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A generalized modelling technique for linearized motions of mechanisms with flexible parts

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Abstract

Analysis and control of vibrations of agricultural machines to improve machine performance and vibration comfort of the operator is a major concern of manufacturers these days. In this paper, an analytical method to build the linearized equations of motion of an elastic tree structured mechanism, is presented. The method is based on the principle of virtual work resulting in a set of parameterized linear equations that are functions of the mechanical parameters and the geometry and the interconnection structure of different bodies in the mechanism. The rigid-body motions of the mechanical system are represented by Lagrangian generalized co-ordinates while elastic deformations are described by nodal co-ordinates from a finite element formulation. Explicit expressions for external distributed and concentrated forces and internal concentrated forces acting on the mechanism are given.

1. Introduction

Modern agriculture, with competitive prices for marketable arable crops, forces farmers and contractors to apply faster and more powerful mobile agricultural machines to reduce labour costs in field crop production. However, with increased operating speed, dominant peaks in the power spectrum of soil unevenness, shift toward higher frequency bands, imposing excessive vibration levels on the machine. Heavy vibrations cause malfunctioning of machine and implement and discomfort to the driver or operator. Manufacturers and research groups put in a great deal of effort to reduce machine vibration in order to improve comfort and to increase efficiency of field operations.

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The vibration phenomena of a complex mechanical system are completely characterized by the fundamental modes of the mechanism, also called modal parameters or fundamental parameters [1,2] including resonance frequencies, damping ratios and mode shapes of the system. Modal parameters can be extracted from the linearized equations of motion of a structure which are derived analytically or experimentally. In the method of experimental modal analysis, the response of a structure to known input excitation forces, is measured at several locations. From the input–output measurements, the frequency response functions (FRFs) are determined. These FRFs can be used to derive a dynamic model of the structure by estimating the modal parameters in the frequency domain or in the time domain [1]. Because experimental models are accurate and easy to build, they have shown to be very useful to analyze the vibrational behaviour of operational mechanical structures such as ships, air planes, cars, trains, offshore platforms, agricultural mobile machines, etc. [3,4]. For design studies, however, experimental models can hardly be applied as they are unstructured black box models lacking a sufficient flexibility in use owing to the absence of explicitly defined mechanical parameters (as masses, centres of mass, moments and products of inertia, damping and stiffness coefficients of applied dampers and springs, etc.) in the model [1]. Analytical methods used to build the differential equations of motion of a mechanism, are based upon the principle of virtual work [5], the method of Newton–Euler [6], the method of Lagrange [6,7] or Kane's method [8] (i.e., the principle of the virtual power). The resulting parameterized equations of motion, also called white box models, are functions of the mechanical parameters and the geometry and the interconnection structure of different bodies in the mechanism. Although white box models are commonly less accurate than black box models, they are much better suited for design purposes thanks to their parameterized structure.

Linearized analytical models have widely been used as a tool to predict vibration levels on agricultural machines and implements [9–19]. At the design stage, machine performance and ride comfort are improved by optimizing diverse mechanical parameters and the configuration of the mechanical structure as well as through the assistance of the models within the constraints set by other requirements [13,19–21]. However, the applied linear models are generic and can only be employed in the application they are derived for. In addition, most studies are concentrated on mechanical systems with only rigid-body motions. In just a few cases, finite element techniques are used to describe small elastic deformations of flexible parts in the mechanism [18].

Most control designs, including these on agricultural machinery [22–26], are based upon the use of a design model. The relationship between models and the reality they represent is subtle since no single model can respond exactly like the true plant. As a result, control strategies must be able to account for the inevitable inadequacy of design models. To cope with modelling errors, mathematical system theoreticians and control engineers developed robust control theories among which linear robust control offers today a pallet of well established design methods [27–33]. Linear robust controllers guarantee robust stability and performance of the feedback system in the presence of model uncertainties within well-defined boundaries (e.g., uncertainties arising from variations in the mechanical parameters and material properties, or from unmodelled non-linearities and truncated high-frequency dynamics) and thus created a renewed interest to apply linear white box models for control system design.

2. Objectives

A generalized methodology, based upon the principle of the virtual work, to build systematically the linearized equations of motion of a tree structured, non-gyroscopic and time-invariant rigid multibody has been developed [34]. The equations of motion, composed of a set of coupled second order differential equations, are formulated as a function of independent Lagrangian generalized co-ordinates. This indicates that the total number of differential equations is minimal in the course of which each differential equation represents just one degree of freedom of the mechanism.

In this paper, the methodology is extended to multibodies with flexible parts. The rigid-body motions are again described by independent Lagrangian generalized co-ordinates while the flexible deformations of the multibody are represented by nodal co-ordinates from a finite element formulation. This generalized modelling methodology is validated in another paper on an elastic spray boom of which the final differential equations of motion are evaluated on a laboratory experiment [35].

3. Description of the degrees of freedom of an elastic multibody

Motions of an elastic multibody Q, composed of n deformable bodies $Q_1, Q_2, \ldots, Q_n$, are a combination of rigid-body displacements and flexible body deformations. The linearized equations of motion of a tree structured mechanical system with elastic parts, will be presented in hybrid co-ordinates: independent Lagrangian generalized co-ordinates (collected in the vector $\mathbf{q}$) and independent nodal (or modal) co-ordinates (introduced by applying finite elements for the elastic deformations and collected in the vector $\mathbf{u}$) will be used to describe the rigid-body motion and the flexible body deformation of the mechanism.

Remember that a multibody has a tree structure (i.e., multibody ordered in an open chain) if and only if a selected body Q_i in Q can be reached through only one single path when proceeding from any body in Q or from Q_0 to Q_i along a sequence of bodies and hinges in such a way that no hinge is passed more than once.

3.1. Nodal co-ordinates

In finite element analysis, an elastic body Q_i is divided into say m_i finite elements or subregions with finite size and having simpler geometries than the original structure (Fig. 1). In each finite element a number of nodes is selected in which the elastic deformation of Q_i is derived. All nodes in Q_i have a certain number of degrees of freedom (d.o.f.) (up to a maximum of six d.o.f.: three translational and three rotational), each represented by a time-dependent nodal co-ordinate [36,37]. As a consequence, elastic deformations of Q_i with h_i flexible d.o.f. are described by the h_i independent nodal co-ordinates $u_{i1}, \ldots, u_{ih_i}$, collected in the nodal displacement vector $\mathbf{u}_i$. The total number of flexible d.o.f. of Q, composed of n elastic bodies, is represented by the overall vector of independent nodal co-ordinates

$$\mathbf{u} = [\mathbf{u}_1^{\mathrm{T}} \;\ldots\; \mathbf{u}_i^{\mathrm{T}} \;\ldots\; \mathbf{u}_n^{\mathrm{T}}]^{\mathrm{T}} = [u_{11} \;\ldots\; u_{1h_1} \;\ldots\; u_{i1} \;\ldots\; u_{ih_i} \;\ldots\; u_{n1} \;\ldots\; u_{nh_n}]^{\mathrm{T}}.$$

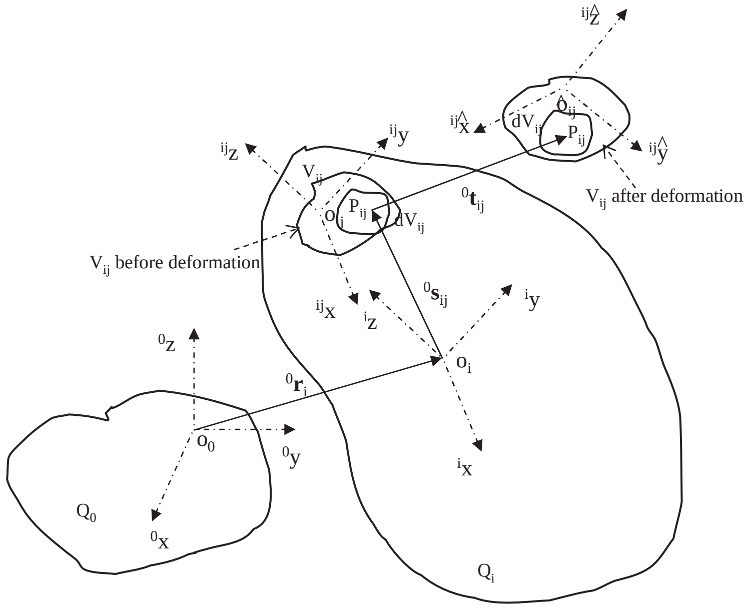

Fig. 1. Description of the position of a point in space.

Contrary to other methods (Rayleigh–Ritz method, the assumed mode method, the method of the weighted residuals), **u** is a collection of physical co-ordinates, which is an interesting feature of the finite element method [38].

3.2. Deformation co-ordinates

Suppose a selected point P_{ij} located in volume element dV_{ij} which is part of element j enclosing volume V_{ij} (Fig. 1). To describe the position of P_{ij} in Q_i, moving with respect to the inertial body Q_0, several co-ordinate systems should be introduced (Fig. 1). The (absolute) reference co-ordinate system (^ox, ^oy, ^oz, ^oo) (or global co-ordinate system, or inertial co-ordinate system) is fixed to Q_0. The floating reference co-ordinate system (^ix, ^iy, ^iz, ^io) (or body co-ordinate system) follows the rigid-body motion of Q_i i.e., when Q_i moves as an undeformable structure. However, due to the elastic nature of ^io, the origin ^io cannot be associated with a physical point of Q_i. By this, the term "floating axes" marks the floating nature of the axes relative to the deformable finite elements. Finite element j in Q_i is located with regard to (^ix, ^iy, ^iz, ^io) by two co-ordinate systems: an element reference frame ($^{ij}\hat{x}$, $^{ij}\hat{y}$, $^{ij}\hat{z}$, $^{ij}\hat{o}$) (or local reference frame) attached to a fixed point of element j and following the orientation of that element, and an intermediate element reference frame (^{ij}x, ^{ij}y, ^{ij}z, ^{ij}o) of which the origin ^{ij}o remains fixed with respect to ^io and whose axes retain a fixed orientation with respect to (^ix, ^iy, ^iz) whether Q_i performs a rigid-body motion or an elastic deformation. When Q_i is in undeformed state, the intermediate element reference frame and element reference frame coincide.

The position of P_{ij} after rigid-body motion and elastic deformation of Q_i, is stated by the vector $\mathbf{s}_{ij}$ representing the rigid-body motion and the variable element displacement co-ordinate vector

[38] $\mathbf{t}_{ij}$ (or deformation co-ordinate vector) representing the elastic deformation (Fig. 1). The vector $\mathbf{s}_{ij}$ is defined in $({}^ix, {}^iy, {}^iz, {}^io)$. However, since the finite element formulation requires local co-ordinates in the shape functions, $\mathbf{t}_{ij}$ is defined in $({}^{ij}x, {}^{ij}y, {}^{ij}z, {}^{ij}o)$. All components of $\mathbf{t}_{ij}$ are time-dependent deformation co-ordinates (or deformation variables). The deformation variables express the displacement of P_{ij} from its initial position to its end position after deformation.

Transformation matrices provide a relationship between $\mathbf{t}_{ij}$ and $\mathbf{u}_i$ [38]:

$$\mathbf{t}_{ij} = \mathbf{\Phi}_{ij}^{\mathrm{T}}\mathbf{u}_i. \tag{1}$$

As the floating reference co-ordinate system follows rigid-body motions of Q_i, displacements due to elastic deformations can fully be described in this frame. By this, the vector $\mathbf{u}_i$ is described in the floating reference frame of Q_i. The matrix $\mathbf{\Phi}_{ij}^{\mathrm{T}}({}^{ij}x, {}^{ij}y, {}^{ij}z)$ is a function of the shape functions (or basic functions, Hermitian interpolation functions) of every finite element j of Q_i [38].

3.3. Lagrangian generalized co-ordinates

Supposing an arbitrary hinge, a_a, linking two contiguous bodies Q_i and Q_j in the multibody Q, preserves n_a translational and/or rotational rigid-body degrees of freedom. As an example, the two adjacent bodies Q_i and Q_j in Fig. 2, connected by the single cylindrical joint a_a, allow one translational and one rotational rigid-body d.o.f. by which n_a becomes two. The n_a independent Lagrangian generalized co-ordinates $q_{a1}, \ldots, q_{an_a}$, collected in the vector $\mathbf{q}_a$, describe the n_a rigid-body d.o.f. in hinge a_a. A tree structured multibody Q composed of n bodies, $Q_1, \ldots, Q_i, \ldots, Q_n$, is linked by an equal number of hinges $a_1, \ldots, a_a, \ldots, a_n$. The overall vector of generalized co-ordinates $\mathbf{q}$, representing all rigid-body d.o.f. of the n-body system Q becomes then

$$\mathbf{q} = [\mathbf{q}_1^{\mathrm{T}} \;\ldots\; \mathbf{q}_a^{\mathrm{T}} \;\ldots\; \mathbf{q}_n^{\mathrm{T}}]^{\mathrm{T}} = [q_{11} \;\ldots\; q_{1n_1} \;\ldots\; q_{a1} \;\ldots\; q_{an_a} \;\ldots\; q_{n1} \;\ldots\; q_{nn_n}]^{\mathrm{T}}.$$

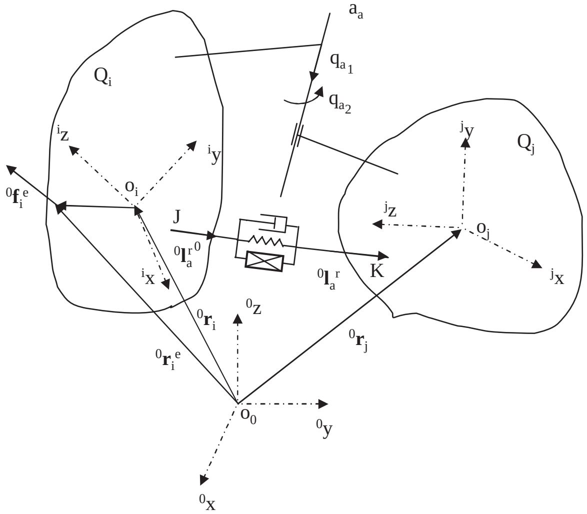

Fig. 2. Lagrangian generalized co-ordinates.

4. Kinematic analysis of a multibody system

Kinematic analysis is the process of defining the position, velocity and acceleration of a specified multibody design.

4.1. The overall angular acceleration vector

The inertial angular velocity of Q_i, is a vector quantity, commonly presented (i.e., projected) in $({}^o x, {}^o y, {}^o z, {}^o o)$ as ${}^o\boldsymbol{\omega}_i$ or in $({}^i x, {}^i y, {}^i z, {}^i o)$ as $\boldsymbol{\omega}_i = {}^o\mathbf{A}_i^{\mathrm{T}}\,{}^o\boldsymbol{\omega}_i$ in which ${}^o\mathbf{A}_i$ is the co-ordinate or rotation transformation matrix from $({}^i x, {}^i y, {}^i z, {}^i o)$ to $({}^o x, {}^o y, {}^o z, {}^o o)$. The right superscript T of a matrix means the transpose of this matrix. It can be proved for linear time-invariant systems that the overall angular acceleration vector $\dot{\boldsymbol{\omega}}$ can be expressed as [39]

$$\dot{\boldsymbol{\omega}} = \mathbf{M}_{11}\begin{bmatrix}\ddot{\mathbf{q}}\\ \ddot{\mathbf{u}}\end{bmatrix} \tag{2}$$

in which for the n-body system Q

$$\dot{\boldsymbol{\omega}} = [\dot{\boldsymbol{\omega}}_1^{\mathrm{T}}\ \ldots\ \dot{\boldsymbol{\omega}}_i^{\mathrm{T}}\ \ldots\ \dot{\boldsymbol{\omega}}_n^{\mathrm{T}}]^{\mathrm{T}}.$$

$\mathbf{M}_{11}$ is a constant matrix with appropriate dimensions.

Example. In space, the instantaneous orientation of a rigid-body Q_1 with three rotational d.o.f. with respect to Q_O can be described with the Tait–Bryan angles $\theta(t)$, $\phi(t)$ and $\psi(t)$ representing body roll, pitch and yaw of Q_1 around its co-ordinate axes ${}^1x$, ${}^1y$ and ${}^1z$. In this case for linear motions,

$${}^o\mathbf{A}_1 = \begin{bmatrix}1 & -\psi & \phi\\ \psi & 1 & -\theta\\ -\phi & \theta & 1\end{bmatrix} \tag{3}$$

and

$${}^o\boldsymbol{\omega}_1 = \begin{bmatrix}{}^o\omega_{1_x}\\ {}^o\omega_{1_y}\\ {}^o\omega_{1_z}\end{bmatrix} = \begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\\ \ddot{\psi}\end{bmatrix}. \tag{4}$$

4.2. The variation of the overall angular orientation

Finite rotations do not obey the commutative law of addition and therefore cannot be considered as vectors [40]. However, infinitesimal rotations and the variation of finite rotations meet the three necessary attributes characterizing vectors [40] and can be treated as such.

The inertial variation of small angular displacements of Q_i with respect to Q_0 can again be presented in Q_0 by the vector ${}^o\boldsymbol{\pi}_i$ or in Q_i by the vector $\boldsymbol{\pi}_i = {}^o\mathbf{A}_i^{\mathrm{T}}\,{}^o\boldsymbol{\pi}_i$. After collecting these vectors in the overall vector of the variation of the angular orientation $\boldsymbol{\pi}$, one obtains for a linear-time invariant n-body system [39]

$$\delta\boldsymbol{\pi} = (\mathbf{M}_{11} + \mathbf{M}_{12}(\mathbf{q}, \mathbf{u}))\begin{bmatrix} \delta\mathbf{q} \\ \delta\mathbf{u} \end{bmatrix}, \tag{5}$$

in which

$$\delta\boldsymbol{\pi} = [\delta\boldsymbol{\pi}_1^{\mathrm{T}} \ \dots \ \delta\boldsymbol{\pi}_i^{\mathrm{T}} \ \dots \ \delta\boldsymbol{\pi}_n^{\mathrm{T}}]^{\mathrm{T}}.$$

The elements in the matrix $\mathbf{M}_{12}(\mathbf{q}, \mathbf{u})$ are linear functions of $\mathbf{q}$ and $\mathbf{u}$.

Example. The variation of the overall angular acceleration of Q_1 in the example of the previous subsection is calculated as

$${}^{o}\delta\boldsymbol{\pi}_1 = \begin{bmatrix} {}^{o}\delta\pi_{1_x} \\ {}^{o}\delta\pi_{1_y} \\ {}^{o}\delta\pi_{1_z} \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\psi & \phi \\ \psi & 0 & -\theta \\ -\phi & \theta & 0 \end{bmatrix} \right\} \begin{bmatrix} \delta\theta \\ \delta\phi \\ \delta\psi. \end{bmatrix} \tag{6}$$

4.3. The overall acceleration vector of the origin of the body co-ordinate systems

The second derivative of the inertial position vector ${}^{o}\mathbf{r}_i$ of the origin ${}^{i}o$ of Q_i gives the inertial acceleration vector of ${}^{i}o$ which can be presented in (${}^{o}x$, ${}^{o}y$, ${}^{o}z$, ${}^{o}o$) as ${}^{o}\ddot{\mathbf{r}}_i$ or in (${}^{i}x$, ${}^{i}y$, ${}^{i}z$, ${}^{i}o$) as $\ddot{\mathbf{r}}_i = {}^{o}\mathbf{A}_i^{\mathrm{T}\,o}\ddot{\mathbf{r}}_i$. Consequently,

$$\ddot{\mathbf{r}} = \mathbf{M}_{21}\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{u}} \end{bmatrix}, \tag{7}$$

$\ddot{\mathbf{r}} = [\ddot{\mathbf{r}}_1^{\mathrm{T}} \ \dots \ \ddot{\mathbf{r}}_i^{\mathrm{T}} \ \dots \ \ddot{\mathbf{r}}_n^{\mathrm{T}}]^{\mathrm{T}}$, is the inertial overall acceleration vector of the origin of the floating reference frames. As already explained, each acceleration vector $\mathbf{r}_i$ is represented in the corresponding floating reference frame of Q_i and $\mathbf{M}_{21}$ is a constant matrix with appropriate dimensions.

Example. Suppose that in the example of the previous subsections, a rigid-body Q_2 is connected to Q_1 with a hinge retaining three translational d.o.f. that are represented by the Lagrangian generalized co-ordinates x, y and z traced along the body co-ordinate axes ${}^{2}x$, ${}^{2}y$ and ${}^{2}z$ of Q_2. In this case, the overall acceleration vector of the origin ${}^{2}0$ of the body co-ordinate system for linear motions becomes

$${}^{o}\mathbf{r}_1 = \begin{bmatrix} {}^{o}r_{2_x} \\ {}^{o}r_{2_y} \\ {}^{o}r_{2_z} \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}. \tag{8}$$

4.4. The variation of the overall position vector of the origin of the body co-ordinate systems

The procedure to derive $\ddot{\mathbf{r}}_i$ or ${}^o\ddot{\mathbf{r}}_i$ can be used to calculate $\delta^o\mathbf{r}_i$ or $\delta\mathbf{r}_i = {}^o\mathbf{A}_i^{\mathrm{T}}\delta^o\mathbf{r}_i$ such that [39]

$$\delta\mathbf{r} = [\mathbf{M}_{21} + \mathbf{M}_{22}(\mathbf{q},\mathbf{u})]\begin{bmatrix}\delta\mathbf{q}\\ \delta\mathbf{u}\end{bmatrix}, \tag{9}$$

$\delta\mathbf{r} = [\delta\mathbf{r}_1^{\mathrm{T}}\ \dots\ \delta\mathbf{r}_i^{\mathrm{T}}\ \dots\ \delta\mathbf{r}_n^{\mathrm{T}}]^{\mathrm{T}}$, is the variation of the overall position vector of the origin of the floating reference frames. Again, the variation of each individual inertial position vector $\delta\mathbf{r}_i$ is represented in the corresponding floating reference frame (${}^ix$, ${}^iy$, ${}^iz$, ${}^io$) and the elements in the matrix $M_{22}(\mathbf{q},\mathbf{u})$ are linear functions of $\mathbf{q}$ and $\mathbf{u}$.

Example.

$$\delta^o\mathbf{r}_2 = \begin{bmatrix}\delta^o r_{2_x}\\ \delta^o r_{2_y}\\ \delta^o r_{2_z}\end{bmatrix} = \left\{\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} + \begin{bmatrix}0&-y&z&0&-\psi&\phi\\ x&0&-z&\psi&0&-\theta\\ -x&y&0&-\phi&\theta&0\end{bmatrix}\right\}\begin{bmatrix}\delta\theta\\ \delta\phi\\ \delta\psi\\ \delta x\\ \delta y\\ \delta z\end{bmatrix}. \tag{10}$$

5. Dynamic analysis of a multibody system

5.1. The principle of virtual work for a flexible multibody

For a multibody system, composed of n elastic bodies, the principle of virtual work can be written in the form [38–43]

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{m_i}\int_{V_{ij}}\rho_{ij}\delta^o\mathbf{p}_{ij}^{\mathrm{T}}\,{}^o\ddot{\mathbf{p}}_{ij}\,\mathrm{d}V_{ij}\right\} = -\sum_{i=1}^{n}\left\{\sum_{j=1}^{m_i}\int_{V_{ij}}\delta\boldsymbol{\varepsilon}_{ij}^{\mathrm{T}}\boldsymbol{\sigma}_{ij}\,\mathrm{d}V_{ij}\right\} + \sum_{i=1}^{n}\left\{\sum_{j=1}^{m_i}(\delta W_{ij}^{df} + \delta W_{ij}^{cf})\right\} \tag{11}$$

explaining that the virtual work owing to forces of inertia is equal to the sum of the virtual work performed by internal and external forces, where n is the number of bodies in the mechanism, m_i the number of finite elements in Q_i, $\boldsymbol{\varepsilon}_{ij}$ the strain vector defined in (${}^{ij}x$, ${}^{ij}y$, ${}^{ij}z$, ${}^{ij}o$), $\delta\boldsymbol{\varepsilon}_{ij}$ the virtual strain vector defined in (${}^{ij}x$, ${}^{ij}y$, ${}^{ij}z$, ${}^{ij}o$), $\boldsymbol{\sigma}_{ij}$ the normal stress vector defined in (${}^{ij}x$, ${}^{ij}y$, ${}^{ij}z$, ${}^{ij}o$), V_{ij} the volume of finite element j in body Q_i, ρ_{ij} the density of the elementary volume element $\mathrm{d}V_{ij}$, ${}^o\mathbf{p}_{ij}$

the inertial position vector of point P_{ij}, $\delta^o\mathbf{p}_{ij}$ the inertial virtual displacement vector of P_{ij}, ${}^o\ddot{\mathbf{p}}_{ij}$ the inertial acceleration vector of P_{ij}, δW_{ij}^{df} the virtual work done by the external distributed body and surface forces on $\mathrm{d}V_{ij}$, and δW_{ij}^{cf} the virtual work executed by all concentrated forces in $\mathrm{d}V_{ij}$.

With the aid of the Kelvin–Voight model for visco-elastic materials [43], one can prove that [39]

$$\delta\boldsymbol{\varepsilon}_{ij}^{\mathrm{T}}\boldsymbol{\sigma}_{ij}\,\mathrm{d}V_{ij} = \delta\mathbf{u}_{ij}^{\mathrm{T}}\,\mathrm{d}\mathbf{K}_{ij}\mathbf{u}_{ij} + \delta\mathbf{u}_{ij}^{\mathrm{T}}\,\mathrm{d}\mathbf{C}_{ij}\dot{\mathbf{u}}_{ij}, \tag{12}$$

$\mathrm{d}\mathbf{K}_{ij}$ and $\mathrm{d}\mathbf{C}_{ij}$ are respectively the stiffness matrix and the damping matrix of the elementary volume element $\mathrm{d}V_{ij}$.

From Fig. 1 it directly follows that

$${}^o\mathbf{p}_{ij} = {}^o\mathbf{r}_i + {}^o\mathbf{s}_{ij} + {}^o\mathbf{t}_{ij} \tag{13}$$

or after substitution of Eq. (1) into Eq. (13)

$${}^o\mathbf{p}_{ij} = {}^o\mathbf{r}_i + {}^o\mathbf{A}_i(\mathbf{s}_{ij} + \boldsymbol{\Phi}_{ij}^{\mathrm{T}}\mathbf{u}_i). \tag{14}$$

For linear time-invariant non-gyroscopic mechanisms, the variation and the second derivative of Eq. (14) lead to the expressions

$$\delta^o\mathbf{p}_{ij}^{\mathrm{T}} = \delta^o\mathbf{r}_i^{\mathrm{T}} + \delta^o\boldsymbol{\pi}_i^{\mathrm{T}\,o}\mathbf{A}_i\tilde{\mathbf{s}}_{ij}{}^o\mathbf{A}_i^{\mathrm{T}} + \delta\mathbf{u}_i^{\mathrm{T}}\boldsymbol{\Phi}_{ij}{}^o\mathbf{A}_i^{\mathrm{T}} \tag{15}$$

and

$${}^o\ddot{\mathbf{p}}_{ij} = {}^o\ddot{\mathbf{r}}_{ij} - {}^o\mathbf{A}_i\tilde{\mathbf{s}}_{ij}{}^o\mathbf{A}_i^{\mathrm{T}\,o}\dot{\boldsymbol{\omega}}_i + {}^o\mathbf{A}_i\boldsymbol{\Phi}_{ij}^{\mathrm{T}}\ddot{\mathbf{u}}_i. \tag{16}$$

One should remark that for a vector $\mathbf{a} = [a_x\ \ a_y\ \ a_z]^{\mathrm{T}}$, $\tilde{\mathbf{a}}$, pronounced a-tilde, represents the second order tensor or the (3×3) skew-symmetric matrix

$$\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

Substituting Eqs. (12), (15) and (16) in Eq. (11) and taking into account that $\mathbf{r} = {}^o\mathbf{A}^{\mathrm{T}\,o}\mathbf{r}$ and $\boldsymbol{\omega} = {}^o\mathbf{A}^{\mathrm{T}\,o}\boldsymbol{\omega}$, result in

$$\begin{aligned}[][\delta\mathbf{r}^{\mathrm{T}}\ \ \delta\boldsymbol{\pi}^{\mathrm{T}}\ \ \delta\mathbf{u}^{\mathrm{T}}]\left\{\begin{bmatrix} [\mathbf{m}_b] & \mathbf{B}^{\mathrm{T}} & \mathbf{E}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{J} & \mathbf{G}^{\mathrm{T}} \\ \mathbf{E} & \mathbf{G} & \mathbf{M} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}} \\ \ddot{\mathbf{u}} \end{bmatrix}\right\} \\ +\ \delta\mathbf{u}^{\mathrm{T}}[\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u}] - [\delta W^{df} + \delta W^{cf}] = 0. \end{aligned} \tag{17}$$

The matrix

$$\begin{bmatrix} [\mathbf{m}_b] & \mathbf{B}^{\mathrm{T}} & \mathbf{E}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{J} & \mathbf{G}^{\mathrm{T}} \\ \mathbf{E} & \mathbf{G} & \mathbf{M} \end{bmatrix} = \mathbf{M}_{se} \tag{18}$$

is the embryonal form of the overall mass matrix of the mechanism under consideration.

$[\mathbf{m}_b] = {}^o\mathbf{A}[\mathbf{m}_b]{}^o\mathbf{A}^{\mathrm{T}} = \mathrm{diag}[(m_{b_i}\mathbf{I}_3)_i] = \mathrm{diag}(m_{b_1}\mathbf{I}_3, \ldots, m_{b_i}\mathbf{I}_3, \ldots, m_{b_n}\mathbf{I}_n)$ is a block diagonal matrix in which m_{b_i} is the mass of Q_i and $\mathbf{I}_3$ a (3×3)-identity matrix. ${}^o\mathbf{A} = \mathrm{diag}[({}^o\mathbf{A}_i)_i]$, $\mathbf{B} = \mathrm{diag}[(\mathbf{B}_i)_i]$,

$\mathbf{E} = \mathrm{diag}[(\mathbf{E}_i)_i]$, $\mathbf{J} = \mathrm{diag}[(\mathbf{J}_i)_i]$, $\mathbf{G} = \mathrm{diag}[(\mathbf{G}_i)_i]$, $\mathbf{C} = \mathrm{diag}[(\mathbf{C}_i)_i]$ and $\mathbf{K} = \mathrm{diag}[(\mathbf{K}_i)_i]$ all have an equivalent block diagonal structure.

In these matrices one can recognize [42]

$$m_{b_i} = \sum_{j=1}^{m_i} \int_{V_{ij}} \rho_{ij} \,\mathrm{d}V_{ij}, \tag{19}$$

the total mass of body Q_i;

$$\mathbf{B}_i = \sum_{j=1}^{m_i} \int_{V_{ij}} \tilde{\mathbf{s}}_{ij} \rho_{ij} \,\mathrm{d}V_{ij}, \tag{20}$$

the inertia coupling between the translation and rotation of Q_i;

$$\mathbf{B}_i = \mathbf{O}_3 \tag{21}$$

when Q_i has a centroidal floating reference frame (the origin ${}^io$ is placed in the center of gravity of Q_i); $\mathbf{O}_3$ is a (3×3)-zero matrix;

$$\mathbf{E}_i = \sum_{j=1}^{m_i} \int_{V_{ij}} \mathbf{\Phi}_{ij} \rho_{ij} \,\mathrm{d}V_{ij}, \tag{22}$$

the inertia coupling between the translation and the elastic deformation of Q_i;

$$\mathbf{G}_i = -\sum_{j=1}^{m_i} \int_{V_{ij}} \mathbf{\Phi}_{ij} \tilde{\mathbf{s}}_{ij} \rho_{ij} \,\mathrm{d}V_{ij}, \tag{23}$$

the inertia coupling between the rotation and the elastic deformation of Q_i;

$$\mathbf{J}_i = -\sum_{j=1}^{m_i} \int_{V_{ij}} \tilde{\mathbf{s}}_{ij} \tilde{\mathbf{s}}_{ij} \rho_{ij} \,\mathrm{d}V_{ij}, \tag{24}$$

the inertia matrix (or inertia tensor) with respect to $({}^ix, {}^iy, {}^iz, {}^io)$. $\mathbf{J}_i$ is a (3×3)-matrix. Its diagonal elements are the mass moments of inertia and the off-diagonal elements the mass products of inertia. When the latter are zero, the former are called the principal moments of inertia and the body co-ordinate axes of Q_i become the principal axes.

$\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{K}_i$ are the mass matrix, the damping matrix and the stiffness matrix of body Q_i which correspond to the finite element modelling of Q_i. These matrices are given by

$$\mathbf{M}_i = \sum_{j=1}^{m_i} \int_{V_{ij}} \mathbf{\Phi}_{ij} \mathbf{\Phi}_{ij}^{\mathrm{T}} \rho_{ij} \,\mathrm{d}V_{ij}, \tag{25}$$

$$\mathbf{C}_i = \sum_{j=1}^{m_i} \mathbf{L}_{ij}^{\mathrm{T}} \mathbf{C}_{ij} \mathbf{L}_{ij}, \tag{26}$$

$$\mathbf{K}_i = \sum_{j=1}^{m_i} \mathbf{L}_{ij}^{\mathrm{T}} \mathbf{K}_{ij} \mathbf{L}_{ij}, \tag{27}$$

$\mathbf{L}_{ij}$ is called the locator (or label) matrix [38] (or Boolean matrix since its elements only consists of 0 or 1). $\mathbf{L}_{ij}$ relates the nodal co-ordinates of finite element j with the nodal displacement vector $\mathbf{u}_i$.

Since $[\mathbf{m}_b]$, $\mathbf{J}$ and $\mathbf{M}$ are symmetric matrices, it is obvious that $\mathbf{M}_{se} = \mathbf{M}_{se}^{\mathrm{T}}$ which is in accordance with the basic principles of kinetics.

When only rigid-body motion is considered, nodal co-ordinates vanish and Eq. (17) becomes

$$[\delta\mathbf{r}^{\mathrm{T}} \;\; \delta\boldsymbol{\pi}^{\mathrm{T}}]\left\{\begin{bmatrix}[\mathbf{m}_b] & \mathbf{B}^{\mathrm{T}}\\ \mathbf{B} & \mathbf{J}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{r}}\\ \dot{\boldsymbol{\omega}}\end{bmatrix}\right\} - [\delta W^{df} + \delta W^{cf}] = 0. \tag{28}$$

Neglecting rigid-body motion results in the equation

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \delta W^{df} + \delta W^{cf} \tag{29}$$

which is the equation of motion of a mechanical system in structural dynamics.

5.2. Derivation of expressions for the forces acting on a mechanism

A force ${}^{o}\mathbf{f}_i$, acting on a body Q_i can be divided into a dynamic part ${}^{o}\mathbf{f}_i^{d}(t)$ and a static part ${}^{o}\mathbf{f}_i^{o}$. The three components of ${}^{o}\mathbf{f}_i^{d}$ are known time functions (except for the spring-damper hinge forces which are functions of the generalized co-ordinates $\mathbf{q}$ and $\mathbf{u}$ or their derivatives) and should be small, to assure a linear rigid-body motion and/or elastic deformation of Q_i. In addition, any product of a nodal or Lagrangian generalized co-ordinate with a dynamic force always vanishes as it concerns here an infinite small quantity of higher order.

Static forces have a constant and finite magnitude and a fixed direction with regard to a body reference frame or to the absolute reference frame. From this, it is obvious that the product of ${}^{o}\mathbf{f}_i^{o}$ with one or more nodal or Lagrangian generalized co-ordinates, which may not be neglected, can manifest in the equations of motion (see, e.g., the vertical pendulum).

All forces can further be partitioned in distributed forces (body or surface forces) and concentrated forces (internal hinge forces and external forces). These forces are known with regard to a body reference frame (hinge forces, etc.) or with regard to the absolute reference frame (gravitational forces, etc.).

In the following sections distributed forces, concentrated hinge forces (translational spring–damper–actuator) and concentrated absolute and relative forces which are mostly used, will be treated.

5.2.1. External distributed forces

The virtual work executed by the distributed body forces ${}^{o}\mathbf{F}_{b_{ij}}$ and the distributed surface forces ${}^{o}\mathbf{F}_{s_{ij}}$ in element j of Q_i can be expressed as [41]

$$\delta W_{ij}^{df} = \int_{V_{ij}} \delta^{o}\mathbf{p}_{ij}^{\mathrm{T}\,o}\mathbf{F}_{b_{ij}}(t)\,\mathrm{d}V_{ij} + \int_{S_{ij}} \delta^{o}\mathbf{p}_{ij}^{\mathrm{T}\,o}\mathbf{F}_{s_{ij}}(t)\,\mathrm{d}S_{ij}, \tag{30}$$

V_{ij} and S_{ij} are the volume and the surface area of element j in Q_i. If ${}^{o}\mathbf{F}_{b_{ij}}(t)$ and ${}^{o}\mathbf{F}_{s_{ij}}(t)$ are known with regard to the floating reference frame of Q_i, a summation of Eq. (30) over all finite elements of all bodies in the mechanism gives [38]

$$\delta W^{df} = [\delta \mathbf{q}^{\mathrm{T}} \;\; \delta \mathbf{u}^{\mathrm{T}}] \left[\mathbf{M}_{21}^{\mathrm{T}} \;\; \mathbf{M}_{11}^{\mathrm{T}} \quad \begin{bmatrix} \mathbf{O} \\ \mathbf{I} \end{bmatrix} \right] \begin{bmatrix} \mathbf{f}_b + \mathbf{f}_s \\ \mathbf{n}_b + \mathbf{n}_s \\ \mathbf{F}_b + \mathbf{F}_s \end{bmatrix}, \tag{31}$$

$\mathbf{O}$ is a square zero matrix in which the number of rows corresponds with the number of components in $\mathbf{q}$ and $\mathbf{I}$ is a unit matrix with equal number of rows as the number of components in $\mathbf{u}$. The vectors in the right hand side of Eq. (31) are

$$\mathbf{f}_b = [\mathbf{f}_{b_1}^{\mathrm{T}} \; \ldots \; \mathbf{f}_{b_i}^{\mathrm{T}} \; \ldots \; \mathbf{f}_{b_n}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathbf{n}_b = [\mathbf{n}_{b_1}^{\mathrm{T}} \; \ldots \; \mathbf{n}_{b_i}^{\mathrm{T}} \; \ldots \; \mathbf{n}_{b_n}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathbf{F}_b = [\mathbf{F}_{b_1}^{\mathrm{T}} \; \ldots \; \mathbf{F}_{b_i}^{\mathrm{T}} \; \ldots \; \mathbf{F}_{b_n}^{\mathrm{T}}]^{\mathrm{T}},$$

$$\mathbf{f}_s = [\mathbf{f}_{s_1}^{\mathrm{T}} \; \ldots \; \mathbf{f}_{s_i}^{\mathrm{T}} \; \ldots \; \mathbf{f}_{s_n}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathbf{n}_s = [\mathbf{n}_{s_1}^{\mathrm{T}} \; \ldots \; \mathbf{n}_{s_i}^{\mathrm{T}} \; \ldots \; \mathbf{n}_{s_n}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathbf{F}_s = [\mathbf{F}_{s_1}^{\mathrm{T}} \; \ldots \; \mathbf{F}_{s_i}^{\mathrm{T}} \; \ldots \; \mathbf{F}_{s_n}^{\mathrm{T}}]^{\mathrm{T}},$$

in which

$$\mathbf{f}_{b_i} = \sum_{j=1}^{n_i} \int_{V_{ij}} \mathbf{F}_{b_{ij}} \, \mathrm{d}V_{ij} \tag{32}$$

is the resultant force of all distributed body forces acting on Q_i with regard to the floating reference frame of Q_i;

$$\mathbf{n}_{b_i} = \sum_{j=1}^{n_i} \int_{V_{ij}} \tilde{\mathbf{s}}_{ij} \mathbf{F}_{b_{ij}} \, \mathrm{d}V_{ij} \tag{33}$$

is the resultant moment of all distributed body forces acting on Q_i with regard to the origin of the floating reference frame of Q_i;

$$\mathbf{F}_{b_i} = \sum_{j=1}^{n_i} \int_{V_{ij}} \mathbf{\Phi}_{ij} \mathbf{F}_{b_{ij}} \, \mathrm{d}V_{ij}. \tag{34}$$

By replacing the right subscript b with s and integrating over S_i for all i, expressions for $\mathbf{f}_s$, $\mathbf{n}_s$ and $\mathbf{F}_s$ are immediately deduced.

5.2.2. Internal concentrated hinge forces

In Fig. 2, a translational spring–damper–actuator, built in hinge a_a is attached to Q_i and Q_j through the contact points J and K.

For the virtual work, executed by a hinge force ${}^{o}\mathbf{f}_a^r$, the following sign convention is introduced [44]: when two contiguous bodies are drifting (obviously, the variation of the length $\|{}^{o}\mathbf{l}_a^r\| = l_a^r$ of the device is positive) and the force is tending to pull the bodies together, then f_a^r is taken positive. Consequently, the virtual work done on the bodies by ${}^{o}\mathbf{f}_a^r$ is negative as the angle between $\delta^{o}\mathbf{l}_a^r$ and ${}^{o}\mathbf{f}_a^r$ is 180°. By this, a negative sign should be introduced in the expression of the virtual work

$$\delta W_a^r = -\delta l_a^r f_a^r. \tag{35}$$

Consequently, for a translational spring–damper–actuator in hinge a_a, one can write

$$\delta W_a^r = -\delta l_a^r \{ k_a^r (l_a^r - l_a^{r^o} + \delta_a^{r^o}) + c_a^r \dot{l}_a^r + {}_c f_a^r \}. \tag{36}$$

The first term in Eq. (36) is the spring force (k_a^r = spring constant), the second term represents the damping force (c_a^r = damping coefficient), and the third term is the actuator force that contains a dynamic part ${}_c f_a^{r^d}$ and a static part ${}_c f_a^{r^o}$. l_a^r, which is always $\geqslant 0$, is the distance between the attachment points J and K on time t, whereas $l_a^{r^o}$ describes the same distance in static equilibrium position of the mechanism. $\delta_a^{r^o}$ represents the static deformation of the spring. Without dynamic forces $k_a^r \delta_a^{r^o}$ and ${}_c f_a^{r^o}$ keep the system in a static equilibrium. Through Eq. (36), the total virtual work $\delta W^r = \sum_{a=1}^n \delta W_a^r$ done by all hinge forces in the multibody, can be transformed into

$$\delta W^r = -\delta \mathbf{l}^{r^{\mathrm{T}}} \{\mathbf{K}^r(\mathbf{l}^r - \mathbf{l}^{r^o} + \delta^{r^o}) + \mathbf{C}^r \dot{\mathbf{l}}^r + {}_c\mathbf{f}^r\}, \tag{37}$$

$$\mathbf{K}^r = \mathrm{diag}[(k_a^r)_a], \tag{38}$$

$$\mathbf{C}^r = \mathrm{diag}[(c_a^r)_a], \tag{39}$$

$$\mathbf{l}^r = [l_1^r \; \ldots \; l_a^r \; \ldots \; l_n^r]^{\mathrm{T}}, \quad \mathbf{l}^{r^o} = [l_1^{r^o} \; \ldots \; l_a^{r^o} \; \ldots \; l_n^{r^o}]^{\mathrm{T}},$$

$$\dot{\mathbf{l}}^r = [\dot{l}_1^r \; \ldots \; \dot{l}_a^r \; \ldots \; \dot{l}_n^r]^{\mathrm{T}},$$

$$\delta^{r^o} = [\delta_1^{r^o} \; \ldots \; \delta_a^{r^o} \; \ldots \; \delta_n^{r^o}]^{\mathrm{T}}, \quad {}_c\mathbf{f}^r = [{}_c f_1^r \; \ldots \; {}_c f_a^r \; \ldots \; {}_c f_n^r]^{\mathrm{T}}.$$

Through a Taylor series expansion about $\mathbf{q} = \mathbf{o}$ and $\mathbf{u} = \mathbf{o}$, Eq. (37) can be expressed as a linear function in $\mathbf{q}$ and $\mathbf{u}$ [39]:

$$\begin{aligned} \delta W^r = & -[\delta \mathbf{q}^{\mathrm{T}} \;\; \delta \mathbf{u}^{\mathrm{T}}] \begin{bmatrix} \mathbf{B}^{r^{\mathrm{T}}} \\ \mathbf{G}^{r^{\mathrm{T}}} \end{bmatrix} \left\{ {}^o\mathbf{E}_k^{r^o} [\mathbf{B}^r \;\; \mathbf{G}^r] \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} + {}^o\mathbf{E}_c^{r^o} [\mathbf{B}^r \;\; \mathbf{G}^r] \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{u}} \end{bmatrix} \right\} \\ & - [\delta \mathbf{q}^{\mathrm{T}} \;\; \delta \mathbf{u}^{\mathrm{T}}] \begin{bmatrix} \mathbf{B}^{r^{\mathrm{T}}} \\ \mathbf{G}^{r^{\mathrm{T}}} \end{bmatrix} \{ {}^o\mathbf{E}^{r^o} (\mathbf{K}^r \delta^{r^o} + {}_c\mathbf{f}^r)\}. \end{aligned} \tag{40}$$

Explicit expressions for the matrices in Eq. (40) for a general multibody configuration are found in [39].

5.2.3. External concentrated forces

Generally, any external force ${}^o\mathbf{f}_i^e$ acting on body Q_i (Fig. 2) can be split in a dynamic part ${}^o\mathbf{f}_i^{e^d}(t)$ and a static part ${}^o\mathbf{f}_i^{e^o}$. The virtual work done by the external force ${}^o\mathbf{f}_i^e$ is

$$\delta W^e = \sum_{i=1}^n \delta W_i^e = \sum_{i=1}^n \{\delta^o \mathbf{r}_i^{e^{\mathrm{T}}} {}^o\mathbf{f}_i^e\} = \delta^o \mathbf{r}^{e^{\mathrm{T}}} {}^o\mathbf{f}^e, \tag{41}$$

$${}^o\mathbf{r}^e = [{}^o\mathbf{r}_1^{e^{\mathrm{T}}} \; \ldots \; {}^o\mathbf{r}_i^{e^{\mathrm{T}}} \; \ldots \; {}^o\mathbf{r}_n^{e^{\mathrm{T}}}]^{\mathrm{T}}, \quad {}^o\mathbf{f}^e = [{}^o\mathbf{f}_1^{e^{\mathrm{T}}} \; \ldots \; {}^o\mathbf{f}_i^{e^{\mathrm{T}}} \; \ldots \; {}^o\mathbf{f}_n^{e^{\mathrm{T}}}]^{\mathrm{T}}.$$

From Fig. 2 and after some manipulations, $\delta^o \mathbf{r}^e$ becomes [39]

$$\delta^o \mathbf{r}^e = \mathbf{B}^e \delta \mathbf{q} + \mathbf{G}^e \delta \mathbf{u} + \mathbf{H}^e(\mathbf{q}, \mathbf{u}) \begin{bmatrix} \delta \mathbf{q} \\ \delta \mathbf{u} \end{bmatrix}. \tag{42}$$

The elements of the matrix $H^e(\mathbf{q}, \mathbf{u})$ are linear functions in the nodal and Lagrangian generalized co-ordinates. After neglecting all products of any generalized co-ordinate with the dynamic part of all external forces, for linear time invariant systems δW^e will become

$$\delta W^e = [\delta \mathbf{q}^{\mathrm{T}} \;\; \delta \mathbf{u}^{\mathrm{T}}] \left\{ \begin{bmatrix} \mathbf{B}^{e^{\mathrm{T}}} \\ \mathbf{G}^{e^{\mathrm{T}}} \end{bmatrix} {}^{o}\mathbf{f}^{e^d} + \mathbf{H}_1^e({}^{o}\mathbf{f}^{e^o}) \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} \right\}, \tag{43}$$

$${}^{o}\mathbf{f}^{e^d} = [{}^{o}\mathbf{f}_1^{e^{d\mathrm{T}}} \;\; \dots \;\; {}^{o}\mathbf{f}_i^{e^{d\mathrm{T}}} \;\; \dots \;\; {}^{o}\mathbf{f}_n^{e^{d\mathrm{T}}}]^{\mathrm{T}}, \quad {}^{o}\mathbf{f}^{e^o} = [{}^{o}\mathbf{f}_1^{e^{o\mathrm{T}}} \;\; \dots \;\; {}^{o}\mathbf{f}_i^{e^{o\mathrm{T}}} \;\; \dots \;\; {}^{o}\mathbf{f}_n^{e^{o\mathrm{T}}}]^{\mathrm{T}}.$$

Again, explicit expressions for the matrices used in Eq. (43) can be derived. If a force ${}^{o}\mathbf{f}_i^{e}$ acting on Q_i is known or can be described with respect to an arbitrary body Q_k (e.g., the gravity force of any body is described in Q_0), ${}^{o}\mathbf{f}_i^{e^d}$ and ${}^{o}\mathbf{f}_i^{e^o}$ in Eq. (43) should be replaced by ${}^{k}\mathbf{A}_i^{o\mathrm{T}}\mathbf{f}_i^{e^d}$ and ${}^{k}\mathbf{A}^{o\mathrm{T}}\mathbf{f}_i^{e^o}$. The matrix ${}^{k}\mathbf{A}_i^{o}$ is the co-ordinate transformation matrix between Q_i and Q_k when the mechanical system is in static equilibrium or ${}^{k}\mathbf{A}_i^{o}$ describes the initial orientation of Q_i with regard to Q_k. Because the components of ${}^{o}\mathbf{f}^{e^o}$ are finite quantities, products of components of this vector with nodal or Lagrangian generalized co-ordinates are not negligible, leading to the second part of the right hand side of Eq. (43). $\mathbf{H}_1^e({}^{o}\mathbf{f}^{e^o})$ is a matrix whose elements are linear functions of the components of ${}^{o}\mathbf{f}^{e^o}$.

Example. A vertical pendulum (Fig. 3) consisting of a point mass m mounted at the tip of a massless pendulum rod with length l, has one rotational degree of freedom represented by the Lagrangian generalized co-ordinate θ. Forces acting on the point mass are the constant gravity force ${}^{o}\mathbf{f}^{e^o} = [0 \;\; -mg]^{\mathrm{T}}$ and a small dynamic force ${}^{o}\mathbf{f}^{e^d}(t) = [{}^{o}f_x^{e^d}(t) \;\; {}^{o}f_y^{e^d}(t)]^{\mathrm{T}}$. After some small

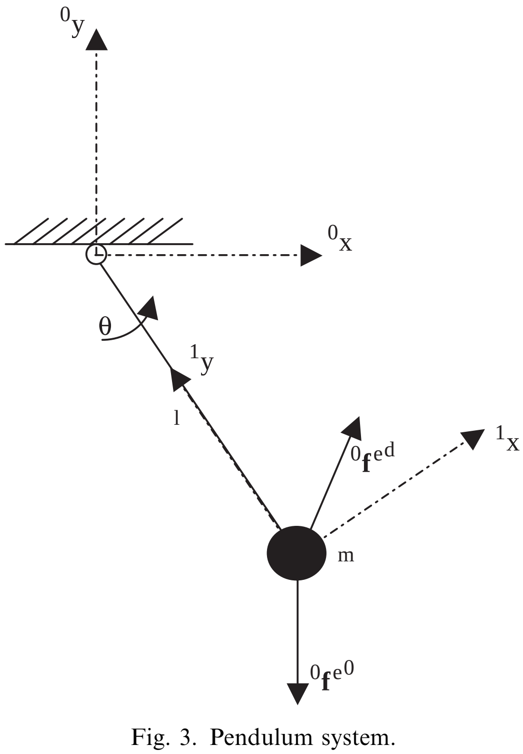

Fig. 3. Pendulum system.

calculations, the virtual work performed by these two forces is expressed as

$$\delta W^e = \delta\theta\{l^o f_x^{e^d}(t) + mgl\theta\} \tag{44}$$

or written in the form of Eq. (43):

$$\delta W^e = \delta\theta\{[l \;\; o]^o \mathbf{f}^{e^d}(t) + [o \;\; -l]^o \mathbf{f}^{e^o}\theta\}. \tag{45}$$

5.3. The equation of motion of an elastic multibody

Since $\delta W^{cf} = \delta W^r + \delta W^e$, a substitution of Eqs. (2), (5), (7), (9), (31), (40) and (43) in Eq. (17) leads to a vector second order system expressed in hybrid co-ordinates:

$$\mathbf{M}_s \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{u}} \end{bmatrix} + \mathbf{C}_s \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{u}} \end{bmatrix} + \mathbf{K}_s \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} = \mathbf{V}_s \mathbf{v} + \mathbf{W}_s \mathbf{w}, \tag{46}$$

where $\mathbf{M}_s$ is the mass matrix, $\mathbf{C}_s$ the damping matrix, $\mathbf{K}_s$ the stiffness matrix, $\mathbf{V}_s$ the control distribution matrix, $\mathbf{W}_s$ the disturbance distribution matrix, $\mathbf{v}$ the vector of generalized control forces, and $\mathbf{w}$ the vector of generalized disturbance forces.

To develop control algorithms, it is necessary to split all forces acting on the multibody into control force vectors (e.g., the actuator forces ${}_c\mathbf{f}^r$) which can be manipulated in feedback or feedforward systems, and disturbance force vectors which cannot be changed by control actions or human operators. The control and disturbance force vectors are collected into the vectors $\mathbf{v}$ and $\mathbf{w}$ respectively. $\mathbf{v}$ and $\mathbf{w}$ consist of forces and/or torques.

The vectors $\mathbf{v}$ and $\mathbf{w}$ still contain generalized static forces which are ${}_c\mathbf{f}^{r^o}$, $\mathbf{K}^r\delta^{r^o}$, $\mathbf{f}_b^o$, $\mathbf{f}_s^o$, $\mathbf{n}_b^o$, $\mathbf{n}_s^o$, $\mathbf{F}_b^o$, $\mathbf{F}_s^o$, $\mathbf{f}^{e^o}$. It is clear that in these cases $\mathbf{q}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in Eq. (46) should be replaced by $\mathbf{q} + \mathbf{q}^o$, $\mathbf{u} + \mathbf{u}^o$, $\mathbf{v} + \mathbf{v}^o$ and $\mathbf{w} + \mathbf{w}^o$. When the mechanism is in static equilibrium, Eq. (46) becomes

$$\mathbf{K}_s \begin{bmatrix} \mathbf{q}^o \\ \mathbf{u}^o \end{bmatrix} = \mathbf{V}_s \mathbf{v}^o + \mathbf{W}_s \mathbf{w}^o \tag{47}$$

which indicates that the sum of all static forces = 0 when the multibody is in static equilibrium.

After elimination of Eq. (47) in Eq. (46), the latter equation still holds for linear systems in which $\mathbf{q}^o = \mathbf{o}$, $\mathbf{u}^o = \mathbf{o}$ and only dynamic forces have to be taken into account.

After the rigid-body equations of motion, described by Lagrangian generalized co-ordinates and the flexible body equations of motion, described by nodal co-ordinates, are joined together in Eq. (46), it can happen that some rigid-body motions are counted twice, once by the finite element formulation and once by the rigid-body formulation. The superfluous rigid-body nodal co-ordinates should be eliminated from the finite element formulation by introducing as many constraint equations as there are rigid-body nodal co-ordinates. The constraint equations express a linear relationship between the rigid-body nodal co-ordinates and the Lagrangian generalized co-ordinates and are used to eliminate the rigid-body nodal co-ordinates from Eq. (46). After the substitution process, Eq. (46) should be reduced by elimination of appropriate rows and columns such that each Lagrangian generalized co-ordinate occurs only once.

6. Conclusions

A modelling methodology has been proposed to derive the linear equation of motions of complex mechanisms, executing combined non-gyroscopic small rigid-body motions and elastic deformations. The final equation of motion (46) is presented in hybrid co-ordinates. Lagrangian generalized co-ordinates describe the rigid-body motions of the mechanism while nodal co-ordinates from a finite element analysis are used to describe the elastic deformations of the mechanism.

The application of hybrid co-ordinates has advantages even when rigid-body motions of the mechanism are approximated by linear equations:

(1) A clear distinction exists between flexible body and rigid-body motions: the former is described by independent nodal or co-ordinates, the latter by independent Lagrangian generalized co-ordinates.
(2) The elastic multibody can easily be separated into main operating groups or individual components. Each sub-system is then studied, tested and modelled separately.
(3) Active elements with mechanical, electric or hydraulic devices, controlled by a regulator are easily built in.
(4) The number of degrees of freedom for the whole mechanism can be reduced, which is an essential premise for utilizing control algorithms.

Appendix A. Nomenclature

c_a^r	damping coefficient of the damper in a_a (N s/m)
$\mathbf{f}_{b_i}$	resulting force of all distributed body forces acting on Q_i with regard to $({}^ix, {}^iy, {}^iz, {}^io)$ (N)
$\mathbf{f}_i^e$	external concentrated force acting on Q_i (N)
$\mathbf{f}_i^{e^o}$	static part of $\mathbf{f}_i^e$ (N)
$\mathbf{f}_i^{e^d}$	dynamic part of $\mathbf{f}_i^e$ (N)
${}^o\mathbf{f}_a^r$	internal concentrated hinge force in a_a (N)
${}_c\mathbf{f}_a^r$	actuator force in a_a (N)
$\mathbf{f}_{s_i}$	resulting force of all distributed surface forces acting on Q_i with regard to $({}^ix, {}^iy, {}^iz, {}^io)$ (N)
${}^o\mathbf{f}_i$	force acting on Q_i (N)
${}^o\mathbf{f}_i^d(t)$	dynamic part of ${}^o\mathbf{f}_i$ (N)
${}^o\mathbf{f}_i^o$	static part of ${}^o\mathbf{f}_i$ (N)
h_i	number of flexible d.o.f. in Q_i
k_a^r	spring constant of the spring in a_a (N/m)
m_i	number of finite elements in Q_i
m_{b_i}	total mass of Q_i (kg)
l_a^r	spring–damper–actuator length in a_a (m)
$l_a^{r^o}$	initial spring–damper–actuator length in a_a (m)

n	number of bodies in the mechanism
$\mathbf{n}_{b_i}$	resulting moment of all distributed body forces acting on Q_i with regard to $({}^ix, {}^iy, {}^iz, {}^io)$ (N m)
$\mathbf{n}_{s_i}$	resulting moment of all distributed surface forces acting on Q_i with regard to $({}^ix, {}^iy, {}^iz, {}^io)$ (N m)
${}^o\mathbf{p}_{ij}$	inertial position vector of point P_{ij} (m)
$\delta^o\mathbf{p}_{ij}$	inertial virtual displacement vector of P_{ij} (m)
${}^o\ddot{\mathbf{p}}_{ij}$	inertial acceleration vector of P_{ij} (m/s^2)
$\mathbf{q}$	overall vector of Lagrangian generalized co-ordinates
$\mathbf{q}_a$	vector of Lagrangian generalized co-ordinates in hinge a_a
q_{ai}	Lagrangian generalized co-ordinate i in hinge a_a
$\mathbf{r}_i$	inertial position vector of the origin of the floating reference frame of Q_i (m)
$\mathbf{r}_i^e$	position vector of the point of action of $\mathbf{f}_i^e$ (m)
$\mathbf{s}_{ij}$	rigid-body displacement vector in Q_i (m)
$\mathbf{t}_{ij}$	variable element displacement vector in Q_i (m)
$\mathbf{u}$	overall vector of independent nodal co-ordinates in Q
$\mathbf{u}_i$	overall vector of independent nodal co-ordinates in Q_i
$\mathbf{v}$	vector of generalized control forces
$\mathbf{w}$	vector of generalized disturbance forces
$({}^ox, {}^oy, {}^oz, {}^oo)$	(absolute) reference co-ordinate system
$({}^ix, {}^iy, {}^iz, {}^io)$	floating reference co-ordinate system of Q_i
$({}^{ij}x, {}^{ij}y, {}^{ij}z, {}^{ij}o)$	intermediate element reference co-ordinate system of finite element j in Q_i
$({}^{ij}\hat{x}, {}^{ij}\hat{y}, {}^{ij}\hat{z}, {}^{ij}\hat{o})$	element reference co-ordinate system of finite element j in Q_i
${}^o\mathbf{A}_i$	co-ordinate transformation matrix from $({}^ix, {}^iy, {}^iz, {}^io)$ to $({}^ox, {}^oy, {}^oz, {}^oo)$
$\mathbf{B}_i$	inertia coupling between the translation and rotation of Q_i
$\mathbf{C}$	damping matrix of Q from a finite element modelling
$\mathbf{C}_i$	damping matrix of Q_i from a finite element modelling
$\mathbf{C}_s$	damping matrix of Q
$\mathbf{E}_i$	inertia coupling between the translation and elastic deformation of Q_i
${}^o\mathbf{F}_{b_i}$	resultant body force acting on Q_i (N)
${}^o\mathbf{F}_{b_{ij}}$	body force acting on finite element j of Q_i (N/m^3)
${}^o\mathbf{F}_{s_i}$	resultant surface force acting on Q_i (N)
${}^o\mathbf{F}_{s_{ij}}$	surface force acting on finite element j of Q_i (N/m^2)
$\mathbf{G}_i$	inertia coupling between the rotation and elastic deformation of Q_i
$\mathbf{J}_i$	inertia tensor of Q_i with respect to $({}^ix, {}^iy, {}^iz, {}^io)$
$\mathbf{K}$	stiffness matrix of Q from a finite element modelling
$\mathbf{K}_i$	stiffness matrix of Q_i from a finite element modelling
$\mathbf{K}_s$	stiffness matrix of Q
$\mathbf{L}_{ij}$	locator matrix for finite element j in Q_i
$[\mathbf{m}_b]$	mass matrix for the rigid-body part of the multibody
$\mathbf{M}$	mass matrix of Q from a finite element modelling
$\mathbf{M}_i$	mass matrix of Q_i from a finite element modelling
$\mathbf{M}_s$	mass matrix of Q
$\mathbf{O}$	matrix of zeros

P_{ij}	selected point in $\mathrm{d}V_{ij}$
Q	multibody
Q_i	body i in Q
S_{ij}	surface of finite element j in Q_i (m^2)
V_{ij}	volume of finite element j in Q_i (m^3)
$\mathrm{d}V_{ij}$	selected elementary volume in V_{ij} (m^3)
V_s	control distribution matrix of Q
W_s	disturbance distribution matrix of Q
V_{ij}	volume of finite element j in body Q_i (m^3)
δW^{cf}	the virtual work executed by all concentrated forces in Q (N m)
δW_{ij}^{cf}	the virtual work executed by all concentrated forces in $\mathrm{d}V_{ij}$ (N m)
δW^{df}	the virtual work done by the external distributed body and surface forces on Q (N m)
δW_{ij}^{df}	the virtual work done by the external distributed body and surface forces on $\mathrm{d}V_{ij}$ (N m)
δW^e	virtual work done by the external concentrated forces in Q_i (N m)
δW_i^e	virtual work done by all external concentrated forces in Q (N m)
δW_a^r	virtual work executed by the internal concentrated hinge forces in a_a (N m)
δW^r	virtual work executed by all hinge forces in Q (N m)
$\boldsymbol{\varepsilon}_{ij}$	strain vector defined in $({}^{ij}x, {}^{ij}y, {}^{ij}z, {}^{ij}o)$
$\delta_a^{r^o}$	initial deformation in a_a (m)
$\delta\boldsymbol{\varepsilon}_{ij}$	virtual strain vector defined in $({}^{ij}x, {}^{ij}y, {}^{ij}z, {}^{ij}o)$
$\theta(t)$, $\phi(t)$, $\psi(t)$	Tait–Bryan angles representing body roll, pitch and yaw
$\boldsymbol{\pi}$	overall vector of small angular displacements of Q
$\boldsymbol{\pi}_i$	vector of small angular displacements of Q_i
ρ_{ij}	density of the elementary volume element $\mathrm{d}V_{ij}$ (kg/m^2)
$\boldsymbol{\sigma}_{ij}$	normal stress vector $({}^{ij}x, {}^{ij}y, {}^{ij}z, {}^{ij}o)$ (N/m)
$\boldsymbol{\Phi}_{ij}$	matrix of shape fuctions
$\boldsymbol{\omega}$	overall angular velocity vector of Q
$\boldsymbol{\omega}_i$	angular velocity vector of Q_i

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