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Response of a strongly non-linear oscillator to narrowband random excitations

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Abstract

The principal resonance of a van der Pol–Duffing oscillator subject to narrowband random excitations has been studied. By introducing a new expansion parameter $\varepsilon = \varepsilon(\bar{\varepsilon}, u_0)$ the method of multiple scales is adapted for the strongly non-linear system. The behavior of steady state responses, together with their stability, and the effects of system damping and the detuning, and magnitude of the random excitation on steady state responses are analyzed in detail. Theoretical analyses are verified by some numerical results. It is found that when the random noise intensity increases, the steady state solution may change form a limit cycle to a diffused limit cycle, and the system may have two different stable steady state solutions for the same excitation under certain conditions. The results obtained for the strongly non-linear oscillator complement previous results in the literature for weakly non-linear systems.

1. Introduction

The study on the response of non-linear systems to narrowband random excitations is of great importance. For example, the excitation of a secondary system would be a narrowband random process if the primary system could be modelled as a single-degree-of-freedom system with light damping subject to wide-band excitations. In the theory of non-linear random vibration, most results obtained so far are attributed to the response of non-linear oscillators to wide-band

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random excitations, while results of the effect of narrowband excitations on non-linear oscillators are quite limited.

Some researchers have used the equivalent linearization method [1–7], or the quasi-static method [8,9], or the stochastic averaging method [10], or the path integral method [11], or the digital simulation method [12–13], to study the response of a single-degree-of-freedom weakly non-linear system subject to narrowband random excitations [1–13]. However, the response of a strongly non-linear system to a narrowband excitation has not been studied. In this paper, the principal resonance of a strongly non-linear van der Pol–Duffing oscillator under narrowband random excitations is studied. Theoretical and numerical results show that when the random noise intensity increases, the steady state solution may change from a limit cycle to a diffused limit cycle, and under certain conditions the non-linear system may have two different steady state solutions for the same random excitation.

2. General analysis

Considering a van der Pol–Duffing oscillator subject to a random excitation

$$\ddot{u} + 2\bar{\varepsilon}\beta\dot{u} + u + \bar{\varepsilon}\alpha_1 u^3 + \bar{\varepsilon}\alpha_2\dot{u}^2 u = 2\bar{\varepsilon}\xi(t), \tag{1}$$

where dot indicates differentiation with respect to time t, the positive parameter $\bar{\varepsilon}$ may be "small" or not, β is the damping coefficient, α_1 and α_2 represent the intensity of the non-linear terms, and $\xi(t)$ is a random process governed by the following equation [14]:

$$\xi(t) = h\cos\phi(t), \quad \dot{\phi}(t) = \Omega + \gamma\dot{W}(t). \tag{2}$$

Thus, the external dynamic force $\xi(t)$ is modelled by a cosine function with deterministic amplitude h and a random phase angle $\phi(t)$, whose rotation speed is expressed by a constant Ω superimposed by white noise $\dot{W}(t)$ with intensity γ. According to Wedig [14], the power spectrum $S_\xi(\omega)$ of $\xi(t)$ is

$$S_\xi(\omega) = \frac{1}{2}\frac{h^2\gamma^2(\Omega^2 + \omega^2 + \gamma^4/4)}{(\Omega^2 - \omega^2 + \gamma^4/4)^2 + \omega^2\gamma^4}. \tag{3}$$

From Eq. (3) one can see that the generalized fluctuation model (2) covers both two extreme limiting cases. Obviously, for one extreme limiting case, $\gamma \to 0$, the fluctuation spectrum $S_\xi(\omega)$ is vanishing over the entire frequency range except at the singular frequency $\omega = \pm\Omega$, where $S_\xi(\pm\Omega)$ goes to infinity. This is a typical spectrum for narrowband random noise. Besides, it is worthwhile mentioning that a second order filtered white-noise model, usually used to describe narrowband random processes, possess two main disadvantages compared with the generalized fluctuation model (2). The former model needs two state variables and a certain transient time for reaching the stationary behavior. In this paper only the case for small γ is discussed, so $\xi(t)$ is bound to narrowband random noise.

A van der Pol–Duffing oscillator is a typical model in non-linear analysis. It has been shown by Dowell [15] and Holmes and Rand [16] that this model may represent the motion of a thin panel under supersonic airflow. This model may also describe the dynamics of a single-model laser with a saturable absorber as pointed out by Velarde and Antoranz [17]. Furthermore, according to

Knobloch and Proctor [18], this model is suitable for describing the evolution of the dominant velocity mode in an overstable convection when the frequency of oscillation is low. Rajan and Davies [19], Nayfeh and Serhan [20], and the authors of Refs. [21,22] studied the weakly non-linear cases for $\bar{\varepsilon} \ll 1$ by the method of multiple scales [23]. However, when $\bar{\varepsilon}$ is not small, the response problem of system (1) subject to narrowband excitations has not been studied. And at times there are situations of "strongly non-linear" systems, in which $\bar{\varepsilon}$ is not small, or $\bar{\varepsilon}\xi(t)$ is not small. This paper is devoted to developing a variation of multiple scales method applicable to "strongly non-linear" systems.

3. A modified method of multiple scales

While studying deterministic responses in a strongly non-linear system, Burton [24] suggested a modified version of perturbation technique. Now we put forward his idea to deal with random responses in a strongly non-linear van der Pol–Duffing system. Different from the conventional version, the modified one defines a new expansion parameter $\varepsilon = \varepsilon(\bar{\varepsilon}, u_0)$, and introduces the detuning parameter σ into the expression for Ω^2, rather than for Ω. Then, the first step is to redefine the time by introducing $T = \Omega t$, so that Eq. (1) can be rewritten as

$$\Omega^2\ddot{u} + 2\bar{\varepsilon}\beta\Omega\dot{u} + u + \bar{\varepsilon}\alpha_1 u^3 + \bar{\varepsilon}\alpha_2\Omega^2\dot{u}^2 u = 2\bar{\varepsilon}\xi(T), \tag{4}$$

where dot indicates the differentiation with respect to "time" T. This step accommodates the Ω^2 in the inertia term in Eq. (4). Now we may expect the fundamental harmonic in a steady state response having the amplitude u_0, for $\gamma = 0$. And a new expansion parameter $\varepsilon = \varepsilon(\bar{\varepsilon}, u_0)$ may be defined by u_0 along with $\bar{\varepsilon}$ as follows:

$$\varepsilon = \frac{\bar{\varepsilon}u_0^2}{4 + 3\bar{\varepsilon}u_0^2}. \tag{5}$$

It is obvious that $\varepsilon < \frac{1}{3}$ for all $\bar{\varepsilon}u_0^2$. In a weakly non-linear case the limit for which $\bar{\varepsilon}u_0^2 \to 0$, then we have $\varepsilon \to \frac{1}{4}\bar{\varepsilon}u_0^2$. Therefore, ε is a small parameter no matter whether $\bar{\varepsilon}$ is small or large. In terms of ε the original parameter $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \frac{4\varepsilon}{u_0^2(1 - 3\varepsilon)}. \tag{6}$$

The detuning parameter σ is now introduced into the expression for Ω^2 as follows:

$$\Omega^2 = \frac{1 + \sigma\varepsilon}{1 - 3\varepsilon}. \tag{7}$$

By Eqs. (2), (5) and (6), Eq. (4) can be rewritten as follows:

$$(1 + \sigma\varepsilon)\ddot{u} + 2\mu\varepsilon\dot{u} + u + \varepsilon\left[\frac{4}{u_0^2}(\alpha_1 u^3 + \varepsilon\alpha_2\Omega^2\dot{u}^2 u) - 3u\right] = \frac{4\varepsilon}{u_0^2}2h\cos(T + \gamma W(T)), \tag{8}$$

where the damping coefficient has been redefined as $\mu = 4\beta\Omega/u_0^2$. The next step is to put the Eq. (8) into non-dimensional form by letting $v = u/u_0$, then we have

$$\begin{aligned}(1+\sigma\varepsilon)\ddot{v} + 2\mu\varepsilon\dot{v} + v + \varepsilon[4(\alpha_1 v^3 + \alpha_2\Omega^2\dot{v}^2 v) - 3v]\\ = \frac{8\varepsilon}{u_0^3} h\cos(T + \gamma W(T)),\end{aligned} \tag{9}$$

where presumably $v = O(1)$ at most. It must be noted that by the above construction the amplitude of the fundamental harmonic of a steady state response v must be unity, for $\gamma = 0$.

From now on the usual steps in the method of multiple scales can be further applied. Then, we look for a uniformly approximate solution of Eq. (9) in the form

$$v(T,\varepsilon) = v_0(T_0,T_1) + \varepsilon v_1(T_0,T_1) + \cdots, \tag{10}$$

where $T_0 = T, \quad T_1 = \varepsilon T$ are the fast and the slow time scales, respectively.

By denoting $D_0 = \partial/\partial T_0$, $D_1 = \partial/\partial T_1$ the ordinary-time derivatives can be transformed into partial derivatives as

$$\frac{\mathrm{d}}{\mathrm{d}T} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}T^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots. \tag{11}$$

By substituting Eqs. (10) and (11) into Eq. (9) and equating coefficients of ε^0 and ε^1 to zero, the following equations can be derived:

$$D_0^2 v_0 + v_0 = 0, \tag{12}$$

$$\begin{aligned}D_0^2 v_1 + v_1 = &-\sigma D_0^2 v_0 - 2D_0 D_1 v_0 - 2\mu D_0 v_0\\ &- 4[\alpha_1 v_0^3 + \alpha_2\Omega^2 (D_0 v_0)^2 v_0] + 3v_0\\ &+ \frac{8h}{u_0^3}\cos(T + \gamma W(T)).\end{aligned} \tag{13}$$

The general solution of Eq. (12) can be written as

$$v_0(T_0,T_1) = \frac{a}{2}\mathrm{e}^{\mathrm{i}(T_0+\varphi)} + \text{c.c.}, \tag{14}$$

where c.c. represents the complex conjugate of its preceding terms, and a and φ are functions of the slow time scale. Eq. (13) then becomes

$$\begin{aligned}D_0^2 v_1 + v_1 = \mathrm{e}^{\mathrm{i}(T_0+\varphi)}\Bigg[&\frac{\sigma a}{2} - \mathrm{i}a' + a\varphi' - \mathrm{i}\mu a - \frac{3a^3}{2}\left(\alpha_1 + \frac{1}{3}\Omega^2\alpha_2\right)\\ &+ \frac{3}{2}a + \frac{4h}{u_0^3}\mathrm{e}^{\mathrm{i}(\gamma W(T_1)-\varphi)}\Bigg]\\ &- \frac{a^3}{2}(\alpha_1 - \Omega^2\alpha_2)\mathrm{e}^{3\mathrm{i}(T_0+\varphi)} + \text{c.c.},\end{aligned} \tag{15}$$

where prime stands for the derivative with respect to T_1. In order to eliminate secular terms, it is required that a and φ vary in the slow time scale according to

$$a' = -\mu a - \frac{4h}{u_0^3}\sin(\varphi - \gamma W(T_1)),$$
$$a\varphi' = -\frac{\sigma a}{2} - \frac{3}{2}a(1 - \alpha a^2) - \frac{4h}{u_0^3}\cos(\varphi - \gamma W(T_1)), \tag{16}$$

where $\alpha = \alpha_1 + \frac{1}{3}\Omega^2\alpha_2$ Once a and η are obtained, the first order uniform expansion for the steady state solution of Eq. (1) is given by

$$u = a(\varepsilon T)\cos(T + \varphi(\varepsilon T)) + O(\varepsilon).$$

4. Steady state solutions and their stability

The response of system (16) for $\gamma = 0$ is determined firstly. In this case Eq. (16) can be written as

$$a' = -\mu a - \frac{4h}{u_0^3}\sin\varphi,$$
$$a\varphi' = -\frac{\sigma a}{2} - \frac{3}{2}a(1 - \alpha a^2) - \frac{4h}{u_0^3}\cos\varphi. \tag{17}$$

Since the steady state value of a is $a = a_0 = 1$, by letting $a' = 0, \varphi' = 0, a = a_0 = 1$ in Eq. (17) the steady state solutions can be found. This leads to the following result:

$$\mu = -\frac{4h}{u_0^3}\sin\varphi,$$
$$\frac{\sigma}{2} + \frac{3}{2}(1 - \alpha) = -\frac{4h}{u_0^3}\cos\varphi. \tag{18}$$

Eliminating $\sin\varphi$ and $\cos\varphi$ from Eq. (18) yields the frequency response relation

$$\sigma = -3(1-\alpha) \pm 2\sqrt{\left(\frac{4h}{u_0^3}\right)^2 - \mu^2}.$$

In terms of the actual excitation frequency Ω, the original parameter $\bar{\varepsilon}$ and the original damping parameter β, the above equation can be written as follows:

$$\Omega^2 = \left(1 + \frac{3}{4}\bar{\varepsilon}u_0^2\right)\left[1 - 3(1-\alpha)\varepsilon \pm \frac{8\varepsilon}{u_0^2}\sqrt{\left(\frac{h}{u_0}\right)^2 - (\beta\Omega)^2}\right]. \tag{19}$$

The steady state solution of Eq. (15) is found to be

$$v_1 = \tfrac{1}{16}(\alpha_1 - \Omega^2\alpha_2)e^{3i(T_0+\varphi)} + \text{c.c.}$$

Then the steady state solution for the original variable u is

$$u(t) = u_0\cos(T+\varphi) + \frac{\varepsilon u_0}{8}(\alpha_1 - \Omega^2\alpha_2)\cos 3(T+\varphi) + O(\varepsilon^2). \tag{20}$$

The local stability of the steady state response may be checked in the usual way. Suppose that

$$a = a_0 + a_1, \quad \varphi = \varphi_0 + \varphi_1, \tag{21}$$

where $a_0 = 1$ and φ_0 are the steady state value and a_1 and φ_1 are arbitrarily small deviations from these values. Substituting Eqs. (21) into Eq. (17) and neglecting the nontrivial terms, one obtains the linearized variation equations for a_0,φ_0

$$\begin{bmatrix} a_1' \\ \varphi_1' \end{bmatrix} = \begin{bmatrix} -\mu & \dfrac{\sigma + 3(1-\alpha)}{2} \\ -\dfrac{\sigma + 3(1-3\alpha)}{2} & -\mu \end{bmatrix} \begin{bmatrix} a_1 \\ \varphi_1 \end{bmatrix}. \tag{22}$$

The eigenvalues of the coefficient matrix in Eq. (22) are

$$\lambda_{1,2} = -\mu \pm \tfrac{1}{2}\sqrt{[\sigma + 3(1-\alpha)][\sigma + 3(1-3\alpha)]}. \tag{23}$$

Therefore, the steady state response is locally stable if and only if

$$\mu^2 > \tfrac{1}{4}[\sigma + 3(1-\alpha)][\sigma + 3(1-3\alpha)]. \tag{24}$$

The unstable region is obtained in a similar way for weak non-linear cases [23] and can be interpreted in the same way. As long as there exist three different steady state solutions for a certain excitation frequency Ω, the solution with intermediate amplitude must be unstable.

Next step is to determine the effect of the noise on the deterministic steady state motion, for $\gamma \neq 0$. To this end, let $\eta = \varphi - \gamma W(T_1)$, then Eq. (16) can be rewritten as

$$\begin{aligned} a' &= -\mu a - \frac{4h}{u_0^3}\sin\eta, \\ a\eta' &= -\frac{\sigma a}{2} - \frac{3}{2}a(1-\alpha a^2) - \frac{4h}{u_0^3}\cos\eta + a\gamma W'(T_1). \end{aligned} \tag{25}$$

However, it is difficult to solve Eq. (25) exactly, so we have to make some approximation. In the case when γ is small, i.e., $\xi(t)$ is a narrowband random process, the perturbation method can be used to solve Eq. (25). Let

$$a = a_0 + a_1, \quad \eta = \varphi_0 + \eta_1, \tag{26}$$

where $a_0 = 1$ and φ_0 are defined by Eq. (18)and a_1, η_1 are small terms. Substituting the above equations into Eq. (25), and neglecting the non-linear terms, we obtain the linearized equations as

$$\begin{bmatrix} a_1' \\ \eta_1' \end{bmatrix} = \begin{bmatrix} -\mu & \dfrac{\sigma + 3(1-\alpha)}{2} \\ -\dfrac{\sigma + 3(1-3\alpha)}{2} & -\mu \end{bmatrix} \begin{bmatrix} a_1 \\ \eta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma W'(T_1) \end{bmatrix}. \tag{27}$$

Eq. (27) can be written as the following Ito equations:

$$\begin{aligned} \mathrm{d}a_1 &= \left[-\mu a_1 + \frac{\sigma + 3(1-\alpha)}{2} \eta_1 \right] \mathrm{d}T_1, \\ \mathrm{d}\eta_1 &= \left[-\frac{\sigma + 3(1-3\alpha)}{2} a_1 - \mu \eta_1 \right] \mathrm{d}T_1 + \gamma \,\mathrm{d}W(T_1). \end{aligned} \tag{28}$$

$E[a_1]$ and $E[a_1^2]$ can be obtained by the moment method [25]. For the steady state moments, one has

$$\frac{\mathrm{d}E[a_1]}{\mathrm{d}T_1} = \frac{\mathrm{d}E[\eta_1]}{\mathrm{d}T_1} = 0.$$

Taking expectation on both sides of Eq. (28), one obtains

$$Ea_1 = E\eta_1 = 0. \tag{29}$$

According to Ito's rule, the second order steady state moments, $Ea_1^2, Ea_1\eta_1$ and $E\eta_1^2$ satisfy the following equations:

$$\begin{aligned} \frac{\mathrm{d}Ea_1^2}{\mathrm{d}T_1} &= -2\mu Ea_1^2 + [\sigma + 3(1-\alpha)]Ea_1\eta_1, \\ \frac{\mathrm{d}Ea_1\eta_1}{\mathrm{d}T_1} &= -\frac{\sigma + 3(1-3\alpha)}{2} Ea_1^2 - 2\mu Ea_1\eta_1 + \frac{\sigma + 3(1-\alpha)}{2} E\eta_1^2, \\ \frac{\mathrm{d}E\eta_1^2}{\mathrm{d}T_1} &= -[\sigma + 3(1-3\alpha)]Ea_1\eta_1 - 2\mu E\eta_1^2 + \gamma^2. \end{aligned} \tag{30}$$

For the steady state moments, we have

$$\frac{\mathrm{d}Ea_1^2}{\mathrm{d}T_1} = \frac{\mathrm{d}Ea_1\eta_1}{\mathrm{d}T_1} = \frac{\mathrm{d}E\eta_1^2}{\mathrm{d}T_1} = 0.$$

Using the above equations and Eq. (30), one obtains

$$\begin{aligned} Ea_1^2 &= \frac{[\sigma+3(1-\alpha)]^2}{[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+4\mu^2}\frac{\gamma^2}{8\mu}, \\ Ea_1\eta_1 &= \frac{\sigma+3(1-\alpha)}{[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+4\mu^2}\frac{\gamma^2}{2}, \\ E\eta_1^2 &= \frac{[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+8\mu^2}{[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+4\mu^2}\frac{\gamma^2}{4\mu}. \end{aligned} \tag{31}$$

From Eqs. (31), the necessary conditions for existence of the second order moments of the response are $Ea_1^2 \geqslant 0, \quad E\eta_1^2 \geqslant 0$, i.e.,

$$[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+4\mu^2 > 0. \tag{32}$$

While the necessary and sufficient condition for the second order moments of the response being stable is that the coefficient matrix of Eq. (30)

$$\begin{bmatrix} -2\mu & \sigma+3(1-\alpha) & 0 \\ -\dfrac{\sigma+3(1-3\alpha)}{2} & -2\mu & \dfrac{\sigma+3(1-\alpha)}{2} \\ 0 & -[\sigma+3(1-3\alpha)] & -2\mu \end{bmatrix}$$

must be negative definite. According to Hurwitz rule, the second order moments of the response is stable if and only if

$$[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+\mu^2 > 0. \tag{33}$$

Condition (33) shows that not all the branches given by Eqs. (19) and (31) are stable. If there are three branches, only the top and the bottom ones are stable and realizable, and the jump is the occasionally switching between these two stable branches. However, if γ is small enough, the random noise $\gamma\dot{W}(t)$ is not yet able to change the stability of these branches, i.e., there will be three stationary displacement variances given by Eqs. (19) and (31) and among them only the largest and the smallest ones are stable and realizable.

Combining Eqs. (26), (29) and (31), one obtains

$$Ea = a_0, \quad Ea^2 = a_0^2 + Ea_1^2 = a_0^2 + \frac{\sigma+3(1-\alpha)}{[\sigma+3(1-\alpha)][\sigma+3(1-3\alpha)]+4\mu^2}\frac{\gamma^2}{8\mu}. \tag{34}$$

5. Numerical simulation

For the method of numerical simulation, readers are referred to Zhu [25] and Shinozuka [26,27]. Eq. (2) can be written as the following equations:

$$\begin{aligned} &\xi(t) = h\cos(\phi(t)), \\ &\dot{\phi}(t) = \Omega + \gamma\zeta(t), \quad \zeta(t) = \dot{W}(t). \end{aligned} \tag{35}$$

The formal derivative $\xi(t)$ of the unit Wiener process is Gaussian white noise, which has a uniform power spectrum and is physically unrealized. However, for numerical simulation in this paper, the power spectrum of $\xi(t)$ is taken as

$$S_{\zeta}(\omega) = \begin{cases} 1, & 0<\omega\leq 2\Omega, \\ 0, & \omega > 2\Omega. \end{cases} \tag{36}$$

For numerical simulation it is more convenient to use the pseudo-random signal given by [25]

$$\zeta(t) = \sqrt{\frac{4\Omega}{N}} \sum_{k=1}^{N} \cos\left[\frac{\Omega}{N}(2k-1)t + \varphi_k\right], \tag{37}$$

where φ_k's are mutually independent and uniformly distributed in $(0, 2\pi]$, and N is a large integer number.

By the center limit theorem, it can be proved [25] that when $N \to \infty$, the random process $\xi(t)$ given by Eq. (37) will converge to an ergodic Gaussian stationary process with the same correlation function and spectrum density given by Eq. (36) as that of the expected process.

In the numerical simulation, the parameters in systems (1) and (37) are chosen as follows:

$$\alpha = 1.0, \quad N = 1000, \quad \beta = 0.1, \quad h = 5.0.$$

The governing equation (1) is numerically integrated by the fourth order Runge–Kutta algorithm, and the numerical results are shown in Figs. 1–5. When $\gamma = 0$, $\bar{\varepsilon} = 0.1$, the variations of the steady state response u_0 along with Ω are shown in Fig. 1, and the theoretical results given by Eq. (19) are also shown in Fig. 1 for comparison. When $\gamma = 0$, $\bar{\varepsilon} = 1.0$, the results are shown in Fig. 2. Both Figs. 1 and 2 show that the deterministic response predicted by the modified method of multiple scales is in good agreement with that obtained by numerical integration. Fig. 1 is for the weakly non-linear case when $\bar{\varepsilon} = 0.1$, while Fig. 2 is for the strongly non-linear case when $\bar{\varepsilon} = 1.0$.

Next is to determine the effect of random noise $\gamma \dot{W}(t)$ on the primary resonance. When $\bar{\varepsilon} = 1.0, \Omega = 4.0, \gamma = 0.01$, for different initial conditions, the numerical results for Eq. (1) are shown in

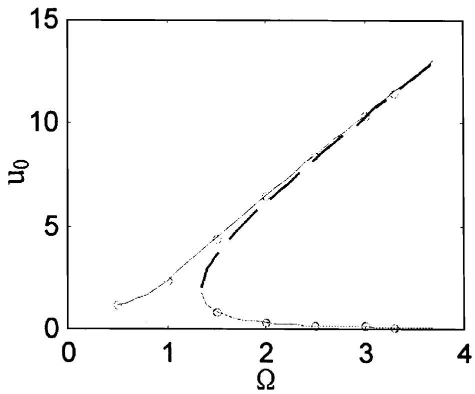

Fig. 1. Frequency response of system (1) ($\bar{\varepsilon} = 0.1$): —, stable solution; - - -, unstable solution; ○ ○ ○, numerical solution.

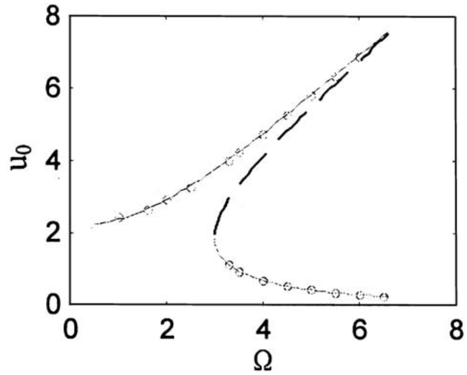

Fig. 2. Frequency response of system (1) ($\bar{\varepsilon} = 1.0$): —, stable solution; - - -, unstable solution; ○ ○ ○, numerical solution.

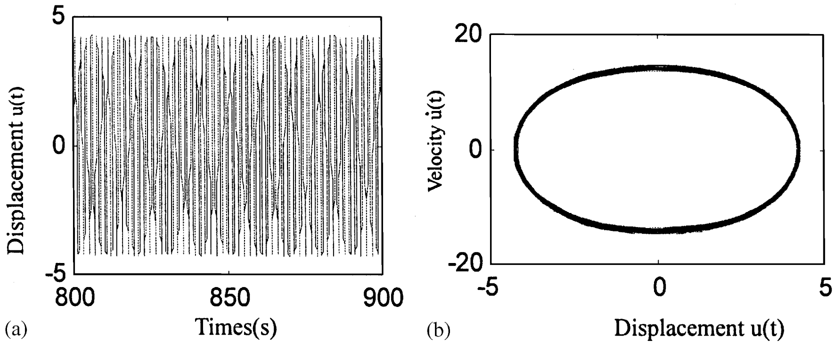

Fig. 3. Numerical results of Eq. (1) ($u(0) = -10.1$, $\dot{u}(0) = -5.5$): (a) Time history of $u(t)$ and (b) phase plot.

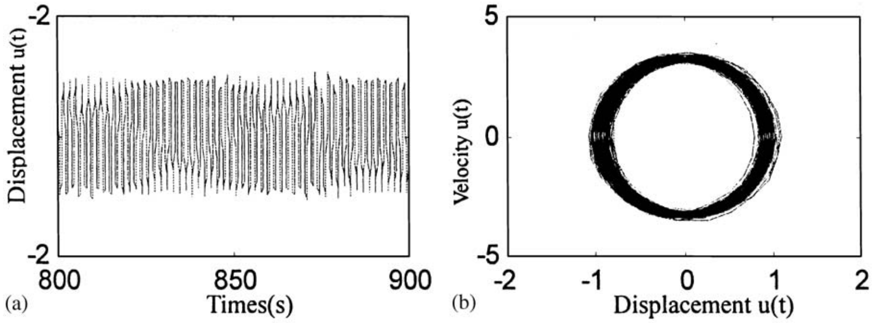

Fig. 4. Numerical results of Eq. (1) ($u(0) = -0.1$, $\dot{u}(0) = -0.5$): (a) Time history of $u(t)$ and (b) phase plot.

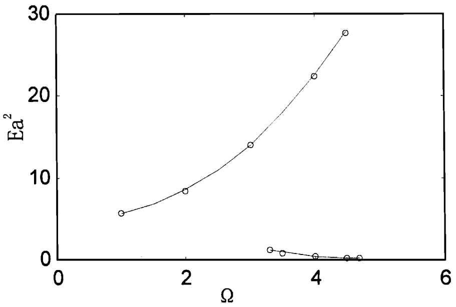

Fig. 5. Frequency response of system (1) ($\bar{\varepsilon} = 1.0$): —, theoretical solution; ○ ○ ○, numerical solution.

Figs. 3 and 4. The initial conditions are $u(0) = -10.1, \dot{u}(0) = -5.5$ for Fig. 3, and $u(0) = -0.1, \dot{u}(0) = -0.5$ for Fig. 4.

Figs. 3 and 4 show that when γ is small enough, in some parameter range of Ω, the stationary variances of the displacement response of system (1) may be different for different initial values. The random noise $\gamma \dot{W}(t)$ may change the steady state response of system (1) from a limit cycle to a diffused limit cycle. Further numerical simulation shows that when the random noise intensity γ increases, the width of the diffused limit cycle increases, too.

When $\bar{\varepsilon} = 1.0$, $\gamma = 0.01$, numerical results of the variations of the steady state response with Ω are shown in Fig. 5, and the theoretical results given by Eq. (34) are also shown there for comparison.

6. Conclusions and discussion

Exact solutions of random response problems of non-linear systems up to now are only available for a few problems. Thus, people have to resort various approximate methods to deal with the remaining problems. In fact, the task is equally difficult even for deterministic response problems of non-linear systems. Hence, a number of approximate methods have been developed and widely used in the analysis of deterministic response problems of non-linear systems. For the state-of-art in this respect, readers are referred to Nayfeh and Mook [28], Hagedorn [29], and Nayfeh [23,30]. Actually, some of the approximate methods for deterministic response problems of non-linear systems can be extended to random response problems. For example, the method of multiple scales has been extended to non-linear systems under random external excitations by Rajan and Davies [19], and Nayfeh and Serhan [20], and to non-linear systems under random parametric excitations by the present authors [21,22]. In recent years, several researchers successfully extended the classical perturbation methods to deterministic response problems in certain strongly non-linear systems (see Refs. [31–34], to mention a few). This kind of modified methods may be also extended to random response problems of non-linear systems, as shown in this paper.

By the modified perturbation technique and multiple scale method, the principle resonance of a van der Pol–Duffing system is reanalyzed for the large non-linearity case. Our theoretical analyses and numerical simulations show that under a narrowband random excitation, given by Eq. (2), when the random noise intensity γ is small enough, in some parameter range of Ω , the stationary variances of the response of system (1) may be different for different initial conditions. The random noise $\gamma \dot{W}(t)$ may change the steady state response of system (1) from a limit cycle to a diffused limit cycle. When the random noise intensity γ increases, the width of the diffused limit cycle will increase too. These analytical results are in accordance with the following physical instinct. When γ is small enough, the deterministic harmonic term $h \cos \Omega t$ will still play a decisive role in the response of system (1), so that phenomenon of multiple-valued steady state responses can be observed within some parameter range. Nayfeh and Serhan [20] also observed the similar phenomenon in the response of a Duffing–Rayleigh oscillator under combined deterministic and random excitations.

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