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Letter to the Editor

On the vibration analysis of rectangular clamped plates using the virtual work principle

Jorge P. Arenas*

Institute of Acoustics, Universidad Austral de Chile, P.O. Box 567, Valdivia, Chile

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1. Introduction

Considerable attention has been paid to the solution of the vibration problems of rectangular plates. In recent years with the practical application of active sound and vibration control, several studies have been devoted to plate response subject to different kinds of excitation and with various boundary conditions. The solution for simply supported plates is easy to obtain. It is much more difficult to obtain solutions for other boundary conditions. In addition, there is a great motivation to develop techniques for rapid, global inspection of vibrating structures. Classical methods to find the surface response of a plate with complex boundary conditions include the superposition and Ritz methods using trial polynomials and trigonometric functions. A large amount of information has been compiled by Leissa [1,2]. These methods require the solution of simultaneous equations or high order matrices, making the calculations of sound power radiated by vibrating structures even more difficult. An approach using the virtual work principle has been extensively used by Sung and co-workers [3–5]. This approach provides an easier methodology for calculating the surface response of a plate. It appears that the fundamentals of the method were first introduced by Vlasov [6]. A set of valuable references can be found in the comments made by Laura [7]. However, there is a relative scarcity of information in the literature for rectangular, as opposed to square, fully clamped plates. It can be concluded that an efficient method to predict the low-frequency sound radiated from a vibrating structure will require a computationally fast method to solve the vibration part of the problem combined with a method that, when possible, avoids the integration of the sound pressure field [8].

The final aim of this letter is to summarize the application of the virtual work principle to fully clamped plates, reporting the results for natural frequencies for plates of arbitrary aspect ratio and an estimate for the modal density, results that are not reported in the previous works. This can be useful in evaluating the accuracy of the method by comparing with previous numerical and experimental results reported in the literature.

*Tel.: 56-63-221-012; fax: 56-63-221-013.
E-mail address: jparenas@uach.cl (J.P. Arenas).

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2. Theory

For thin plate theory the wave equation for transverse vibration of an isotropic, undamped plate subjected to a concentrated load at point (x', y') is [2]

$$B\nabla^4\xi + \rho h \frac{\partial^2 \xi}{\partial t^2} = F(t)\delta(x - x')\delta(y - y'), \tag{1}$$

where $\xi(x, y, t)$ is the instantaneous transverse displacement, ρ is the density of the plate, h is the plate thickness, $B = Eh^3/12(1 - \nu^2)$ is the stiffness of the plate in bending, ν is the Poisson ratio, E is the modulus of elasticity (Young's modulus), $F(t)$ is the dynamic amplitude of the external force referring to unit surface area of the plate, and $\delta(\cdot)$ is the delta function.

Now, let a rectangular plate be defined in the region $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$ and consider the clamped–clamped boundary condition, i.e., $\xi = 0$ and $\partial\xi/\partial n = 0$, where n represents the normal direction from the clamped edges.

Application of the virtual work principle to Eq. (1), implies that the steady state amplitude response $\xi_0(x, y)$ of the plate subjected to a harmonic point force of amplitude F_0 and frequency ω can be expressed as [3]

$$\xi_0(x, y) = F_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Psi_{mn}(x, y)\Psi_{mn}(x', y')}{B(I_1 I_2 + 2I_3 I_4 + I_5 I_6) - \rho_s \omega^2 I_2 I_6}, \tag{2}$$

where the shape functions are decomposed in the form of the product

$$\Psi_{mn}(x, y) = \vartheta_m(x)\zeta_n(y), \tag{3}$$

ρ_s is the surface density of the plate, and

$$\begin{aligned} I_1 &= \int_0^a \vartheta_m'''' \vartheta_m \,\mathrm{d}x, \quad I_2 = \int_0^b \zeta_n^2 \,\mathrm{d}y, \quad I_3 = \int_0^a \vartheta_m'' \vartheta_m \,\mathrm{d}x, \\ I_4 &= \int_0^b \zeta_n'' \zeta_n \,\mathrm{d}y, \quad I_5 = \int_0^b \zeta_n'''' \zeta_n \,\mathrm{d}y, \quad I_6 = \int_0^a \vartheta_m^2 \,\mathrm{d}x. \end{aligned} \tag{4}$$

The eigenfunctions $\vartheta_m(x)$ and $\zeta_n(y)$ can be arbitrarily chosen as long as they are *quasi-orthogonal* and both of them satisfy the boundary condition. Eq. (2) shows that the natural frequencies are given by

$$\omega_{mn} = \sqrt{\frac{B}{\rho_s}} \sqrt{\frac{I_1 I_2 + 2I_3 I_4 + I_5 I_6}{I_2 I_6}}. \tag{5}$$

2.1. Solution for a clamped–clamped rectangular plate

Sung and co-workers [3–5] have used an approach for the calculation of the vibration distribution and natural frequencies of a clamped–clamped plate. For this, they define the functions $\mathscr{J}(s) = \cosh(s) - \cos(s)$ and $\mathscr{H}(s) = \sinh(s) - \sin(s)$. Then, the eigenfunctions $\vartheta_m(x)$ and

$\zeta_n(y)$ can be defined as

$$\vartheta_m(x) = \mathscr{J}(\beta_m x/a) - \frac{\mathscr{J}(\beta_m)}{\mathscr{H}(\beta_m)}\mathscr{H}(\beta_m x/a),$$
$$\zeta_n(y) = \mathscr{J}(\beta_n y/b) - \frac{\mathscr{J}(\beta_n)}{\mathscr{H}(\beta_n)}\mathscr{H}(\beta_n y/b), \tag{6}$$

where β_m and β_n are the roots for the equation $\cosh(\beta)\cos(\beta) = 1$. It is noticed that for large values of the integer i then $\beta_i \to (2i+1)\pi/2$. In an earlier work Crocker [9] used a similar approach.

Sung and co-workers did not present explicit formulae for a clamped–clamped plate. Therefore, after integration of Eq. (4) it is useful to define

$$\begin{aligned}\mathscr{Q}_i = &\tfrac{1}{4}(1+\mathscr{D}_i^2)\sinh(2\beta_i) + \sinh(\beta_i)[2\mathscr{D}_i\sin(\beta_i) - (1-\mathscr{D}_i^2)\cos(\beta_i)]\\ &- (1+\mathscr{D}_i^2)\sin(\beta_i)\cosh(\beta_i) + \tfrac{1}{2}(1-\mathscr{D}_i^2)\sin(\beta_i)\cos(\beta_i) + \beta_i\\ &- \tfrac{1}{2}\mathscr{D}_i[1+\cosh(2\beta_i)] + \mathscr{D}_i\cos^2(\beta_i),\end{aligned} \tag{7}$$

and

$$\begin{aligned}\mathscr{R}_i = &\tfrac{1}{4}(1+\mathscr{D}_i^2)\sinh(2\beta_i) - \tfrac{1}{2}\mathscr{D}_i\cosh(2\beta_i) - \tfrac{1}{2}(1-\mathscr{D}_i^2)\sin(\beta_i)\cos(\beta_i)\\ &- \mathscr{D}_i\cos^2(\beta_i) - \mathscr{D}_i^2\beta_i + \tfrac{3}{2}\mathscr{D}_i,\end{aligned} \tag{8}$$

where $\mathscr{D}_i = \mathscr{J}(\beta_i)/\mathscr{H}(\beta_i)$. Now, the following products can be calculated:

$$I_2 I_6 = \frac{ab}{\beta_m\beta_n}\mathscr{Q}_m\mathscr{Q}_n \quad \text{and} \quad I_3 I_4 = \frac{\beta_m\beta_n}{ab}\mathscr{R}_m\mathscr{R}_n. \tag{9}$$

It is observed that $I_1 = I_6(\beta_m/a)^4$ and $I_5 = I_2(\beta_n/b)^4$. Therefore, the natural frequencies for a rectangular clamped–clamped plate are given by

$$\omega_{mn} = \sqrt{\frac{B}{\rho_s}}\sqrt{\left(\frac{\beta_m}{a}\right)^4 + \left(\frac{\beta_n}{b}\right)^4 + 2\left(\frac{\beta_m\beta_n}{ab}\right)^2\frac{\mathscr{R}_m\mathscr{R}_n}{\mathscr{Q}_m\mathscr{Q}_n}}. \tag{10}$$

2.2. Natural frequencies and modal density

Numerical results for the dimensionless frequency parameter $\lambda_{mn} = \omega_{mn}a^2\sqrt{\rho_s/B}$ were computed using Eq. (10) for rectangular plates of arbitrary a/b ratio. These results are summarized in Table 1. Double-precision arithmetic was used in the computations. Comparing the values presented in Table 1 with the numerical results presented in the classical literature [1,2,10,11] it is observed that they compare favorably with more sophisticated and accurate methods. Importantly, comparing the values shown in Table 1 with the available experimental results presented by other authors [12], it can be seen that the differences in the calculations for the natural frequencies using Eq. (10) do not exceed 2%. In order to obtain better approximations it could be suggested to apply some weighting constant to the overestimated values given by Eq. (10). However, it has to be noticed the limitations of the experimental methods. Most of the classical theories have been developed assuming light fluid loading, so that the plate response is not affected by the surrounding environment, which acts as added mass and also provides radiation damping. Loading by fluid significantly lowers the natural frequencies of flat plates, the

Table 1
Values of the dimensionless frequency parameter $\lambda_{mn} = \omega_{mn} a^2 \sqrt{\rho_s / B}$ for a fully clamped rectangular plate of arbitrary a/b ratio

m	n	Aspect ratio a/b									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	1	22.4419	22.6599	23.0621	23.7026	24.6480	25.9694	27.7322	29.9888	32.7747	36.1087
	2	22.6335	23.4941	25.1664	27.9146	31.9618	37.4354	44.3734	52.7604	62.5608	73.7372
	3	22.9426	24.9258	28.9480	35.5549	44.9729	57.1935	72.1298	89.6963	109.8289	132.4831
	4	23.3833	27.0804	34.6999	46.8896	63.6595	84.8361	110.2715	139.8698	173.5726	211.3440
	5	23.9674	30.0501	42.5109	61.8203	87.7238	119.9624	158.3821	202.8982	253.4637	310.0511
	6	24.7592	34.0446	52.5793	80.4547	117.2381	162.6589	216.5817	278.9382	349.6918	428.8221
2	1	61.7650	62.0456	62.5265	63.2269	64.1724	65.3936	66.9248	68.8021	71.0616	73.7372
	2	62.0188	63.0815	64.9312	67.6720	71.4252	76.3107	82.4316	89.8657	98.6632	108.8499
	3	62.4187	64.7402	68.8605	75.0681	83.6327	94.7485	108.5206	124.9831	144.1270	165.9226
	4	62.9727	67.0788	74.4940	85.7589	101.2462	121.1171	145.3876	174.0072	206.9086	244.0295
	5	63.6835	70.1336	81.9379	99.8784	124.3104	155.2695	192.6628	236.3780	286.3212	342.4226
	6	64.6279	74.1964	91.7307	118.1030	153.5158	197.8604	250.9752	312.7283	383.0274	461.8103
3	1	121.0042	121.3086	121.8224	122.5554	123.5205	124.7341	126.2152	127.9850	130.0663	132.4831
	2	121.2811	122.4258	124.3715	127.1710	130.8917	135.6092	141.4014	148.3422	156.4978	165.9226
	3	121.7158	124.1918	128.4400	134.6185	142.9067	153.4771	166.4747	182.0060	200.1383	220.9063
	4	122.3153	126.6459	134.1488	145.1578	159.9983	178.9224	202.0875	229.5671	261.3756	297.4941
	5	123.0803	129.8041	141.5611	158.9179	182.3292	212.0596	248.2086	290.7734	339.7038	394.9359
	6	124.0938	133.9883	151.3420	176.9099	211.1674	254.2950	306.2926	367.0857	436.5845	514.7084
4	1	199.9652	200.2835	200.8179	201.5741	202.5600	203.7856	205.2626	207.0045	209.0262	211.3440
	2	200.2554	201.4498	203.4622	206.3242	210.0769	214.7693	220.4553	227.1909	235.0315	244.0295
	3	200.7104	203.2851	207.6461	213.8893	222.1322	232.5018	245.1219	260.1031	277.5372	297.4941
	4	201.3371	205.8228	213.4639	224.4720	239.0839	257.5238	279.9770	306.5780	337.4113	372.5196
	5	202.1353	209.0700	220.9513	238.1619	261.0822	290.0181	325.1731	366.6544	414.4982	468.6958
	6	203.1922	213.3671	230.8357	256.1429	289.7556	331.9795	382.9631	442.7434	511.2958	588.5685
5	1	298.6644	298.9917	299.5398	300.3124	301.3148	302.5537	304.0370	305.7739	307.7749	310.0511
	2	298.9631	300.1900	302.2491	305.1607	308.9526	313.6582	319.3159	325.9670	333.6547	342.4226
	3	299.4313	302.0723	306.5187	312.8339	321.0987	331.4054	343.8511	358.5310	375.5334	394.9359
	4	300.0756	304.6690	312.4302	323.5018	338.0548	356.2680	378.3097	404.3243	434.4257	468.6958
	5	300.8957	307.9831	320.0039	337.2234	359.9359	388.4169	422.8922	463.5243	510.4145	563.6139
	6	301.9814	312.3675	330.0054	355.2833	388.5900	430.2432	480.4609	539.3654	607.0060	683.3838
6	1	417.1048	417.4475	418.0203	418.8259	419.8681	421.1514	422.6815	424.4650	426.5091	428.8221
	2	417.4177	418.7013	420.8499	423.8775	427.8025	432.6482	438.4412	445.2111	452.9898	461.8103
	3	417.9079	420.6682	425.2976	431.8375	440.3423	450.8759	463.5080	478.3108	495.3552	514.7084
	4	418.5824	423.3778	431.4369	442.8520	457.7387	476.2243	498.4374	524.4983	554.5129	588.5685
	5	419.4403	426.8299	439.2756	456.9493	480.0536	508.7947	543.3609	583.9084	630.5556	683.3838
	6	420.5760	431.3948	449.6211	475.5002	509.2995	551.2656	601.5970	660.4348	727.8661	803.9348

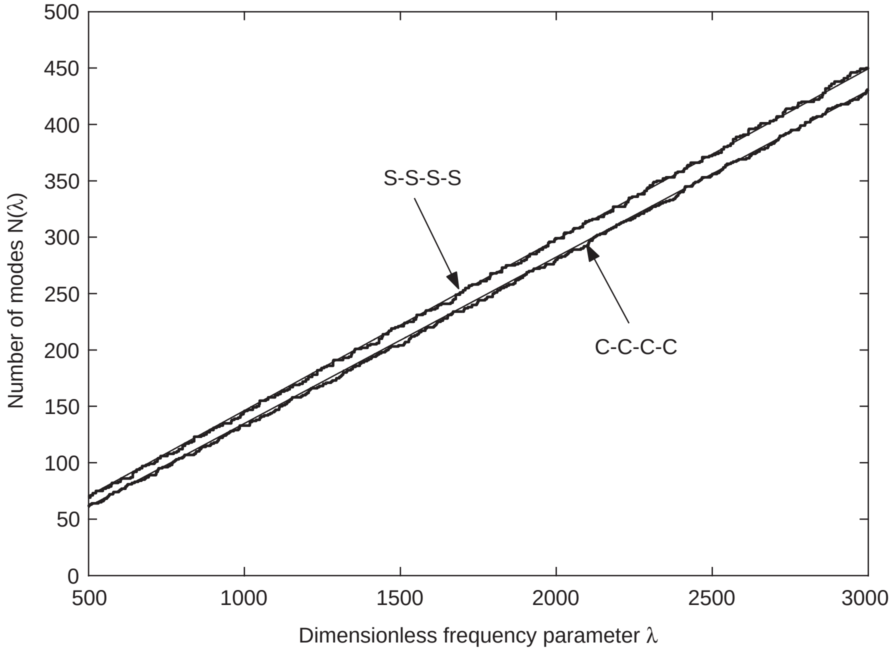

Fig. 1. Number of modes as a function of the dimensionless frequency parameter λ for both clamped and simply supported plate of $a/b = 0.5$.

effect decreasing with increasing mode order. In addition, differences between the numerical and experimental approaches are due to imperfections in the experimental fixture, the damping which couples the modes (non-proportional damping), and some non-linearities, among others.

On the other hand, the modal density is often used to study the sound radiation from a plate, in particular when statistical methods are used [13]. As an example, Fig. 1 shows the results for the cumulative mode count, $N(\lambda)$, as a function of the dimensionless frequency parameter for a fully clamped plate of aspect ratio $a/b = 0.5$, computed using Eq. (10). The results for an equivalent simply supported plate are plotted for comparison. $N(\lambda)$ represents the number of modes which can be excited in the range from zero up to λ. It can be observed in Fig. 1 that the number of modes for the clamped plate can be approximated by

$$N(\lambda) \approx \frac{b}{4\pi a}\lambda + C, \tag{11}$$

where C is a real number that depends on the aspect ratio. The average modal density $n(\lambda)$, which is the number of modes that can be excited in a narrow frequency band, is obtained from the derivative of Eq. (11). Then

$$n(\lambda) = \frac{\mathrm{d}N}{\mathrm{d}\lambda} \approx \frac{b}{4\pi a}. \tag{12}$$

Eq. (12) is exactly the expression for the modal density of a simply supported plate, which is independent of frequency. Eq. (12) confirms that the modal density depends not too strongly on the boundary conditions. However, Eq. (11) is a good approximation for plates of not too high aspect ratio and for $\lambda > 2\lambda_{11}$.

2.3. Velocity response to multi-point force excitation of the plate

Most of the practical applications which use active control of vibration are developed by placing piezoceramic actuators on the surface of the plate and by applying point forces with electromagnetic shakers. In such cases, a matrix equation is very useful in predicting the velocity response of the plate to these forces. In addition, the response of the plate to a point moment can be obtained since a point moment can be treated as two point forces F_1 at (x_1, y_1) and F_2 at (x_2, y_2) with the same magnitude separated by a small distance $\Delta = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ but oriented in opposite directions [4]. Multi-point excitation can be found also in the case of a machine installed on a plate with several mounting positions [14].

For multi-point force excitation (several forces of amplitude $F_1, F_2, \ldots, F_k$ applied to a set of corresponding co-ordinates $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$) on the plate, the total displacement of the plate is obtained using the superposition of the responses induced by each of the forces, since the system is assumed to be linear. Then, including the response of modes up to the mode (M, N) the total displacement is

$$\xi_0(x, y) = \frac{1}{\rho_s ab} \sum_{i=1}^{k} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{F_i \Psi_{mn}(x, y) \Psi_{mn}(x_i, y_i)}{\gamma(\omega_{mn}^2 - \omega^2)}, \tag{13}$$

where ω_{mn} is computed using Eq. (10), and γ is

$$\gamma = \frac{1}{ab} \int_0^b \int_0^a \Psi_{mn}^2(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{14}$$

Now, it is useful to write the velocity V on the plate for any co-ordinate point (x, y) in matrix form. Let $\mathbf{F}$ be a $k \times k$ complex diagonal matrix of forces defined as $\mathbf{F} = \mathrm{diag}(F_1, F_2, \ldots, F_k)$. Then, the velocity can be expressed as

$$V(x, y, \omega, t) = \frac{\omega}{2\pi\kappa} \mathrm{trace}(\mathbf{F}\mathbf{V}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{P}) \exp\left(\omega t + \frac{\pi}{2}\right), \tag{15}$$

where $\kappa = 2\pi\rho_s ab\gamma$, $\mathbf{V}$ is a $M \times k$ matrix, $\mathbf{P}$ is a $N \times k$ matrix and $\boldsymbol{\Omega}$ is a $M \times N$ matrix. The entries for the matrices in Eq. (15) are

$$V_{mk} = \vartheta_m(x)\vartheta_n(x_k), \quad P_{nk} = \zeta_n(y)\zeta_n(y_k), \quad \Omega_{mn} = 4\pi^2/(\omega_{mn}^2 - \omega^2). \tag{16}$$

3. Concluding remarks

The use of the virtual work principle produces good approximations for the natural frequencies and modal density of a fully clamped rectangular plate to within acceptable limits, at least for acoustical requirements. The method summarized in the present letter is very useful for fast calculations of the sound radiation characteristics of fully clamped plates, where the velocity distribution on the plate is needed, since it does not require the solution of simultaneous equations. This is particularly true when the computational cost of formulating and solving the system of equations to predict the sound radiation can become prohibitive. In addition, the method avoids the symmetric eigenvalue problem that results when the Rayleigh–Ritz, or other more sophisticated method is used. Moreover, all the formulae are relatively simple to convert into a computational code when numerical software approaches are used.

The theory may be extended to other boundary conditions simply by selecting quasi-orthogonal shape functions that satisfy the boundary conditions. Further work can be carried out to estimate the sound radiation from free plates and for mixed boundary conditions, by combining the method described in this article with the surface resistance matrix [15]. However, the lack of experimental results for the vibration of rectangular plates of arbitrary aspect ratio still remains.

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