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An alternative derivation of dynamic admittance matrix of piezoelectric cantilever bimorph

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Abstract

In this paper, the derivation method used in (J. Microelectromech. Systems 3 (1994) 105) and the solutions of dynamic admittance matrix of a piezoelectric device derived from the method are reviewed. By solving the problem of dynamic responses of a piezoelectric cantilever bimorph with mode analysis method, an alternative approach in the derivation of the dynamic admittance matrix and other related parameters of a piezoelectric system, which can be expressed explicitly in terms of series resonance characteristics of the structure, is presented. It is shown that this form of solutions may offer some conveniences in studying mechanical and electrical properties of the system in the vicinity of resonance frequencies.

1. Introduction

Admittance of an elastic piezoelectric transducer derived from vibration analysis is often used to calculate the parameters of equivalent circuit model and to study electromechanical coupling behaviour of the transducer in the vicinity of resonance frequencies. Therefore, considerable efforts have been made to obtain dynamic admittances of various piezoelectric structures in the literature [1–5].

For a piezoelectric cantilever bimorph, the analytic expressions of its dynamic admittance matrix have been obtained and discussed in Refs. [1,2]. The derivation method used in Ref. [1] has been followed by some researchers to derive admittance matrices of other piezoelectric structures [5]. In the derivations, the general solutions of dynamic responses are obtained from the governing

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equation of free vibration. The applied force and moment at the free end of the cantilever are incorporated into the boundary conditions. Therefore, to derive the dynamic responses for different exciting forces, the respective coefficients in the general solutions have to be determined separately due to the different boundary conditions defined. For the forces acting on the positions other than boundaries, specific derivations also need to be taken to obtain corresponding special solutions added to the general solutions. This kind of treatment is effective for a simple problem discussed, and exact and concise analytical results can be obtained. For a complex problem, however, this method may not be effective and convenient. For example, the related analytic special solutions are usually not easy to be obtained for complex applied forces, and the separate derivations of the dynamic responses corresponding to the different excitations are not suitable for unified treatments, which are important for the problems with complex applied forces or structures.

In addition, since the solutions in Ref. [1] are not expressed explicitly based on resonance properties of the structure, it may not be very convenient to be used for the discussion of resonance behaviours of the structure. Additional treatments usually have to be taken to study mechanical and electrical properties of the transducers in the vicinities of the resonance frequencies [6].

In view of the discussions mentioned above, the mode analysis method, or mode summation method, is used in this paper to derive the dynamic admittance matrices by taking piezoelectric cantilever bimorph as an example. In the derivations, the natural frequencies and corresponding normal modes of the cantilever bimorph are obtained first through mode analysis of free vibration. Since the vibrations of the dynamic system for different applied forces are studied under the same boundary conditions, the corresponding responses of the system can then be treated uniformly and can be expressed in unified general form based on the summation of the contributions of the series normal modes. This treatment may be more convenient as long as a problem with complex structure or applied forces is involved. From the general solutions, the dynamic admittance matrix of the piezoelectric cantilever bimorph can be determined following a standard procedure. Although the solutions obtained by the mode summation method are not as concise as those given in Ref. [1], the resonance properties of the structure, however, are included into the expressions explicitly. Since the dynamic properties at resonance frequencies are of most interest for a piezoelectric transducer or a piezoelectric resonator, it would be convenient with the form of the solutions to discuss mechanical and electrical properties of the transducers in the vicinity of the resonance frequencies.

2. Forced vibration of cantilever beam

Consider a piezoelectric cantilever bimorph acted by a moment M at the tip, a force F at the tip, a distributed force p, and a voltage V, respectively, as shown in Fig. 1. The geometries of the bimorph are length L, width b, and thickness of each strip h. The constitutive equations of the strips are given by [1]

$$S_1 = s_{11}^E T_1 + d_{31} E_3,$$

$$D_3 = d_{31} T_1 + \varepsilon_{33}^T E_3, \tag{1}$$

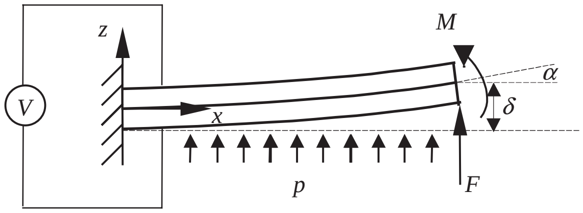

Fig. 1. The cantilever bimorph and applied forces [1].

where S_1 and T_1 are the strain and stress in length direction of the piezoelectric strips; D_3 and E_3 are the electric displacement and electric field; s_{11}^E, d_{31} and ε_{33}^T are the compliance at constant electric field, the piezoelectric strain constant, and dielectric constant under constant stress, respectively. The equation of motion for the lateral vibration of the bimorph beam, including the effect of damping, can be written as

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + c\frac{\partial u(x,t)}{\partial t} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t), \tag{2}$$

where $u(x,t)$ is the transverse displacement, $f(x,t)$ the applied force, ρ the density of the material, c the equivalent damping coefficient, $A = 2bh$ the cross-sectional area of the bimorph, and EI the bending rigidity of the bimorph, which is given by

$$EI = \frac{2}{3}\frac{bh^3}{s_{11}^E}. \tag{3}$$

In Eq. (2), the damping force is assumed to be proportional to the particle velocity in the piezoelectric material [7,8].

In addition to the form of the solutions given in Ref. [1], the general solutions of the motion of Eq. (2) can also be obtained following mode summation method often used in mechanical and structural engineering [9].

According to mode summation method, the general solutions of Eq. (2) can be expanded based on normal modes of the cantilever, which are obtained from its mode analysis of free vibration [9]. By applying orthogonal conditions of the normal modes, the uncoupled ordinary differential equations for the generalized co-ordinates, $q_i(t)$, can be obtained as

$$\ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = f_i(t), \quad i = 1, 2, \ldots, \tag{4}$$

where ω_i is the ith order natural frequency of the cantilever; ζ_i is defined as damping factor, M_i the generalized mass, and $f_i(t)$ the generalized force, which are given by

$$\zeta_i = \frac{c}{2\rho A\omega_i}, \quad M_i = \int_0^L \rho A\varphi_i^2(x)\,\mathrm{d}x = \rho AL, \quad f_i(t) = \frac{1}{M_i}\int_0^L f(x,t)\varphi_i(x)\,\mathrm{d}x. \tag{5}$$

In Eq. (5), $\varphi_i(x)$ is the normal mode corresponding the natural frequency, ω_i, and have the form

$$\omega_i = \frac{(\lambda_i L)^2}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad \varphi_i(x) = \cosh\lambda_i x - \cos\lambda_i x - \beta_i(\sinh\lambda_i x - \sin\lambda_i x), \tag{6}$$

and the series eigenvalues $\lambda_i L$ for the cantilever are obtained as $\lambda_1 L = 1.875$, $\lambda_2 L = 4.694$, etc. The constants β_i in $\varphi_i(x)$ are given in Eq. (A.1) of Appendix A. It is noted that the solutions given in Ref. [1] can only provide approximate mode shapes as the applied frequencies approach series resonance frequencies [2]. It cannot determine the mode shapes, or normal modes, exactly at the resonance frequencies since the solutions are infinite at these frequencies.

If, instead of distributed loads, a concentrated force $F(a,t)$ and a concentrated moment $M(a,t)$ are acted at some point $x = a$, the generalized force for such loads is

$$f_i(t) = \frac{1}{M_i}[F(a,t)\varphi_i(a) + M(a,t)\varphi_i'(a)]. \tag{7}$$

Therefore, the dynamic responses of the system can be treated uniformly for different applied forces.

For harmonic excitation $f(x,t) = \tilde{f}(x)\mathrm{e}^{\mathrm{j}\omega t}$ with the exciting frequency ω, the generalized force $f_i(t)$ can be further written as

$$f_i(t) = \tilde{f}_i \mathrm{e}^{\mathrm{j}\omega t}, \quad \tilde{f}_i = \frac{1}{M_i}\int_0^L \tilde{f}(x)\varphi_i(x)\,\mathrm{d}x. \tag{8}$$

Substituting Eq. (8) into Eq. (4), the steady state response $q_i(t)$ can be obtained as

$$q_i(t) = \frac{\tilde{f}_i}{\omega_i^2} H_i(\omega)\mathrm{e}^{\mathrm{j}\omega t} = Q_i \mathrm{e}^{\mathrm{j}(\omega t - \alpha_i)}, \tag{9}$$

where

$$H_i(\omega) = \frac{1}{1 - (\omega/\omega_i)^2 + \mathrm{j}2\zeta_i\omega/\omega_i} = |H_i(\omega)|\mathrm{e}^{-\mathrm{j}\alpha_i}, \quad \alpha_i = \arctan\frac{2\zeta_i\omega/\omega_i}{1-(\omega/\omega_i)^2}, \tag{10}$$

$$|H_i(\omega)| = \frac{1}{\sqrt{[1-(\omega/\omega_i)^2]^2 + (2\zeta_i\omega/\omega_i)^2}}, \quad Q_i = \frac{\tilde{f}_i}{\omega_i^2}|H_i(\omega)|.$$

Therefore, the general steady state response of the cantilever excited by either mechanical or electric forces with the applied frequency ω can be written as

$$u(x,t) = \sum_i \varphi_i(x)q_i(t) = \sum_i \frac{\tilde{f}_i}{\omega_i^2}H_i(\omega)\varphi_i(x)\mathrm{e}^{\mathrm{j}\omega t} = \sum_i Q_i\varphi_i(x)\mathrm{e}^{\mathrm{j}(\omega t-\alpha_i)}. \tag{11}$$

It is noted that the general solution (11) has satisfied the boundary conditions of the cantilever bimorph without any unknowns to be determined. For an exciting force, the corresponding response can be obtained simply by substituting the force into Eq. (8) to determine $\tilde{f}_i$. Therefore, the method is suitable for systematic treatments for any type of applied forces.

If the applied frequency is very close to one of the natural frequencies of the beam, say ω_m, solution (11) can be approximated as

$$u(x,t) = \frac{\tilde{f}_m}{\omega_m^2}H_m(\omega)\varphi_m(x)\mathrm{e}^{\mathrm{j}\omega t} = \frac{\tilde{f}_m\varphi_m(x)\mathrm{e}^{\mathrm{j}\omega t}}{\omega_m^2 - \omega^2 + \mathrm{j}2\zeta_m\omega\omega_m}. \tag{12}$$

If the damping effect is not considered, Eq. (12) can be further simplified as

$$u(x,t) = \frac{\tilde{f}_m \varphi_m(x) \mathrm{e}^{\mathrm{j}\omega t}}{\omega_m^2 - \omega^2}. \tag{13}$$

It should be pointed out that solutions (11) are essentially same as the expressions obtained by analytical method in Ref. [1] but with different forms. They are expressed explicitly based on the expansion of the normal modes at resonance frequencies. In addition, the solutions can also be approximated as a simple form (12) or (13) in the vicinity of a resonance, which contains the dynamic parameters of the system only at the resonance. Therefore, it is convenient to discuss the resonance behaviour and to derive the equivalent circuit parameters of the transducer with the expressions.

3. Admittance matrix

In this section, the elements of the admittance matrix **B** defined in Ref. [1] are derived with the general solutions given in the preceding section. The derivations are proceed under the following applied forces:

- cantilever bimorph subjected to a concentrated moment $M = M_0 \mathrm{e}^{\mathrm{j}\omega t}$ at $x = L$;
- cantilever bimorph subjected to a concentrated force $F = F_0 \mathrm{e}^{\mathrm{j}\omega t}$ at $x = L$;
- cantilever bimorph subjected to a distributed force $q(x) = q_0 \mathrm{e}^{\mathrm{j}\omega t}$;
- cantilever bimorph subjected to an electric voltage $V(x) = V_0 \mathrm{e}^{\mathrm{j}\omega t}$.

3.1. *Cantilever bimorph subjected to mechanical forces*

If a cantilever bimorph is excited by a dynamic moment $M = M_0 \mathrm{e}^{\mathrm{j}\omega t}$ at its tip, the related generalized force $\tilde{f}_i$ can be determined from Eqs. (8) and (7) as

$$\tilde{f}_i = \frac{1}{M_i} M_0 \varphi_i'(L) = \frac{M_0}{\rho A L} \varphi_i'(L). \tag{14}$$

Substituting Eq. (14) into the general solution (11), the deflection of the cantilever acted under the moment at the tip can be obtained as

$$u(x,t) = \frac{1}{\rho A L} \sum_i \frac{H_i(\omega)}{\omega_i^2} \varphi_i'(L) \varphi_i(x) M_0 \mathrm{e}^{\mathrm{j}\omega t}, \tag{15}$$

where $\varphi_i(x), \varphi_i'(L)$ and $H_i(\omega)$ are given by Eqs. (6), (A.2) and (10), respectively.

By using the relations given in Appendix A, the deflection δ at the tip of the cantilever due to the action of the moment is obtained as

$$\begin{aligned} \delta = u(L,t) &= \frac{1}{\rho A L} \sum_i \frac{H_i(\omega)}{\omega_i^2} \varphi_i'(L) \varphi_i(L) M_0 \mathrm{e}^{\mathrm{j}\omega t} \\ &= \frac{4}{\rho A L} \sum_i \frac{\lambda_i \beta_i}{\omega_i^2} H_i(\omega) M_0 \mathrm{e}^{\mathrm{j}\omega t}. \end{aligned} \tag{16}$$

Similarly, the slope α at the tip is given by

$$\alpha = \left.\frac{\mathrm{d}u(x,t)}{\mathrm{d}x}\right|_{x=L} = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\varphi_i'(L)M_0\mathrm{e}^{\mathrm{j}\omega t}$$
$$= \frac{4}{\rho AL}\sum_i \frac{\lambda_i^2\beta_i^2}{\omega_i^2}H_i(\omega)M_0\mathrm{e}^{\mathrm{j}\omega t} \tag{17}$$

and the volume displacement v can be obtained by

$$v = \int_0^b\int_0^L u(x,t)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\Phi_i(L)M_0\mathrm{e}^{\mathrm{j}\omega t}$$
$$= \frac{4b}{\rho AL}\sum_i \frac{\beta_i\eta_i}{\omega_i^2}H_i(\omega)M_0\mathrm{e}^{\mathrm{j}\omega t}. \tag{18}$$

Therefore, expressions (17), (16) and (18) fill up the elements (1,1), (2,1) and (3,1) of the admittance matrix **B**.

The elements of the admittance matrix corresponding to the tip force and the distributed force can be obtained similarly. The results are listed in Appendix B.

3.2. Cantilever bimorph subjected to an electric voltage $V(x) = V_0\mathrm{e}^{\mathrm{j}\omega t}$

The effect of the applied electric voltage can be equivalent to a moment acted at the end section of the cantilever bimorph [1]:

$$M = -2b\int_0^h \frac{d_{31}}{s_{11}^E}E_3 z\,\mathrm{d}z = \frac{bhd_{31}}{2s_{11}^E}V, \tag{19}$$

where the electric field E_3 has been replaced by the relation

$$E_3 = -\frac{V}{2h}. \tag{20}$$

By replacing the mechanical moment in expression (15) by the equivalent moment (19), the deflection of the cantilever under the electric voltage can be obtained as

$$u(x,t) = \frac{bhd_{31}}{2s_{11}^E}\frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\varphi_i(x)V_0\mathrm{e}^{\mathrm{j}\omega t}. \tag{21}$$

The corresponding slope α, deflection δ, and volume displacement v are obtained as

$$\alpha = \frac{bhd_{31}}{2s_{11}^E}\frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\varphi_i'(L)V_0\mathrm{e}^{\mathrm{j}\omega t}$$
$$= \frac{bhd_{31}}{s_{11}^E}\frac{2}{\rho AL}\sum_i \frac{\lambda_i^2\beta_i^2}{\omega_i^2}H_i(\omega)V_0\mathrm{e}^{\mathrm{j}\omega t}, \tag{22}$$

$$\begin{aligned}\delta &= \frac{bhd_{31}}{2s_{11}^E}\frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\varphi_i(L)V_0\mathrm{e}^{\mathrm{j}\omega t}\\ &= \frac{bhd_{31}}{s_{11}^E}\frac{2}{\rho AL}\sum_i \frac{\lambda_i\beta_i}{\omega_i^2}H_i(\omega)V_0\mathrm{e}^{\mathrm{j}\omega t},\end{aligned} \tag{23}$$

$$\begin{aligned}v &= \frac{bhd_{31}}{2s_{11}^E}\frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\Phi_i(L)V_0\mathrm{e}^{\mathrm{j}\omega t}\\ &= \frac{bhd_{31}}{s_{11}^E}\frac{2b}{\rho AL}\sum_i \frac{\beta_i\eta_i}{\omega_i^2}H_i(\omega)V_0\mathrm{e}^{\mathrm{j}\omega t}.\end{aligned} \tag{24}$$

The electric charge Q in the electrodes is given by

$$Q = \int_0^b \int_0^L D_3\,\mathrm{d}x\,\mathrm{d}y, \tag{25}$$

in which the electric displacement D_3 can be determined from the constitutive relations (1)

$$D_3 = \frac{d_{31}}{s_{11}^E}S_1 - \frac{d_{31}^2}{s_{11}^E}E_3 + \varepsilon_{33}^T E_3, \tag{26}$$

where the strain S_1 on the surface the bimorph is given by

$$S_1 = -h\frac{\partial^2 u}{\partial x^2}. \tag{27}$$

By substituting Eqs. (20), (21), (27) and (26) into Eq. (25), the electric charge Q in the electrodes due to the action of the voltage is obtained as

$$\begin{aligned}Q &= -\frac{bhd_{31}}{s_{11}^E}\int_0^L \frac{\partial^2 u}{\partial x^2}\mathrm{d}x + bL\left(\varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E}\right)E_3\\ &= -\left[\frac{bL}{2h}\left(\varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E}\right) + \frac{1}{2}\left(\frac{bhd_{31}}{s_{11}^E}\right)^2\frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i'(L)\varphi_i'(L)\right]V_0\mathrm{e}^{\mathrm{j}\omega t}\\ &= -\left[\frac{bL}{2h}\left(\varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E}\right) + \left(\frac{bhd_{31}}{s_{11}^E}\right)^2\frac{2}{\rho AL}\sum_i \frac{\lambda_i^2\beta_i^2}{\omega_i^2}H_i(\omega)\right]V_0\mathrm{e}^{\mathrm{j}\omega t}.\end{aligned} \tag{28}$$

Expressions (22)–(24) and (28) fill up the elements (1,4), (2,4), (3,4) and (4,4) of the admittance matrix **B**. Due to the symmetry property, the elements (4,1), (4,2) and (4,3) in the matrix are same as the elements (1,4), (2,4) and (3,4), respectively. Therefore, all elements of the dynamic admittance matrix **B** defined by

$$\{\alpha, \delta, v, Q\}^{\mathrm{T}} = \mathbf{B}\{M, F, p, V\}^{\mathrm{T}} \tag{29}$$

have been obtained.

By comparing with the expressions given in Ref. [1], the elements of the matrix obtained here are expressed explicitly according to the summation of the dynamic properties of the structure, and the damping effects are also included. With the expressions, it might be easier to study mechanical and electrical properties of the transducer in the vicinities of series resonance frequencies. Furthermore, since the damping effects can be included into the expressions conveniently, the resistance in equivalent circuit model can also be determined.

4. Parameters of equivalent circuit model

With the dynamic admittance matrix of the cantilever bimorph obtained, the corresponding resonance and antiresonance properties as well as parameters of the lumped-element equivalent circuit model can be calculated.

If the frequency of the applied dynamic voltage ω is very close to one of the natural frequencies, ω_m, of the cantilever bimorph, the cantilever is nearly under the resonance vibration of the mode. In the case, the electric charge (28) in the electrodes can be reduced to

$$Q = \left[\frac{bL}{2h}\left(\varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E}\right) + \frac{2}{\rho AL}\left(\frac{bhd_{31}}{s_{11}^E}\right)^2 \frac{\lambda_m^2 \beta_m^2}{\omega_m^2} H_m(\omega)\right] V_0\, \mathrm{e}^{\mathrm{j}\omega t}. \tag{30}$$

Therefore, the total current flowing into the electrode surface is

$$I = \frac{\partial Q}{\partial t} = \mathrm{j}\omega Q, \tag{31}$$

and the admittance is given by

$$Y = \frac{I}{V_0 \mathrm{e}^{\mathrm{j}\omega t}} = \mathrm{j}\omega C, \tag{32}$$

where

$$C = \frac{bL}{2h}\left(\varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E}\right) + \frac{2}{\rho AL}\left(\frac{bhd_{31}}{s_{11}^E}\right)^2 \frac{\lambda_m^2 \beta_m^2}{\omega_m^2} H_m(\omega), \tag{33}$$

is the dynamic capacitance of the bimorph. The first term of Eq. (33) is the capacitance when the applied frequency is far from the resonant frequencies of the bimorph

$$C_0 = \frac{bL}{2h}\left(\varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E}\right). \tag{34}$$

To determine other parameters of the equivalent circuit model, the complex frequency response $H_m(\omega)$ in Eq. (33) should be simplified first. Since the applied frequency ω is very nearly equal to the resonant frequency ω_m, by defining $\Delta\omega_m = \omega_m - \omega$, we have

$$\frac{1}{\omega_m^2} H_m(\omega) = \frac{1}{\omega_m^2 - \omega^2 + \mathrm{j}2\zeta_m \omega\omega_m} \approx \frac{1}{2\omega_m} \frac{\Delta\omega_m - \mathrm{j}\zeta_m\omega_m}{\Delta\omega_m^2 + \zeta_m^2\omega_m^2}. \tag{35}$$

By substituting Eq. (35) into Eq. (33) and defining

$$k_m = \frac{1}{\rho AL}\left(\frac{bhd_{31}}{s_{11}^E}\right)^2 \frac{\lambda_m^2 \beta_m^2}{\omega_m}, \quad \vartheta = \zeta_m\omega_m = \frac{c}{2\rho A}, \tag{36}$$

the admittance related to the dynamic part, Y_m, in Eq. (32) can be written as

$$Y_m = \mathrm{j}\omega k_m \frac{\Delta\omega_m - \mathrm{j}\vartheta}{\Delta\omega_m^2 + \vartheta^2}. \tag{37}$$

The reciprocal of the admittance Y_m is the impedance Z_m, which is

$$Z_m = \frac{1}{Y_m} = \frac{1}{\omega k_m}(\vartheta - \mathrm{j}\Delta\omega_m). \tag{38}$$

Therefore, the resistance is given by [7]

$$R_m = \mathrm{Re}(Z_m) = \frac{\vartheta}{\omega_m k_m} = \frac{cL}{2b^2h^2}\left(\frac{s_{11}^E}{d_{31}\lambda_m\beta_m}\right)^2. \tag{39}$$

The reactive part of Z_m is given by

$$X_m = \mathrm{Im}(Z_m) = -\frac{\Delta\omega_m}{\omega_m k_m} = -\frac{2\rho L}{bh}\left(\frac{s_{11}^E}{d_{31}\lambda_m\beta_m}\right)^2\Delta\omega_m. \tag{40}$$

On the other hand, the reactance of a circuit consisting of L and C in series is [7]

$$X = \omega L - \frac{1}{\omega C} = \frac{\omega^2 LC - \omega_0^2 LC}{\omega C} = \frac{(\omega+\omega_0)(\omega-\omega_0)L}{\omega} \approx -2\Delta\omega L. \tag{41}$$

With the relation, the reactance X_m behaves as if it were an inductance L_m having the value

$$L_m = \frac{\rho L}{bh}\left(\frac{s_{11}^E}{d_{31}\lambda_m\beta_m}\right)^2, \tag{42}$$

in series with a capacitance having the value

$$C_m = \frac{1}{\omega_m^2 L_m} = \frac{3b}{hL}\frac{d_{31}^2}{s_{11}^E}\frac{\beta_m^2}{\lambda_m^2}. \tag{43}$$

Therefore, the parameters of the equivalent circuit model have been obtained. They are given by Eqs. (34), (39), (42) and (43), respectively.

The resonant and antiresonant frequencies can be obtained from relation (32). If the damping effect is not considered, Eq. (32) can be simplified as

$$Y = \mathrm{j}\omega\left[\frac{bL\varepsilon_{33}^T}{2h}(1-k_{31}^2) + \frac{bh}{\rho L}\frac{\varepsilon_{33}^T k_{31}^2}{s_{11}^E}\frac{\lambda_m^2\beta_m^2}{\omega_m^2 - \omega^2}\right], \tag{44}$$

where k_{31}^2 is the piezoelectric coupling factor defined by

$$k_{31}^2 = \frac{d_{31}^2}{s_{11}^E\varepsilon_{33}^T}. \tag{45}$$

According to the definition of resonance and antiresonance, the resonant frequency can be determined by letting $Y \to \infty$, and the antiresonant frequency by $Y = 0$. Therefore, it can be found from Eq. (44) that the resonant frequency ω_r is equal to the natural frequency ω_m of the bimorph, and the related antiresonant frequency ω_a is obtained as

$$\omega_a = \left(\omega_m^2 + \frac{2h^2}{\rho L^2} \frac{k_{31}^2}{1 - k_{31}^2} \frac{\lambda_m^2 \beta_m^2}{s_{11}^E} \right)^{1/2}. \tag{46}$$

Therefore, for series resonant frequencies ω_m ($m = 1, 2, \ldots$), the corresponding antiresonant frequencies can be determined approximately from Eq. (46) without necessity for solving a related equation as usually done in the literature.

With one of the definitions for electromechanical coupling coefficient k_{eff} defined by [10,8]

$$k_{eff}^2 = \frac{\omega_a^2 - \omega_r^2}{\omega_a^2}, \tag{47}$$

the electromechanical coupling coefficients of the bimorph at different resonance modes can be estimated from relation (47).

5. Discussions and concluding remarks

The general expressions of dynamic responses and admittance matrix for a piezoelectric cantilever bimorph based on mode analysis method are obtained in the present paper. The solutions can also be regarded as expansion of the form of the solutions given in Ref. [1] with the infinite series vibration modes of the cantilever. In general, for the problems of which analytical solutions exist, the two forms of the solutions are essentially same. For the problems without analytical solutions, however, the approximate solutions obtained by the two methods, respectively, could be or could not be same depending on the approximate treatments used in the derivations.

In case that two forms of the solutions are same, which of them are used will depend on the convenience of the problems discussed. In determining the dynamic responses, for instance, the concise expressions in Ref. [1] can provide exact results conveniently. For the form of the solutions expressed by the expansion of infinite series, however, extra efforts have to be taken to obtain the expressions of the summation of the infinite series.

On the other hand, the solutions of the mode analysis method are expressed explicitly according to the superposition of series normal mode vibrations, which are the intrinsic characteristics of the vibration system at resonances. Therefore, it is convenient to use the solutions for the study of dynamic properties of the system in the vicinity of resonances, which are of most interest for a piezoelectric transducer working in the range of resonance frequency. For example, the solutions can be approximated as the simple form (12) or (13) in the vicinity of a resonance. The corresponding properties such as equivalent circuit parameters and electromechanical coupling coefficient, which are directly related to the characteristics of the system at the resonance, can be derived straightforward with the expressions. If the solutions in Ref. [1] are used for the derivations instead, additional treatments usually have to be taken to write the solutions in terms of the series resonance frequencies [6].

In addition, by comparing the derivation procedure of the mode summation method with the method used in Ref. [1], it is found that the mode summation method may offer a more unified way to obtain the responses of a vibration system subjected to different excitations, and is easier for the derivations following regular steps. These properties may show some conveniences in the derivation of a complex problem.

The method used and the problem solved in the paper is ordinary. With the dynamic analysis of a piezoelectric cantilever bimorph as an example, this paper is to suggest a different approach in the derivation of dynamic responses, admittance matrices, and equivalent parameters of piezoelectric transducers by the mode summation method. This method has been well used in mechanical and structural engineering for dynamic analysis but relatively less used in electrical engineering. Although the solutions obtained by the method are essentially same as those obtained in the literature, the present expressions could be regarded as complementary to the existing analytic expressions, and as an alternative choice in studying the resonance properties of the piezoelectric systems.

Appendix A

Define the eigenvalue related constants

$$\beta_i = \frac{\sinh\lambda_i L - \sin\lambda_i L}{\cosh\lambda_i L + \cos\lambda_i L}, \quad \gamma_i = \frac{\sinh\lambda_i L \sin\lambda_i L}{\cosh\lambda_i L + \cos\lambda_i L},$$
$$\eta_i = \frac{\cosh\lambda_i L \sin\lambda_i L + \sinh\lambda_i L \cos\lambda_i L}{\cosh\lambda_i L + \cos\lambda_i L}. \tag{A.1}$$

From (10), we have

$$\varphi_i(L) = 2\gamma_i, \quad \varphi_i'(L) = 2\lambda_i\eta_i, \quad \Phi_i(L) = \int_0^b \int_0^L \varphi_i(x)\,\mathrm{d}x\,\mathrm{d}y = \frac{2b}{\lambda_i}\beta_i. \tag{A.2}$$

Therefore, by using the relation $\cos\lambda_i L \cosh\lambda_i L + 1 = 0$, the following expressions can be obtained:

$$\varphi_i^2(L) = 4, \quad \varphi_i(L)\varphi_i'(L) = 4\lambda_i\beta_i, \quad \varphi_i'(L)\varphi_i'(L) = 4\lambda_i^2\beta_i^2,$$
$$\varphi_i(L)\Phi_i(L) = \frac{4b}{\lambda_i}\beta_i\gamma_i, \quad \varphi_i'(L)\Phi_i(L) = 4b\beta_i\eta_i, \quad \Phi_i^2(L) = \frac{4b^2}{\lambda_i^2}\beta_i^2. \tag{A.3}$$

Appendix B

For the cantilever bimorph excited by a concentrated force $F = F_0 e^{j\omega t}$ at its tip, the related generalized force $\tilde{f}_i$ can be determined from Eq. (8) as

$$\tilde{f}_i = \frac{1}{M_i}F_0\varphi_i(L) = \frac{F_0}{\rho AL}\varphi_i(L). \tag{B.1}$$

Substituting (B.1) into the general solution (11), the deflection of the cantilever under the action of the force at the tip can be obtained as

$$u(x,t) = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i(L)\varphi_i(x)F_0 e^{j\omega t}. \tag{B.2}$$

The corresponding slope α, deflection δ, and volume displacement v are obtained as

$$\alpha = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i(L)\varphi_i'(L)F_0 e^{j\omega t} = \frac{4}{\rho AL}\sum_i \frac{\lambda_i\beta_i}{\omega_i^2}H_i(\omega)F_0 e^{j\omega t}, \tag{B.3}$$

$$\delta = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i(L)\varphi_i(L)F_0 e^{j\omega t} = \frac{4}{\rho AL}\sum_i \frac{1}{\omega_i^2}H_i(\omega)F_0 e^{j\omega t}, \tag{B.4}$$

$$v = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\varphi_i(L)\Phi_i(L)F_0 e^{j\omega t} = \frac{4b}{\rho AL}\sum_i \frac{\beta_i\gamma_i}{\lambda_i\omega_i^2}H_i(\omega)F_0 e^{j\omega t}. \tag{B.5}$$

Expressions (B.3), (B.4) and (B.5) fill up the elements (1,2), (2,2) and (3,2) of the admittance matrix **B**.

Similarly, for the cantilever bimorph excited by a distributed force $q(x) = q_0 e^{j\omega t}$, the related generalized force $\tilde{f}_i$ can be determined from Eq. (8) as

$$\tilde{f}_i = \frac{1}{M_i}\int_0^b\int_0^L q_0\varphi_i(x)\,dx\,dy = \frac{q_0}{\rho AL}\Phi_i(L). \tag{B.6}$$

Substituting Eq. (B.6) into the general solution (11), the deflection of the cantilever under the action of the distributed force can be obtained as

$$u(x,t) = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\Phi_i(L)\varphi_i(x)q_0 e^{j\omega t}. \tag{B.7}$$

The corresponding slope α, deflection δ, and volume displacement v are obtained as

$$\alpha = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\Phi_i(L)\varphi_i'(L)q_0 e^{j\omega t} = \frac{4b}{\rho AL}\sum_i \frac{\beta_i\eta_i}{\omega_i^2}H_i(\omega)q_0 e^{j\omega t}, \tag{B.8}$$

$$\delta = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\Phi_i(L)\varphi_i(L)q_0 e^{j\omega t} = \frac{4b}{\rho AL}\sum_i \frac{\beta_i\gamma_i}{\lambda_i\omega_i^2}H_i(\omega)q_0 e^{j\omega t}, \tag{B.9}$$

$$v = \frac{1}{\rho AL}\sum_i \frac{H_i(\omega)}{\omega_i^2}\Phi_i(L)\,\Phi_i(L)q_0 e^{j\omega t} = \frac{4b^2}{\rho AL}\sum_i \frac{\beta_i^2}{\lambda_i^2\omega_i^2}H_i(\omega)q_0 e^{j\omega t}. \tag{B.10}$$

Expressions (B.8), (B.9) and (B.10) fill up the elements (1,3), (2,3) and (3,3) of the admittance matrix **B**.

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