element approximation, the contribution to the damping matrix for the discrete element method is derived by assuming modal damping.

Before assessing the results, the forces applied by the excitation actuators (Fig. 1), need to be transformed to position inputs as the position of the piston rods are fed back through a PID controller. In this case the matrix $\mathbf{J}_s$ of Eq. (2) equals the force input distribution matrix $\mathbf{W}_s$. No matrix $\mathbf{F}_s$ is present and the results of the previous section can be applied immediately, leading to the state-space description of Eqs. (9) and (12). From these equations, the eigenfrequencies, listed in Table 1 are calculated. For both the finite element method and the discrete element method, in which the mass is allocated through Hermitian interpolation functions, convergence of the first five eigenfrequencies is achieved with six elements. Actually, the only difference between the two methods is the way the beam is divided. The formulas to derive the matrices are exactly the same for both methods. Contrary to the discrete element method, in which mass is allocated based on geometrical considerations (Fig. 2) or linear interpolation functions, convergence of the eigenfrequencies is achieved after dividing the beam in 20 elements. Even then the results are worse.

The same conclusion can be drawn by comparing the measured singular values with the singular values computed via the state-space matrices of Eqs. (9) and (12) for the finite and the discrete element method with geometrical mass allocation (Fig. 3). Singular values or principal gains are obtained by a singular value decomposition of the transfer function matrix [28], which is an input output formulation of the state-space description (9, 12) of the system. As the transfer function matrix is function of the frequency, the singular values are also frequency dependent. The principal gains express the amplification characteristics of the system. For a system with one input and one output, the singular value plot corresponds to the magnitude curve of a bode plot. In this case the singular values correspond to the amplification between (anti-)symmetric excitation and (anti-)symmetric response of the spray boom.

The results of the discrete element method with mass allocation based on Hermitian interpolation functions are not shown because they coincide with the singular values of the finite element method. Generally, a good correspondence is demonstrated between measurements and calculations. However, a mismatch is found between the lowest singular value between 2.5 and 3.5 Hz, which can be explained by Coulomb friction in the bearings of the cradle, causing stick slip of the cradle. During the experiments, this phenomenon was clearly visible. Coulomb friction
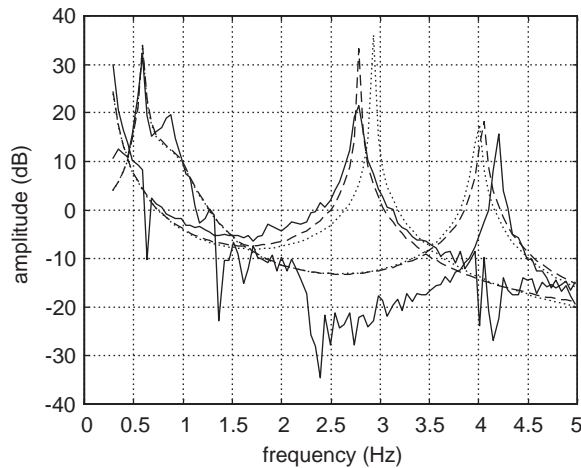
Fig. 3. Singular value plot of the horizontal behaviour of an agricultural spray boom, measured values: solid line, finite element method: dashed line, discrete element method with geometrical mass allocation: dotted line.

is non-linear and can therefore not be modelled by the procedures outlined in this chapter. When the cradle sticks, motion is stopped and the only sensor output is noise. This explains why between 2.5 and 3.5 Hz the measured singular value is below the calculated singular value.

   This example shows that allocation of mass based on geometrical considerations is not a good idea, but it is better to perform it by approximating the deflection of the beam. If the mass is allocated appropriately, the discrete and the finite element method render the same results. However, the finite element method gives more freedom with respect to the division of the beam in elements. Furthermore, in this case, the complexity of the models is similar such that it can be concluded that the finite element method is a better approximation technique for this application. Probably, the application is unfavourable for the discrete element method as the flexible body is a uniform beam. By trying to approximate the deflection shape of the beam, the concept of the finite element method fits better to this application. Actually, in the calculation of the mass matrix, the discrete element method was tailored to the finite element method.

   In case the body consists of lumped masses, interconnected by flexible beams of a small or negligible mass, it may be expected that the discrete element method performs better. Such structures can be found in for example flexible robots of which the rotor of the motor and the payload mass are large with respect to the robot arm [29]. The structure can be considered as a collection of lumped masses, interconnected by leave springs, fitting better the general concept of the discrete element method. Furthermore, the mass matrix can be derived as in case of a lumped system. For the finite element method, problems for deriving the mass matrix may be expected, as an integral [1,24–26] must be evaluated. Therefore, the finite element method needs to be tailored to the discrete element method by deriving the mass matrix as a lumped or inconsistent mass matrix. In many textbooks [2,8,16,17,30,31], results of the finite element method with a lumped and a consistent mass matrix are compared by applying it to a uniform beam. Based on this single application, these authors conclude that the results with the consistent mass matrix are better. Actually, the same is performed in the application here. As there is no difference in the formulas for the stiffness matrix between the discrete and the finite element method, the two distinct mass

allocations in the discrete element method can be seen as a finite element method with a consistent and a lumped mass matrix. The discrete element method, where mass is allocated based on Hermitian interpolation functions, coincides with the finite element method, using a consistent mass matrix, whereas the discrete element method in which mass is allocated based on geometrical considerations can be interpreted as the finite element method with a lumped mass matrix.

5. Conclusions

A general method has been derived to convert linear equations of motion for flexible multibody systems from force to displacement inputs. Both support motions as well as displacements of actuators, linking two bodies, can be used as inputs to the equations of motion. By this, the procedure of Ref. [1] to derive the equations of motion of linear flexible multibody systems, which was valid for tree structured systems can be extended to systems containing closed kinematic chains.

The methodology of Ref. [1] is applied to model the horizontal behaviour of an agricultural spray boom. The flexible boom is approximated by the discrete and finite element method and both methods are compared. As already noted in Ref. [1], the mass and stiffness matrices of the discrete element method are inconsistent. In case of the spray boom, the finite element method is better suited than the discrete element method. The model with the finite element approximation is the smallest and delivers the highest accuracy. A possible reason for this is that the idea of the finite element fits better to the example of the agricultural spray boom, in which the mass is continuously distributed along the boom. Structures consisting of undeformable masses, interconnected by beams with negligible mass, fit better to the discrete element approach.

Appendix A. Derivation of Eqs. (7), (9) and (10)

A.1. Derivation of Eq. (7)

In this Appendix, it is shown how to combine Eqs. (2) and (6) to arrive at expression (7). To split $\mathbf{qu}$ into $\mathbf{qu}_r$ and $\mathbf{qu}_s$, certain permutations need to be carried out. In order to avoid overloading the notations, it is supposed that the permutations have already been performed on Eq. (2) i.e., that the first entries of $\mathbf{qu}$ correspond to $\mathbf{qu}_r$ and the remaining to $\mathbf{qu}_s$. According to the dimensions of $\mathbf{qu}_r$ and $\mathbf{qu}_s$, the matrices of Eq. (2) can be separated in block matrices i.e.,

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{M}_{srr} & \mathbf{M}_{srs} \\ \mathbf{M}_{ssr} & \mathbf{M}_{sss} \end{bmatrix}, \tag{A.1}$$

in which $\mathbf{M}_{srr}$ is an $n_d \times n_d$ matrix, $\mathbf{M}_{srs}$ is an $n_d \times (n_{qu} - n_d)$ matrix, $\mathbf{M}_{ssr}$ is an $(n_{qu} - n_d) \times n_d$ matrix and $\mathbf{M}_{sss}$ is an $(n_{qu} - n_d) \times (n_{qu} - n_d)$ matrix. The same can be performed equivalently for $\mathbf{C}_s$ and $\mathbf{K}_s$. Also the force input distribution matrix F_s can be split in an $(n_d \times n_f)$ matrix $\mathbf{F}_{sr}$

and an $((n_{qu} - n_d) \times n_f)$ matrix $\mathbf{F}_{ss}$ in which n_f is the number of entries in $\mathbf{f}_f$:

$$\mathbf{F}_s = \begin{bmatrix} \mathbf{F}_{sr} \\ \mathbf{F}_{ss} \end{bmatrix} \tag{A.2}$$

Supposing that Eq. (4) is already in permuted form, by using $\mathbf{qu}_r$ and $\mathbf{qu}_s$ instead of $\mathbf{qu}$ and by replacing $\mathbf{M}_s$, $\mathbf{C}_s$, $\mathbf{K}_s$, $\mathbf{F}_s$ and $\mathbf{J}_s$ with their block form, Eq. (4) can be expanded:

$$\delta\mathbf{qu}_r^\mathrm{T}\mathbf{M}_{srr}\ddot{\mathbf{qu}}_r + \delta\mathbf{qu}_r^\mathrm{T}\mathbf{M}_{srs}\ddot{\mathbf{qu}}_s + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{M}_{ssr}\ddot{\mathbf{qu}}_r + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{M}_{sss}\ddot{\mathbf{qu}}_s$$
$$+ \ \delta\mathbf{qu}_r^\mathrm{T}\mathbf{C}_{srr}\dot{\mathbf{qu}}_r + \delta\mathbf{qu}_r^\mathrm{T}\mathbf{C}_{srs}\dot{\mathbf{qu}}_s + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{C}_{ssr}\dot{\mathbf{qu}}_r + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{C}_{sss}\dot{\mathbf{qu}}_s$$
$$+ \ \delta\mathbf{qu}_r^\mathrm{T}\mathbf{M}_{srr}\ddot{\mathbf{qu}}_r + \delta\mathbf{qu}_r^\mathrm{T}\mathbf{K}_{srs}\mathbf{qu}_s + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{K}_{ssr}\mathbf{qu}_r + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{K}_{sss}\mathbf{qu}_s$$
$$= \delta\mathbf{qu}_r^\mathrm{T}\mathbf{F}_{sr}\mathbf{f}_f + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{F}_{ss}\mathbf{f}_f + \delta\mathbf{qu}_r^\mathrm{T}\mathbf{J}_{sr}\mathbf{f}_p + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{J}_{ss}\mathbf{f}_p. \tag{A.3}$$

Inserting Eq. (6) and combining some terms results in:

$$\delta\mathbf{d}_p^\mathrm{T}\mathbf{M}_{dd}\ddot{\mathbf{d}}_p + \delta\mathbf{d}_p^\mathrm{T}\mathbf{M}_{dq}\ddot{\mathbf{qu}}_s + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{M}_{qd}\ddot{\mathbf{d}}_p + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{M}_{qq}\ddot{\mathbf{qu}}_s + \delta\mathbf{d}_p^\mathrm{T}\mathbf{C}_{dd}\dot{\mathbf{d}}_p + \ \delta\mathbf{d}_p^\mathrm{T}\mathbf{C}_{dq}\dot{\mathbf{qu}}_s$$
$$+ \ \delta\mathbf{qu}_s^\mathrm{T}\mathbf{C}_{qd}\dot{\mathbf{d}}_p + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{C}_{qq}\dot{\mathbf{qu}}_s + \delta\mathbf{d}_p^\mathrm{T}\mathbf{K}_{dd}\mathbf{d}_p + \delta\mathbf{d}_p^\mathrm{T}\mathbf{K}_{dq}\mathbf{qu}_s + \ \delta\mathbf{qu}_s^\mathrm{T}\mathbf{K}_{qd}\mathbf{d}_p + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{K}_{qq}\mathbf{qu}_s$$
$$= \delta\mathbf{d}_p^\mathrm{T}\mathbf{F}_d\mathbf{f}_f + \delta\mathbf{qu}_s^\mathrm{T}\mathbf{F}_q\mathbf{f}_f + \delta\mathbf{d}_p^\mathrm{T}\mathbf{f}_p \tag{A.4}$$

in which:

$$\mathbf{M}_{dd} = \mathbf{J}_{sr}^{-1}\mathbf{M}_{srr}(\mathbf{J}_{sr}^\mathrm{T})^{-1}, \tag{A.5}$$

$$\mathbf{M}_{dq} = \mathbf{J}_{sr}^{-1}\mathbf{M}_{srs} - \mathbf{J}_{sr}^{-1}\mathbf{M}_{srr}(\mathbf{J}_{sr}^\mathrm{T})^{-1}\mathbf{J}_{ss}^\mathrm{T}, \tag{A.6}$$

$$\mathbf{M}_{qd} = \mathbf{M}_{ssr}(\mathbf{J}_{sr}^\mathrm{T})^{-1} - \mathbf{J}_{ss}\mathbf{J}_{sr}^{-1}\mathbf{M}_{srr}(\mathbf{J}_{sr}^\mathrm{T})^{-1} \tag{A.7}$$

$$\mathbf{M}_{qq} = \mathbf{M}_{sss} + \mathbf{J}_{ss}\mathbf{J}_{sr}^{-1}\mathbf{M}_{srr}(\mathbf{J}_{sr}^\mathrm{T})^{-1}\mathbf{J}_{ss}^\mathrm{T} - \mathbf{M}_{ssr}(\mathbf{J}_{sr}^\mathrm{T})^{-1}\mathbf{J}_{ss}^\mathrm{T} - \mathbf{J}_{ss}\mathbf{J}_{sr}^{-1}\mathbf{M}_{srs} \tag{A.8}$$

matrices $\mathbf{C}_{dd}$, $\mathbf{C}_{dq}$, $\mathbf{C}_{qd}$, $\mathbf{C}_{qq}$, $\mathbf{K}_{dd}$, $\mathbf{K}_{dq}$, $\mathbf{K}_{qd}$ and $\mathbf{K}_{qq}$ are defined equivalently and $\mathbf{F}_d$ and $\mathbf{F}_q$ are calculated as follows:

$$\mathbf{F}_d = \mathbf{J}_{sr}^{-1}\mathbf{F}_{sr}, \tag{A.9}$$

$$\mathbf{F}_q = \mathbf{F}_{ss} + \mathbf{J}_{ss}\mathbf{J}_{sr}^{-1}\mathbf{F}_{sr}. \tag{A.10}$$

By writing Eq. (A.4) into matrix form, expression (7) is obtained.

A.2. Derivation of Eqs. (9) and (10)

In this Appendix, it is shown how Eq. (8) can be transformed to the state-space equation of Eq. (9). On both sides of the equality sign of Eq. (8), the highest degree of the derivatives is the same. This implies that a direct feed term is present in the state-space equations. To separate this term, Eq. (8) is pre-multiplied on both sides of the equality sign by $\mathbf{M}_{qq}^{-1}$. For ease of notation and to structure the equation, Eq. (8) is transformed in the Laplace domain with Laplace variables:

$$(\mathbf{I}_{qq}s^2 + \mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}s + \mathbf{M}_{qq}^{-1}\mathbf{K}_{qq})\mathbf{qu}_s$$
$$= -(\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd}s^2 + \mathbf{M}_{qq}^{-1}\mathbf{C}_{qd}s + \mathbf{M}_{qq}^{-1}\mathbf{K}_{qd})\mathbf{d}_p + \mathbf{M}_{qq}^{-1}\mathbf{F}_q\mathbf{f}_f. \tag{A.11}$$

The direct feed term can be split by making a term in $\mathbf{d}_p$, which is a multiple of the term in $\mathbf{qu}_s$. Adding terms, cancelling each other performs this. After some manipulations, Eq. (A.11) turns into

$$\mathbf{qu}_s = \mathbf{M}_{qq}^{-1}\mathbf{M}_{qd}\mathbf{d}_p - (\mathbf{I}s^2 + \mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}s + \mathbf{M}_{qq}^{-1}\mathbf{K}_{qq})^{-1}$$
$$\cdot \{[(\mathbf{M}_{qq}^{-1}\mathbf{C}_{qd} - \mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd})s$$
$$+ (\mathbf{M}_{qq}^{-1}\mathbf{K}_{qd} - \mathbf{M}_{qq}^{-1}\mathbf{K}_{qq}\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd})]\mathbf{d}_p - \mathbf{M}_{qq}^{-1}\mathbf{F}_q\mathbf{f}_f\}. \qquad (A.12)$$

The direct feed term does not have any contribution to the states. Therefore, to get better insight how to select the states, the second term of the right hand side of Eq. (A.12), containing the dynamics, is replaced by a vector $\mathbf{qu}_s^d$:

$$\mathbf{qu}_s = -\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd}\mathbf{d}_p + \mathbf{qu}_s^d. \qquad (A.13)$$

By introducing some new symbols:

$$\mathbf{A}_c = \mathbf{M}_{qq}^{-1}\mathbf{C}_{qd} - \mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd}, \qquad (A.14)$$

$$\mathbf{A}_k = \mathbf{M}_{qq}^{-1}\mathbf{K}_{qd} - \mathbf{M}_{qq}^{-1}\mathbf{K}_{qq}\mathbf{M}_{qq}^{-1}\mathbf{M}_{qd} \qquad (A.15)$$

and by applying the inverse Laplace transform, $\mathbf{qu}_s^d$ can be determined from the following differential equation:

$$\ddot{\mathbf{qu}}_s^d + \mathbf{A}_c\dot{\mathbf{d}}_p = -\mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}\dot{\mathbf{qu}}_s^d - \mathbf{M}_{qq}^{-1}\mathbf{K}_{qq}\mathbf{qu}_s^d - \mathbf{A}_k\mathbf{d}_p + \mathbf{M}_{qq}^{-1}\mathbf{F}_q\mathbf{f}_f. \qquad (A.16)$$

From this equation, states must be selected appropriately. A straightforward way is to put Eq. (A.16) in a simulation diagram of Fig. 4. The states that $\mathbf{x}_1$ and $\mathbf{x}_2$ are located after each integral sign and are, respectively, $\mathbf{qu}_s^d$ and $\dot{\mathbf{qu}}_s^d + \mathbf{A}_c\mathbf{d}_p$.

They can be calculated from the following equations:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2 - \mathbf{A}_c\mathbf{d}_p, \qquad (A.17)$$

$$\mathbf{x}_2 = \mathbf{M}_{qq}^{-1}\mathbf{C}_{qq}(\mathbf{x}_2 - \mathbf{A}_c\mathbf{d}_p) - \mathbf{M}_{qq}^{-1}\mathbf{K}_{qq}\mathbf{x}_1 - \mathbf{A}_k\mathbf{d}_p + \mathbf{M}_{qq}^{-1}\mathbf{F}_q\mathbf{f}_f. \qquad (A.18)$$

After writing Eqs. (A.17) and (A.18) into matrix form and by replacing $\mathbf{qu}_s^d$ in Eq. (A.13) by $\mathbf{x}_1$, Eqs. (9) and (10) are obtained.
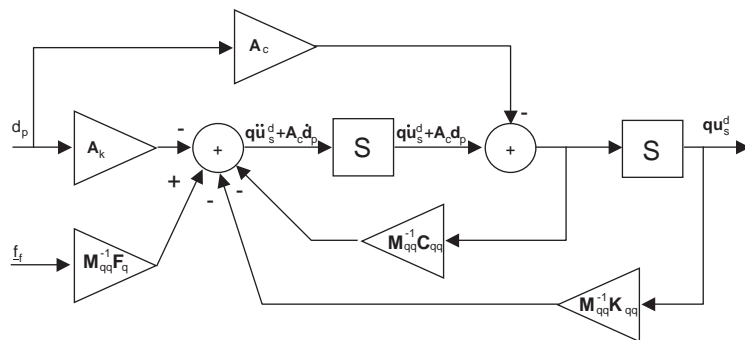

Fig. 4. Simulation diagram of Eq. (A.16).

Appendix B. Design parameters of the agricultural spray boom

Numerical values of all parameters necessary to derive the equations of motion of the horizontal behaviour of an agricultural spray boom are given in Table 2 (See Fig. 5). All parameters are measured, weighted or are material properties provided by the supplier, except the

Table 2
Numerical values for the physical properties of the spray boom

| Symbol	Value	Description
m_1	250 kg	Mass of the table
m_2	50 kg	Mass of the cradle
m_3	35.76 kg	Mass of the beam
I_1	250 kg m^2	Moment of inertia of the table
I_2	50 kg m^2	Moment of inertia of the cradle
I_3	361.38 kg m^2	Moment of inertia of the beam
ρ_3	7800 kg/m^3	Mass density of the beam
d_{cb}	0.07 m	Distance between the revolute joint and the centre of mass of the beam
L_t	1.78 m	Distance between the excitation actuators
L_k	1.318 m	Distance between the spring attachment points
b_{bi}	0.0158 m	See Fig. 5
b_{bu}	0.02 m	See Fig. 5
h_{bi}	0.056 m	See Fig. 5
h_{bu}	0.0603 m	See Fig. 5
k_s	1100 N/m	Spring stiffness
E_b	1.810^{11} N/m^2	Young's modulus of the beam
c_g	160 N s/m	Viscous damping of cradle
c_t	2 N s/rad	Viscous damping of revolute joint
c_b	0.0001	Modal damping coefficient
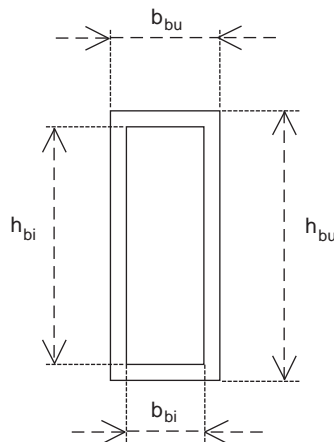
Fig. 5. Cross-section of the beam.

damping coefficients c_g, c_t and c_b, which are determined experimentally. The mass m_1, the moment of inertia I_1 of the platform and the moment of inertia I_2 of the cradle are fictitious values. However they do not have any influence on the dynamics of the system, as the degrees of freedom of the platform are removed by transforming the force inputs of the excitation actuators to position inputs. When computing the mass m_3 and the moment of inertia I_3 of the beam with the values in Table 2, a different m_3 and I_3 will result as listed in Table 2. The reason for this is that in reality the boom is not exactly a beam. For constructional reasons, an extra stiff beam needed to be added, which explains the difference.

References

[1] J. Anthonis, H. Ramon, Comparison between the discrete and finite element methods for modelling an agricultural spray boom—Part 1: Theoretical derivation *Journal of Sound and Vibration* 266 (3) (2003) 515–534, this issue.

[2] D.G. Fertis, *Mechanical and Structural Vibrations*, Wiley, New York, 1995.

[3] N.C. Nigam, S. Narayanan, *Applications of Random Vibrations*, Springer, Narosa Publishing House, 1994.

[4] A.D. Dimarogonas, S. Haddad, *Vibrations for Engineers*, Prentice-Hall, Englewood Cliffs, NJ, 1992.

[5] J.B. Vernon, *Linear Vibration Theory: Generalized Properties and Numerical Methods*, Wiley, New York, 1967.

[6] R.W. Clough, J. Penzien, *Dynamics of Structures*, McGraw-Hill, New York, 1975.

[7] C.L. Davis, G.A. Lesieutre, An actively tuned solid-state vibration absorber using capacitive shunting of piezoelectric stiffness, *Journal of Sound and Vibration* 232 (3) (2000) 601–617.

[8] D.J. Inman, *Engineering Vibrations*, Prentice-Hall, Englewood Cliffs, NJ, 1996.

[9] H.C. Tsai, G.C. Jin, Optimum tuned-mass dampers for minimizing steady-state response of support-excited and damped systems, *Earthquake Engineering and Structural Dynamics* 22 (1993) 957–973.

[10] S.S. Rao, *Mechanical Vibrations*, 2nd Edition, Addison-Wesley, Reading, MA, 1990.

[11] G.B. Warburton, Optimum absorber parameters for various combinations of response and excitation parameters, *Earthquake Engineering and Structural Dynamics* 22 (1982) 957–973.

[12] J.P. Hartog, *Mechanical Vibrations*, 4th Edition, McGraw-Hill, New York, 1956.

[13] J. De Cuyper, D. Coppens, C. Liefooghe, J. Debille, Advanced system identification methods for improved service load simulation on multi axial test rigs, *European Journal of Mechanical and Environmental Engineering* 44 (1) (1999) 27–39.

[14] J. De Cuyper, D. Coppens, C. Liefooghe, J. Debille, M. Verhaegen, Advanced drive file development methods for improved service load simulation on multi axial durability test rigs, in: *Proceedings of the International Acoustics and Vibration Asia Conference*, Singapore, 1998, pp. 339–351.

[15] C. Liefooghe, G. Heltman, Optimisation of drive comfort of a car by means of a road simulation approach, in: *Proceedings of the* 1995 *Noise and Vibration Conference*, Vol. 1, SAE Traverce City (MI), 1995, pp. 345–356.

[16] W. Weaver Jr., S.P. Timoshenko, D.H. Young, *Vibration Problems in Engineering*, 5th Edition, Wiley, New York, 1990.

[17] L. Meirovitch Jr., *Analytical Methods in Vibrations*, The Macmillan, Collier-Macmillan, London, 1967.

[18] H. Kuribara, N. Kobayashi, Dynamic responses of asymmetrical tall mechanical systems under the earthquake, in: *Proceedings of Movic '98 Zurich*, Switzerland, 1998, pp. 335–340.

[19] K. Schlager, H. Irschik, A. Kugi, Control of nonlinear parametrically excited beam vibrations, *Design Engineering Technical Conference* ASME, DE-Vol. 84-1 Vol. 3, Part A, 1995.

[20] J.T. Chen, D.H. Tsaur, H.K. Hong, An alternative method for transient and random responses of structures subject to support motions, *Engineering Structures* 19 (2) (1997) 162–172.

[21] J.T. Chen, H.K. Hong, C.E. Yeh, S.W. Chyuan, Integral representations and regulations for divergent series solution of a beam subjected to support motions, *Earthquake Engineering and Structural Dynamics* 25 (1996) 909–925.

[22] E.J. Haug Jr., *Intermediate Dynamics*, Prentice-Hall, Englewood Cliffs, NJ, 1992.

[23] A. Der Kiureghian, A. Neuenhofer, Response spectrum method for multi-support seismic excitations, *Earthquake Engineering and Structural Dynamics* 21 (1992) 713–740.

[24] J. Anthonis, Design and Development of an Active Horizontal Suspension for Agricultural Spray Booms, Ph.D. Thesis, Faculty of applied sciences, Katholieke Universiteit Leuven, Belgium, 2000.

[25] H. Ramon, A Design Procedure for Modern Control Algorithms on Agricultural Machinery Applied to Active Vibration Control of a Spray Boom, Ph.D. Thesis, No. 213, Department of Agro-Engineering and Economics, Katholieke Universiteit Leuven, Belgium, 1993.

[26] J. Anthonis, I. Hostens, A.M. Mouazen, D. Moshou, H. Ramon, A generalized modelling technique for linearized motions of mechanisms with flexible parts, *Journal of Sound and Vibration* 266 (3) (2003) 553–572, this issue.

[27] J.T. Oden, *Mechanic of Elastic Structures*, McGraw-Hill, New York, 1967.

[28] D.S. Watkins, *Fundamentals of Matrix Computations*, Wiley, New York, 1991.

[29] D.E. Torfs, R. Vuerinckx, J. Swevers, J. Schoukens, Comparison of two feedforward design methods aiming at accurate trajectory tracking of the endpoint of a flexible robot arm, *IEEE Transactions on Control Systems Technology* 6 (1) (1998) 2–14.

[30] A.A. Shabana, *Vibration of Discrete and Continuous Systems*, Mechanical Engineering Series, Springer, New York, 1997.

[31] A.A. Shabana, *Dynamics of Multibody Systems*, 2nd Edition, Cambridge University Press, Cambridge, 1998.