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Letter to the Editor

Natural frequencies of an immersed beam carrying a tip mass with rotatory inertia

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1. Introduction

Beams surrounded by water is the simplest problem modelling fluid–structure interaction (FSI) in offshore and irrigation engineering. Some examples are towers, piles, dams and many research have been conducted in this field. Nagaya [1] and Nagaya and Hai [2] applied elastodynamic theory to solve problems of transient and seismic flexural response of variable cross-section beams with tip inertias and immersed in a fluid, and presented the natural frequencies. Chang and Liu [3] studied the natural frequencies of immersed restrained column subjected to an axial force using transfer matrix method and compared the results with some analytical solutions. Han and Sahglivi [4] studied dynamic response due to wave excitation. Xing et al. [5] derived the eigenvalue equation of the natural vibration of the beam-water system without a tip mass and obtained the exact solutions for each combination of boundary conditions. Calculations showed that for the undisturbed condition at infinity in the water domain, the natural frequencies of the coupled dynamic system are lower than those of the flexible dry beam, indicating that the influence of water on the beam has the effect of an additional mass. Uscilowska and Kołodziej [6] considered an offshore structure having the form of a column with a tip mass partially immersed in a fluid. The effect of added mass on free vibrations in the fluid, the rotatory inertia of the concentrated mass and its eccentricity were all taken into account. Hartnett and Mullarkey [7] outlined a development for linearizing the drag force term in Morison's equation and developed a finite element program for immersed slender members. Zhou et al. [8] used the discrete vortex method incorporating the vortex-in-cell (VIC) technique to study a uniform flow past an elastic circular cylinder. Perov et al. [9] investigated the influence of FSI on vibration modes using the finite element method. Stabel and Ren [10] used different FSI formulations for seismic analysis of fuel storage racks. Yetisir and Weaver [11,12] presented an unsteady theoretical model for fluidelastic instability in an array of tubes in cross-flow. Lever and Rzentkowski [13] presented the dependence of post-stable hysteresis behavior on the number of degrees of freedom of a bundle

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formed by rigid and elastic cylinders. Romberg and Popp [14] considered the influence of isotropic upstream turbulence on the stability behavior of normal and rotated triangular tube arrays of different pitch-to-diameter ratios. Kaye and Maull [15] investigated the response of a flexible cylinder as a function of the ratio of its natural frequency to the wave frequency. Austermann and Popp [16] examined the vibration behavior of one flexibly mounted tube within otherwise fixed bundles with different geometries. Skop and Balasubramanian [17] developed a new twist on an old model for predicting the vortex-excited vibrations of flexible cylindrical structures. Fujarra et al. [18] concerned with experimental study of the vortex-induced vibrations of a flexible cantilever in a fluid flow.

In this study, an Euler–Bernoulli type beam partially immersed in water and carrying a mass at one end is considered. Transverse vibrations of the beam is investigated. The analytical and finite element method are used to calculate natural frequencies. The effects of water height, tip mass and water density are investigated. It is found that an increase in those parameters result in a decrease in frequencies. The rotatory inertia decreases the frequencies sharply than the mass itself.

2. Equations of motion

In many references [1–3,5,6], the equations of motion were given and solved either using transfer matrix methods or some other analytical methods. The equations of motion and boundary conditions in non-dimensional form are as follows:

$$\ddot{w}_1 + w_1^{iv} = 0, \quad \ddot{w}_2 + k^4 w_2^{iv} = 0, \tag{1}$$

$$w_1(0,t) = 0, \quad w_1'(0,t) = 0, \tag{2a}$$

$$w_1(\eta,t) = w_2(\eta,t), \;\; w_1'(\eta,t) = w_2'(\eta,t), \;\; w_1''(\eta,t) = w_2''(\eta,t), \;\; w_1^{iii}(\eta,t) = w_2^{iii}(\eta,t), \tag{2b}$$

$$w_2''(1,t) = -\alpha\varepsilon k^4 \ddot{w}_2(1,t) - (\beta + \alpha\varepsilon^2) k^4 L \ddot{w}_2'(1,t), \quad w_2^{iii}(1,t) = \alpha \ddot{w}_2(1,t) + \alpha\varepsilon L \ddot{w}_2'(1,t), \tag{2c}$$

where dimensions are given by

$$x = \frac{x^*}{L}, \quad w_1 = \frac{w_1^*}{L}, \quad w_2 = \frac{w_2^*}{L}, \quad \eta = \frac{L_1}{L}, \quad \varepsilon = \frac{\varepsilon^*}{L}, \quad t = \frac{t^*}{L^2}\sqrt{\frac{EI}{(\rho_c + \rho_w)A}},$$
$$k^4 = \frac{\rho_c}{\rho_c + \rho_w}, \quad \alpha = \frac{M}{\rho_c A L}, \quad \beta = \frac{J_G}{\rho_c A L^3}. \tag{3}$$

x^* and t^* are the spatial and time variables. w_1^* and w_2^* are the transverse displacements of the beam below and above water level respectively. EI is the flexural rigidity and A is the cross-sectional area of the Euler–Bernoulli beam. ρ_c is beam density and ρ_w is water density. M and J_G are the mass and rotatory inertia with respect to the center of gravity of the tip mass. w_G^* and θ_G^* denote the displacement and slope of the tip mass respectively. Let us assume following functions as the solutions of the equations of motion and boundary conditions

$$w_1(x,t) = A_1 e^{i\omega t} Y_1(x) + cc, \quad w_2(x,t) = A_2 e^{i\omega t} Y_2(x) + cc, \tag{4}$$

where ω, i and cc denote the natural frequency, $\sqrt{-1}$ and complex conjugate. The frequency equation is.

$$
\left|\begin{matrix}
\begin{pmatrix}\cos\lambda\eta\\ -\cosh\lambda\eta\end{pmatrix} & \begin{pmatrix}\sin\lambda\eta\\ -\sinh\lambda\eta\end{pmatrix} & -\cos\lambda k\eta & -\sin\lambda k\eta & -\cosh\lambda k\eta & -\sinh\lambda k\eta \\
\begin{pmatrix}-\sin\lambda\eta\\ -\sinh\lambda\eta\end{pmatrix} & \begin{pmatrix}\cos\lambda\eta\\ -\cosh\lambda\eta\end{pmatrix} & k\sin\lambda k\eta & -k\cos\lambda k\eta & -k\sinh\lambda k\eta & -k\cosh\lambda k\eta \\
\begin{pmatrix}-\cos\lambda\eta\\ -\cosh\lambda\eta\end{pmatrix} & \begin{pmatrix}-\sin\lambda\eta\\ -\sinh\lambda\eta\end{pmatrix} & k^2\cos\lambda k\eta & k^2\sin\lambda k\eta & -k^2\cosh\lambda k\eta & -k^2\sinh\lambda k\eta \\
\begin{pmatrix}\sin\lambda\eta\\ -\sinh\lambda\eta\end{pmatrix} & \begin{pmatrix}-\cos\lambda\eta\\ -\cosh\lambda\eta\end{pmatrix} & -k^3\sin\lambda k\eta & k^3\cos\lambda k\eta & -k^3\sinh\lambda k\eta & -k^3\cosh\lambda k\eta \\
0 & 0 & \begin{pmatrix}(1-\alpha\varepsilon\lambda^2k^2)\sin\lambda k\\ +\alpha\lambda k\cos\lambda k\end{pmatrix} & \begin{pmatrix}(-1+\alpha\varepsilon\lambda^2k^2)\cos\lambda k\\ +\alpha\lambda k\sin\lambda k\end{pmatrix} & \begin{pmatrix}(1+\alpha\varepsilon\lambda^2k^2)\sinh\lambda k\\ +\alpha\lambda k\cosh\lambda k\end{pmatrix} & \begin{pmatrix}(1+\alpha\varepsilon\lambda^2k^2)\cosh\lambda k\\ +\alpha\lambda k\sinh\lambda k\end{pmatrix} \\
0 & 0 & \begin{pmatrix}-(1+\alpha\varepsilon\lambda^2k^2)\cos\lambda k\\ +(\beta+\alpha\varepsilon^2)\lambda^3k^3\sin\lambda k\end{pmatrix} & \begin{pmatrix}-(1+\alpha\varepsilon\lambda^2k^2)\sin\lambda k\\ -(\beta+\alpha\varepsilon^2)\lambda^3k^3\cos\lambda k\end{pmatrix} & \begin{pmatrix}(1-\alpha\varepsilon\lambda^2k^2)\cosh\lambda k\\ -(\beta+\alpha\varepsilon^2)\lambda^3k^3\sinh\lambda k\end{pmatrix} & \begin{pmatrix}(1-\alpha\varepsilon\lambda^2k^2)\sinh\lambda k\\ -(\beta+\alpha\varepsilon^2)\lambda^3k^3\cosh\lambda k\end{pmatrix}
\end{matrix}\right| = 0, \quad (5)
$$

where $\lambda^4 = \omega^2$. The problem (determinant of the matrix) will be solved for different mass and inertia ratios with an eccentricity, water height and density ratios in the numerical analysis section.

3. Numerical solutions

Numerical solutions of frequency equation and some comparisons will be presented in this section.

Firstly, let us assume $\rho_c = 7850\,\mathrm{kg/m^3}$ and $\rho_w = 1000\,\mathrm{kg/m^3}$ ($\rho_{\mathrm{eff}} = 8850\,\mathrm{kg/m^3}$) ($k = 0.9704672$). The beam diameter is 0.3 m and $E = 2.068 \times 10^{11}\,\mathrm{Pa}$. In Table 2 of Ref. [6], it was mentioned that the difference between the results of Refs. [3] and [6] were due to FEM method. In Table 1, the comparison of eigenvalues of fixed–free column with a tip mass is presented

Table 1
Comparisons of eigenvalues of beam partially immersed in water for different tip mass ($\varepsilon^* = 0$), inertia and water height ratios ($\rho_w = 1000\,\text{kg/m}^3$, $\rho_c = 7850\,\text{kg/m}^3$, eigenvalues are made non-dimensional using ρ_c only)

η	β	α	Analytical			FEM			Chang and Liu [3]			Nagaya [1]		
			λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0	0	0	1.87510	4.69409	7.85476	1.87510	4.69409	7.85476	—	—	—	1.875	4.694	7.854
0	0	0.2	1.61640	4.26706	7.31837	1.61640	4.26706	7.31838	—	—	—	1.616	4.266	7.318
0	0	0.5	1.41996	4.11113	7.19034	1.41996	4.11113	7.19034	—	—	—	1.419	4.111	7.190
0	0	1	1.24792	4.03114	7.13413	1.24792	4.03114	7.13414	1.24791	4.03105	7.13373	1.248	4.031	7.134
0	0	2	1.07620	3.98257	7.10265	1.07620	3.98257	7.10265	1.07619	3.98250	7.10227	1.076	3.982	7.103
0	1	1	0.93161	1.84135	4.90087	0.93161	1.84135	4.90087	0.93161	1.84135	4.90076	—	—	—
0	1	2	0.88570	1.68937	4.82936	0.88570	1.68937	4.82936	0.78792	1.59862	4.82488	—	—	—
0.5	0	1	1.24757	3.98756	7.01921	1.24757	3.98756	7.01921	1.24755	3.98642	7.01864	—	—	—
0.5	0	2	1.07603	3.93989	6.98743	1.07603	3.93989	6.98743	1.07602	3.93877	6.98687	—	—	—
0.5	1	1	0.93156	1.84001	4.82539	0.93156	1.84001	4.82539	0.93156	1.83996	4.82388	—	—	—
0.5	1	2	0.88566	1.68866	4.75514	0.88566	1.68866	4.75514	0.78789	1.59794	4.74919	—	—	—
1	0	0	1.81973	4.55547	7.62280	1.81973	4.55547	7.62280	—	—	—	1.819	4.556	7.622
1	0	0.2	1.58931	4.16343	7.12297	1.58931	4.16343	7.12298	1.58929	4.16326	7.12240	1.589	4.164	7.123
1	0	0.5	1.40565	4.00666	6.99065	1.40565	4.00666	6.99066	—	—	—	1.405	4.007	6.991
1	0	1	1.24038	3.92312	6.93087	1.24038	3.92312	6.93087	1.24037	3.92302	6.93047	1.240	3.923	6.931
1	0	2	1.07259	3.87134	6.89693	1.07259	3.87134	6.89693	1.07259	3.87126	6.89655	1.072	3.871	6.897
1	1	1	0.93025	1.82788	4.77259	0.93025	1.82788	4.77259	0.93025	1.82787	4.77247	—	—	—
1	1	2	0.88456	1.68272	4.69752	0.88456	1.68272	4.69752	0.78734	1.59165	4.69268	—	—	—

considering the time parameter in terms of water density only and the results are compared with Refs. [1,3] for the first three modes. The results calculated in the present study by analytical method and FEM (see Ref. [19,20]) are the same. Increasing the water height, mass and inertia ratios again introduce no considerable error to the solutions in FEM. Also the results are in agreement with Refs. [1,3]. Only in Ref. [3] for $\beta = 1$, $\alpha = 2$, the eigenvalues are different in the first and second mode for all water height ratios. Increasing water level and tip mass decreases the frequencies as presented in Table 1.

In Table 2, the eigenvalues of the fixed–free beam with a tip mass are given for $k = 0.9704572$. The analytical and FEM results are compared with Ref. [6]. The solutions in Ref. [3] are correct, but the differences between two studies [3,6] arise from selection of different non-dimensional time parameters. In Ref. [3], the time parameter was made non-dimensional using water density (also in Table 1). But in the present study and in Ref. [6], it is made using the addition of beam and water densities as defined in Eq. (3). The solutions for the tip mass (mass without eccentricity) in Ref. [6] are accurate as shown in Table 2, since k is close to 1.

The dimensional frequencies in rad/s are presented in Table 3 for $k = 0.9704572$, water height ratio $\eta = 1/3$, beam length $L = 15\,\text{m}$, beam diameter $D = 0.3\,\text{m}$. The center of gravity of the mass is $\varepsilon^* = 0.5\,\text{m}$ above the free end of the beam. Different mass and inertia ratios are selected. Analytical and FEM results obtained in the present study and the results in Ref. [6] are compared. The frequency values obtained in the present study are in agreement with each other, but there are some slight differences with Ref. [6]. The closeness of the results of Ref. [6] arises from k-value

Table 2
Comparisons of eigenvalues of beam partially immersed in water for different tip mass ($\varepsilon^* = 0$), inertia and water height ratios ($\rho_w = 1000\,\text{kg/m}^3$, $\rho_c = 7850\,\text{kg/m}^3$, eigenvalues are made non-dimensional using $\rho_c + \rho_w$)

η	β	α	Analytical			FEM			Uscിłowska and Kołodziej [6]		
			λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0	0	1	1.28589	4.15381	7.35123	1.28589	4.15381	7.35123	1.28589	4.15381	7.35123
0	0	2	1.10894	4.10376	7.31877	1.10894	4.10376	7.31879	1.10894	4.10377	7.31879
0	1	1	0.95996	1.89738	5.05001	0.95996	1.89738	5.05000	0.95996	1.89739	5.05001
0	1	2	0.91265	1.74078	4.97632	0.91265	1.74078	4.97632	0.91265	1.74078	4.97632
0.5	0	1	1.28553	4.10890	7.23280	1.28553	4.10890	7.23280	1.28553	4.10890	7.23281
0.5	0	2	1.10878	4.05978	7.20006	1.10878	4.05978	7.20006	1.10878	4.05978	7.20006
0.5	1	1	0.95991	1.89600	4.97223	0.95991	1.89600	4.97223	0.95991	1.98600	4.97223
0.5	1	2	0.91261	1.74004	4.89984	0.91261	1.74004	4.89984	0.91261	1.74004	4.89985
1	0	1	1.27812	4.04250	7.14177	1.27812	4.04250	7.14177	1.27812	4.04250	7.14178
1	0	2	1.10523	3.98914	7.10680	1.10523	3.98914	7.10680	1.10523	3.98914	7.10680
1	1	1	0.95856	1.88350	4.91782	0.95856	1.88350	4.91782	0.95856	1.88350	4.91782
1	1	2	0.91147	1.73393	4.84047	0.91147	1.73393	4.84047	0.91147	1.73393	4.84047

Table 3
Comparisons of frequencies (in rad/s) of beam partially immersed in water for different mass, inertia ratios ($k = 0.9704572$, $\eta = 1/3$, $L = 15\,\text{m}$, $D = 0.3\,\text{m}$, $\varepsilon^* = 0.5\,\text{m}$)

β	α	Analytical			FEM			Uscിłowska and Kołodziej [6]		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	0	6.0125	37.3891	103.247	6.0125	37.3892	103.248	6.0126	37.3900	103.1491
0	0.01	5.8851	36.3450	100.348	5.8851	36.4651	100.349	5.8852	36.4659	100.3509
0	0.1	5.0063	31.6898	88.6396	5.0063	31.6898	88.6397	5.0063	31.6899	88.6405
0	0.5	3.3377	27.0955	81.4991	3.3377	27.0955	81.4991	3.3377	27.0975	81.4999
0.01	0	5.7876	24.4138	57.4074	5.7876	24.4138	57.4074	5.7877	24.4144	57.4087
0.01	0.01	5.67175	24.2562	57.3041	5.67175	24.2562	57.3042	5.6718	24.2567	57.3055
0.01	0.1	4.86553	23.3262	56.7071	4.86553	23.3262	56.7071	4.8655	23.3263	56.7075
0.01	0.5	3.29190	22.1383	55.9713	3.29190	22.1383	55.9713	3.2919	22.1384	55.9717
0.1	0	4.25454	11.9867	51.6503	4.25454	11.9868	51.6503	4.2546	11.9870	51.6516
0.1	0.01	4.20899	11.9472	51.2158	4.20899	11.9472	51.2158	4.2091	11.9475	51.2170
0.1	0.1	3.85291	11.6712	48.3653	3.85291	11.6712	48.3654	3.8529	11.6712	48.3657
0.1	0.5	2.92564	11.1625	43.8848	2.92564	11.1625	43.8848	2.9257	11.1626	43.8851
0.5	0	2.30082	10.0250	51.1875	2.30082	10.0250	51.1875	2.3008	10.0253	51.1887
0.5	0.01	2.29447	9.92528	50.7178	2.29447	9.92528	50.7178	2.2945	9.9255	50.7191
0.5	0.1	2.23903	9.17775	47.6117	2.23903	9.17775	47.6117	2.2390	9.1778	47.6120
0.5	0.5	2.02596	7.48573	42.6599	2.02596	7.48573	42.6599	2.0260	7.4857	42.6607

which is close to 1. If the k-value decreases (fluid density increases) the difference will be large as presented in Table 4. In Table 4, the dimensional frequency values in rad/s of the fixed–free beam partially immersed are presented for different k and η values. Decreasing k means increasing the

Table 4
Comparisons of frequencies (in rad/s) of beam partially immersed in water for different tip mass ($\varepsilon^* = 0.5$), inertia and water height ratios ($k = 0.9704572$, $L = 15\,\text{m}$, $D = 0.3\,\text{m}$)

K	η	Analytical			FEM			Uscilowska and Kołodziej [6]		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0.9704572	1/3	6.0125	37.3891	103.2470	6.0125	37.3892	103.2475	6.0126	37.3900	103.1491
	2/3	5.9464	36.1702	101.6010	5.9464	36.1702	101.6012	5.9465	36.1711	101.6035
0.9554427	1/3	6.0108	37.2148	102.0110	6.0108	37.2148	102.0108	6.9728	43.1709	118.3374
	2/3	5.9081	35.3997	99.5568	5.9081	35.3997	99.5569	6.8536	41.6540	115.4908
0.9306048	1/3	6.0077	36.8975	99.8764	6.0077	36.8975	99.8765	9.7181	59.6858	161.5612
	2/3	5.8393	34.1369	96.1251	5.8393	34.1369	96.1252	9.4457	55.2202	155.4932
0.6930980	1/3	5.9381	30.9678	77.0035	5.9381	30.9678	77.0036	30.3751	158.4103	393.8983
	2/3	4.7136	23.3639	61.7977	4.7136	23.3639	61.7978	24.1116	119.5139	316.1159

density of the water as seen from Eq. (3). The analytical and FEM results of the present study are mostly equal to each other. In Ref. [6], it was mentioned that a decrease in k (increase in water density) results in an increase in frequencies. Increasing water density, since it is an added mass to the system, should decrease the frequencies as presented in Table 4. The values of Ref. [6] increases with an increase in water density (decrease in k). Also it was mentioned in Ref. [6], decreasing the parameter η (decreasing the water height) results in an increase in frequencies. That is correct, because lowering the water height lowers the added mass due to water column around the beam, and this increases the frequencies as presented in Table 4. Also Ref. [5] expressed that, the natural frequencies of the beam in water are less than those of a dry beam (water height is zero). Let us look at these values in Table 4. For $k = 0.9704572$, $\eta = 1/3$, natural frequencies are 6.0125, 37.3891, 103.2470. For $k = 0.9554427$, $\eta = 1/3$, natural frequencies are 6.0108, 37.2148, 102.0110. As can be seen, an increase in fluid density (decrease in k) decreases frequencies. In Ref. [6] the values are as follows. For $k = 0.9704572$, $\eta = 1/3$, natural frequencies are 6.0126, 37.3900, 103.1491. For $k = 0.9554427$, $\eta = 1/3$, natural frequencies are 6.9728, 43.1709, 118.3374. These values show that a decrease in k increases the natural frequencies which means added mass increases frequencies. Added mass should decrease the frequencies.

Tables 5 and 6 present comparisons of dimensional frequencies in rad/s of the beam partially immersed in water for different mass, inertia and water height ratios $\eta = 1/3$ and $2/3$ respectively. $k = 0.9554427$, $L = 15\,\text{m}$, $D = 0.3\,\text{m}$, and the center of gravity of the mass is $\varepsilon^* = 0.5\,\text{m}$ above from top of the beam. The difference of Ref. [6] mentioned in the explanation of Tables 3 and 4 can be seen clearly. A similar comparison is made in Tables 7 and 8 for $k = 0.9306048$. Increase in water density (decrease in k) decreases the frequencies. Increase in water height again decreases the frequencies. Also, an increase in rotatory inertia has a large effect when compared with an increase in mass. For example in Table 5, for $\beta = 0$, $\alpha = 0, 0.01, 0.1, 0.5, 1$, first mode frequencies are 6.01081, 5.88888, 5.04029, 3.39268, 2.61178. For $\alpha = 0$, $\beta = 0, 0.01, 0.1, 0.5, 1$, the frequencies are 6.01081, 5.78616, 4.25415, 2.30078, 1.66843. When a comparison is made between these values, it can be seen that, rotatory inertia has a larger influence on frequencies than the mass

Table 5
Comparisons of frequencies (rad/s) of beam partially immersed in water for different mass, inertia and water height ratios ($k = 0.9554427$, $L = 15\,\text{m}$, $D = 0.3\,\text{m}$, $\varepsilon^* = 0.5\,\text{m}$, $\eta = 1/3$)

β	α	Analytical			Uscilowska and Kołodziej [6]		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	0	6.01081	37.2148	102.0110	6.9728	43.1709	118.3374
0	0.01	5.88888	36.4204	99.7183	6.8251	42.1132	115.0541
0	0.1	5.04029	32.1829	89.9003	5.8065	36.6335	101.7456
0	0.5	3.39268	27.8294	83.1682	3.8716	31.3431	93.5827
0.01	0	5.78616	24.3821	57.0291	6.7122	23.2844	66.1566
0.01	0.01	5.67546	24.2575	56.9017	6.5779	28.0114	66.0420
0.01	0.1	4.89778	23.5000	56.1408	5.6433	27.0215	65.3797
0.01	0.5	3.34520	22.4732	55.1452	3.8185	25.6422	64.5618
0.1	0	4.25415	11.9800	51.2900	4.9350	13.8973	59.4989
0.1	0.01	4.21100	11.9373	50.8581	4.8822	13.8516	59.0041
0.1	0.1	3.87089	11.6371	48.0124	4.4692	13.5327	55.7538
0.1	0.5	2.96530	11.0744	43.4977	3.3937	12.9447	50.6258
0.5	0	2.30078	10.0193	50.8292	2.6690	11.6228	58.9644
0.5	0.01	2.29481	9.91868	50.3668	2.6616	11.5704	58.4292
0.5	0.1	2.24250	9.16357	47.3019	2.5973	10.6419	54.8843
0.5	0.5	2.03929	7.44409	42.3944	2.3502	8.6817	49.2148

Table 6
Comparisons of frequencies (rad/s) of beam partially immersed in water for different mass, inertia and water height ratios ($k = 0.9554427$, $L = 15\,\text{m}$, $D = 0.3\,\text{m}$, $\varepsilon^* = 0.5\,\text{m}$, $\eta = 2/3$)

β	α	Analytical			Uscilowska and Kołodziej [6]		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	0	5.90806	35.3997	99.5568	6.8536	41.6540	115.4908
0	0.01	5.79251	34.6438	97.3722	6.7137	40.0640	112.3695
0	0.1	4.98103	30.5891	87.8738	5.7393	34.8449	99.4839
0	0.5	3.37529	26.3730	81.2165	3.8526	29.7378	91.3822
0.01	0	5.69651	23.8202	55.1921	6.6082	27.6326	64.0256
0.01	0.01	5.59106	23.6838	55.0760	6.4803	27.4355	63.9234
0.01	0.1	4.84441	22.8532	54.3842	5.5828	26.2703	63.3336
0.01	0.5	3.32870	21.7233	53.4826	3.8004	24.7774	62.6903
0.1	0	4.22659	11.7379	49.0826	4.9030	13.6165	56.9383
0.1	0.01	4.18411	11.7016	48.6520	4.8511	13.5779	56.4451
0.1	0.1	3.84920	11.4443	45.8066	4.4444	13.3605	53.1966
0.1	0.5	2.95486	10.9522	41.2613	3.3821	12.7961	48.0398
0.5	0	2.29767	9.77661	48.5938	2.6654	11.3414	56.3712
0.5	0.01	2.29171	9.68468	48.1312	2.6581	11.2359	55.8359
0.5	0.1	2.23949	8.98866	45.0564	2.5939	10.4385	52.2797
0.5	0.5	2.03672	7.36801	40.0971	2.3472	8.5922	46.5509

Table 7
Comparisons of frequencies (rad/s) of beam partially immersed in water for different mass, inertia and water height ratios ($k = 0.9306048$, $L = 15\,\mathrm{m}$, $D = 0.3\,\mathrm{m}$, $\varepsilon^* = 0.5\,\mathrm{m}$, $\eta = 1/3$)

β	α	Analytical			Usciłowska and Kołodziej [6]		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	0	6.00770	36.8975	99.8764	9.7181	59.6858	161.5612
0	0.01	5.88598	36.1219	97.6706	9.5125	58.2465	157.1637
0	0.1	5.03856	31.9682	88.1753	8.0939	50.7563	139.2287
0	0.5	3.39219	27.6758	81.6147	5.3979	43.4787	128.1347
0.01	0	5.78351	24.3236	56.3479	9.3554	39.3462	91.1489
0.01	0.01	5.67298	24.1986	56.2294	9.1685	39.0904	91.0019
0.01	0.1	4.89625	23.4388	55.5214	7.8668	37.5817	90.1501
0.01	0.5	3.34475	22.4091	54.5929	5.3139	35.6556	89.0979
0.1	0	4.25345	11.9675	50.6399	6.8804	19.3587	81.9157
0.1	0.01	4.21030	11.9251	50.2223	6.8067	19.2955	81.2487
0.1	0.1	3.87033	11.6270	47.4635	6.2311	18.8538	76.8551
0.1	0.5	2.96503	11.0678	43.0615	4.7319	18.0392	69.8887
0.5	0	2.30071	10.0087	50.1828	3.7216	16.1901	81.1763
0.5	0.01	2.29474	9.90847	49.7351	3.7114	16.0297	80.4538
0.5	0.1	2.24243	9.15604	46.7596	3.6217	14.8272	75.6550
0.5	0.5	2.03923	7.44083	41.9676	3.2771	12.1007	67.9356

Table 8
Comparisons of frequencies (rad/s) of beam partially immersed in water for different mass, inertia and water height ratios ($k = 0.9306048$, $L = 15\,\mathrm{m}$, $D = 0.3\,\mathrm{m}$, $\varepsilon^* = 0.5\,\mathrm{m}$, $\eta = 2/3$)

β	α	Analytical			Usciłowska and Kołodziej [6]		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	0	5.83927	34.1369	96.1252	9.4457	55.2202	155.4932
0	0.01	5.72787	33.4148	94.0878	9.1736	53.6921	150.8362
0	0.1	4.94086	29.5158	85.1123	7.9396	46.9119	134.4805
0	0.5	3.36334	25.4136	78.6951	5.3538	39.9933	123.5620
0.01	0	5.63622	23.4242	53.6533	9.1172	37.8913	86.7901
0.01	0.01	5.53421	23.2803	53.5509	8.9447	37.6041	86.6675
0.01	0.1	4.80816	22.4027	52.9416	7.7275	35.9047	85.9603
0.01	0.5	3.31736	21.2067	52.1473	5.2819	33.7233	85.0923
0.1	0	4.20761	11.5759	47.3529	6.8063	18.7253	76.3102
0.1	0.01	4.16561	11.5435	46.9348	6.7346	18.6776	75.9311
0.1	0.1	3.83428	11.3132	44.1603	6.1738	18.3405	71.5162
0.1	0.5	2.94766	10.8665	39.6831	4.7050	17.6985	64.4376
0.5	0	2.29553	9.61477	46.8503	3.7133	15.5530	75.7836
0.5	0.01	2.28957	9.52835	46.3998	3.7030	15.4147	75.0586
0.5	0.1	2.23742	8.87045	43.3911	3.6136	14.3642	70.2068
0.5	0.5	2.03495	7.31537	38.4874	3.2702	11.8949	62.3088

itself. The same conclusion can be drawn for upper modes, different water height ratios and water densities.

4. Concluding remarks

In this study, an Euler–Bernoulli type beam partially immersed in water and carrying a mass at one end is considered. Natural frequency equation is presented and the values are calculated for different water height ratios, water densities and tip mass values by using analytical and finite element methods, and some comparisons with other references are made. Increasing water height and tip mass decrease the frequencies due to the added mass on the beam. Similarly increasing the water density decreases the frequencies of oscillations. The decrease in the frequencies due to rotatory inertia is sharper when it is compared with the effect of the tip mass alone.

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