Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 266 (2003) 1109–1116
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Vibration of wires used in electro-discharge machining

S.M. Shahruz

Berkeley Engineering Research Institute, P.O. Box 9984, Berkeley, CA 94709, USA

Received 6 December 2002; accepted 16 December 2002

1. Introduction

Electro-discharge machining (EDM) is a non-contact process of electrically removing (cutting) material from conductive workpieces. In this process, a high potential difference is generated between a wire and a workpiece by charging them positively and negatively, respectively. The potential difference causes sparks between the wire and the workpiece. The extreme heat due to the sparks melts the workpiece. By moving the wire forward and sideways a desired contour can be cut on the workpiece. In this process, the wire is worn off mutually. In order to avoid possible breakage of the wire due to severe localized wear, the wire is moved axially by reeling it off a supply spool and collecting it at a take-up spool; see Fig. 1(a) for a schematic of the EDM process. The set-up in Fig. 1(a) is placed in a dielectric fluid, such as oil or deionized water, to enhance the sparking between the wire and the workpiece, to cool the workpiece, and to flush away the cut particles.

Although the EDM is a fast, accurate, and economical process of manufacturing, it suffers from problems such as: (1) deflection of the wire which can cause inaccuracies in the cut; (2) vibration of the wire which can cause uneven surfaces on the workpiece and possibly the wire rupture. In order to have a better understanding of the EDM, this process has been studied by researchers; see, e.g., Refs. [1–4] and the references therein. Problems of particular interest are the wire dynamics, wire deflection, and wire vibration. To study such problems, mathematical models that (approximately) describe the dynamics of EDM wires have been developed in recent years; see, e.g., Refs. [3,4]. Numerical studies of such models, as well as experimental results, show that for low and intermediate axial speeds of the wire, the straight configuration of the wire is stable, whereas for large axial speeds, the wire becomes unstable.

In this note, using an existing mathematical model of EDM wires, it is rigorously shown that the transversal vibration of the wire decays to zero for wire axial speeds below a critical value. That is, the wire is stable. This fact is proved by showing that an energy-like (Lyapunov) function corresponding to the wire decays to zero exponentially.

E-mail address: shahruz@eecs.berkeley.edu (S.M. Shahruz).

0022-460X/03/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0022-460X(03)00057-9

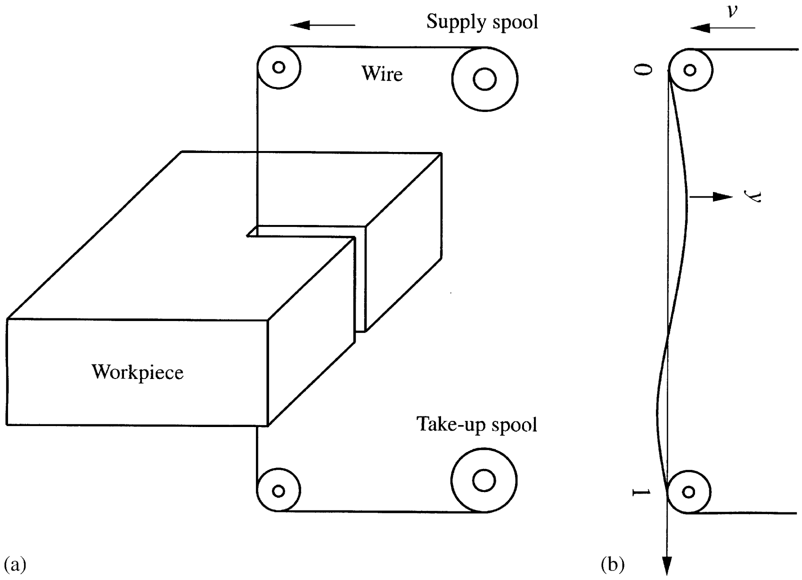

Fig. 1. (a) A schematic of the EDM process. The wire is reeled off a supply spool and is collected at a take-up spool. (b) The transversal displacement of the wire is denoted by y.

2. A mathematical model of EDM wires

The wire used in the EDM process is supported by two pulleys and moves axially. The pulleys are distanced from each other by the unit length; see Fig. 1(b). Moreover, a part of the wire between the two pulleys is heated. Having this simplified model, the dynamics of the wire can be represented by the following non-linear partial differential equation (see Ref. [4]):

$$y_{tt}(x,t) + 2\delta y_t(x,t) + 2v y_{xt}(x,t) = \left(1 - v^2 + kT(x) + b\int_0^1 y_x^2(x,t)\,\mathrm{d}x\right) y_{xx}(x,t) + kT_x(x)y_x(x,t), \tag{1}$$

for all $x \in (0,1)$ and $t \geqslant 0$. In Eq. (1), $y(.,.) \in \mathbb{R}$ denotes the transversal displacement of the wire, $y_t := \partial y/\partial t$, $y_{tt} := \partial^2 y/\partial t^2$, $y_x := \partial y/\partial x$, $y_{xx} := \partial^2 y/\partial x^2$, $y_{xt} := \partial^2 y/\partial x \partial t$; the constant real number $\delta > 0$ corresponds to the damping coefficient of the wire; the constant real number $v \geqslant 0$ is proportional to the wire axial speed; the constant real number $k > 0$ is proportional to the elastic modulus and the thermal expansion coefficient of the wire; the constant real number $b > 0$ is proportional to the elastic modulus of the wire; $T(x)$, where $x \in [0,1]$ is the wire temperature profile; and $T_x := \partial T/\partial x$ represents the temperature change along the wire. In realistic physical situations, $v < 1$.

The boundary conditions of the wire are

$$y(0,t) = y(1,t) = 0, \tag{2}$$

for all $t \geqslant 0$, and the initial displacement and velocity of the wire are, respectively,

$$y(x,0) = f(x), \quad y_t(x,0) = g(x), \tag{3}$$

for all $x \in (0,1)$. It is assumed that $f \in C^1[0,1]$ and that at least one of the functions $f(\cdot)$ or $g(\cdot)$ is not identically equal to zero over $[0,1]$.

On the right-hand side of Eq. (1), there are two terms that account for the dependence of the wire dynamics on the wire temperature profile $x \mapsto T(x)$ and its change $x \mapsto T_x(x)$. If these two terms were absent, then Eq. (1) would present the dynamics of an axially moving Kirchhoff wire; see, e.g., Ref. [5] and the references therein. The temperature $T(x) \geqslant 0$ for all $x \in [0,1]$. The temperature profile $x \mapsto T(x)$ depends on the wire axial speed v. If v is small (respectively, large), then temperatures along the wire are large (small); see Refs. [3,4] for typical wire temperature profiles.

In this note, the goal is to show that the wire is stable, i.e., $y(x,t) \to 0$ as $t \to \infty$ for all $x \in [0,1]$. This goal is achieved by taking an energy approach.

3. Stability of the wire

The plan to establish the stability of the non-linear wire represented by Eqs. (1)–(3) is as follows. An energy-like (Lyapunov) function of time for system (1)–(3) is defined and denoted by $t \mapsto V(t)$. It is then shown that $V(\cdot)$ tends to zero exponentially, from which the stability of the wire is concluded.

The scalar-valued function $V(\cdot)$ is defined as

$$V(t) := E(t) + \int_0^1 [\delta y_t(x,t) y(x,t) + \delta^2 y^2(x,t)]\, \mathrm{d}x, \tag{4}$$

for all $t \geqslant 0$, where

$$E(t) := \frac{1}{2}\int_0^1 \left(y_t^2(x,t) + [1 - v^2 + kT(x)] y_x^2(x,t)\right) \mathrm{d}x + \frac{b}{4}\left(\int_0^1 y_x^2(x,t)\, \mathrm{d}x\right)^2, \tag{5}$$

and $y(\cdot,\cdot)$ satisfies Eqs. (1)–(3). The function $V(\cdot)$ can be written as

$$V(t) = \frac{1}{2}\int_0^1 \left([y_t(x,t) + \delta y(x,t)]^2 + \delta^2 y^2(x,t) + [1 - v^2 + k\ T(x)] y_x^2(x,t)\right) \mathrm{d}x + \frac{b}{4}\left(\int_0^1 y_x^2(x,t)\, \mathrm{d}x\right)^2, \tag{6}$$

for all $t \geqslant 0$. From Eqs. (2)–(5), it is concluded that

$$E(0) = \frac{1}{2}\int_0^1 \left(g^2(x) + [1 - v^2 + kT(x)] f_x^2(x)\right) \mathrm{d}x + \frac{b}{4}\left(\int_0^1 f_x^2(x)\, \mathrm{d}x\right)^2, \tag{7a}$$

$$V(0) = E(0) + \int_0^1 [\delta\ g(x) f(x) + \delta^2\ f^2(x)]\, \mathrm{d}x, \tag{7b}$$

where $f_x(x) := \mathrm{d}f(x)/\mathrm{d}x$.

Now, some properties of $V(\cdot)$ are proved.

Lemma 3.1. *The function $V(\cdot)$ is non-negative and $V(0) > 0$.*

Proof. See Appendix A. □

Next, it is shown that $V(\cdot)$ can be bounded by $E(\cdot)$.

Lemma 3.2. *The function* $V(\cdot)$ *satisfies*

$$V(t) \leqslant KE(t), \tag{8}$$

for all $t \geqslant 0$, *where*

$$K = 1 + \frac{\delta(1 + 2\delta/\pi)}{\pi(1 - v^2)}. \tag{9}$$

Proof. See Appendix A. □

Some useful identities are now established for functions that satisfy Eqs. (2) and (3).

Lemma 3.3. *If* $y(\cdot, \cdot)$ *satisfies Eqs.* (2) *and* (3), *then* (*the argument* (x, t) *of functions is deleted*)

$$2\int_0^1 y_{xt}\, y_t \,\mathrm{d}x = 0, \qquad \int_0^1 y_{xt}\, y \,\mathrm{d}x = -\int_0^1 y_t y_x \,\mathrm{d}x, \tag{10a, b}$$

$$\int_0^1 y_{xx}\, y \,\mathrm{d}x = -\int_0^1 y_x^2 \,\mathrm{d}x, \qquad \int_0^1 (y_{xx}\, y_t + y_{xt}\, y_x) \,\mathrm{d}x = 0, \tag{10c, d}$$

$$\int_0^1 T(x) y_{xx}\, y \,\mathrm{d}x = -\int_0^1 T_x(x) y_x y \,\mathrm{d}x - \int_0^1 T(x) y_x^2 \,\mathrm{d}x, \tag{10e}$$

$$\int_0^1 T(x)(y_{xx}\, y_t + y_{xt}\, y_x) \,\mathrm{d}x = -\int_0^1 T_x(x) y_x y_t \,\mathrm{d}x. \tag{10f}$$

Proof. See Appendix A. □

Up to this point, some properties of the functions $V(\cdot)$ and $y(\cdot, \cdot)$ have been established. Next, it is proved that $V(\cdot)$ tends to zero exponentially, from which the stability of the wire represented by Eqs. (1)–(3) follows.

Lemma 3.4. *Consider system* (1)–(3) *and let*

$$v < v_c := (\sqrt{5} - 1)/2. \tag{11}$$

The function $V(\cdot)$, *along the solution of system* (1)–(3), *tends to zero exponentially*.

Proof. From Eq. (6), it follows that (the argument (x,t) of functions is deleted)

$$\dot{V}(t) = \int_0^1 \big((y_{tt} + \delta y_t)(y_t + \delta y) + \delta^2 y_t y + [1 - v^2 + kT(x)] y_{xt} y_x\big)\,\mathrm{d}x + b\int_0^1 y_x^2\,\mathrm{d}x \int_0^1 y_{xt} y_x\,\mathrm{d}x, \tag{12}$$

for all $t \geqslant 0$. Substituting y_{tt} from Eq. (1) into Eq. (12), it follows that

$$\begin{aligned}\dot{V}(t) = &-\delta\int_0^1 y_t^2\,\mathrm{d}x - 2v\int_0^1 y_{xt} y_t\,\mathrm{d}x - 2\delta v\int_0^1 y_{xt} y\,\mathrm{d}x \\ &+ \left(1 - v^2 + b\int_0^1 y_x^2\,\mathrm{d}x\right)\int_0^1 (y_{xx} y_t + y_{xt} y_x)\,\mathrm{d}x \\ &+ \delta(1 - v^2)\int_0^1 y_{xx} y\,\mathrm{d}x + \delta k\int_0^1 T(x) y_{xx} y\,\mathrm{d}x + \delta b\int_0^1 y_x^2\,\mathrm{d}x\int_0^1 y_{xx} y\,\mathrm{d}x \\ &+ k\int_0^1 T(x)\ (y_{xx} y_t + y_{xt} y_x)\,\mathrm{d}x + k\int_0^1 T_x(x) y_x y_t\,\mathrm{d}x + \delta k\int_0^1 T_x(x) y_x y\,\mathrm{d}x,\end{aligned} \tag{13}$$

for all $t \geqslant 0$. Using Eqs. (10) in Eq. (13), it is concluded that

$$\dot{V}(t) = -\delta\int_0^1 y_t^2\,\mathrm{d}x - \delta\int_0^1 [1 - v^2 + kT(x)] y_x^2\,\mathrm{d}x - \delta\ b\left(\int_0^1 y_x^2\,\mathrm{d}x\right)^2 + 2\delta v\int_0^1 y_t y_x\,\mathrm{d}x, \tag{14}$$

for all $t \geqslant 0$. The last term on the right-hand side of Eq. (14) satisfies the inequality

$$2\delta v\int_0^1 y_t y_x\,\mathrm{d}x \leqslant \delta v\int_0^1 y_t^2\,\mathrm{d}x + \delta v\int_0^1 y_x^2\,\mathrm{d}x. \tag{15}$$

Using inequality (15) in Eq. (14), it follows that

$$\dot{V}(t) \leqslant -\delta(1 - v)\int_0^1 y_t^2\,\mathrm{d}x - \delta[1 - v - v^2 + kT(x)]\int_0^1 y_x^2\,\mathrm{d}x - \delta b\left(\int_0^1 y_x^2\,\mathrm{d}x\right)^2, \tag{16}$$

for all $t \geqslant 0$. Inequality (16) can be written as

$$\dot{V}(t) \leqslant -2\delta\left(\frac{1 - v - v^2}{1 - v^2}\right)E(t) - \delta\left(\left(\frac{v^3}{1 - v^2}\right)\int_0^1 y_t^2\,\mathrm{d}x + \left(\frac{v}{1 - v^2}\right)\int_0^1 kT(x) y_x^2\,\mathrm{d}x. + \frac{b}{2}\left(\frac{1 + v - v^2}{1 - v^2}\right)\left(\int_0^1 y_x^2\,\mathrm{d}x\right)^2\right), \tag{17}$$

for all $t \geqslant 0$. Thus,

$$\dot{V}(t) \leqslant -2\delta\left(\frac{1-v-v^2}{1-v^2}\right)E(t), \tag{18}$$

for all $t \geqslant 0$. By inequality (11), it is clear that $1-v-v^2>0$. Using this fact and inequality (8) in inequality (18), it follows that

$$\dot{V}(t) \leqslant -\frac{2\delta(1-v-v^2)}{K(1-v^2)}V(t), \tag{19}$$

for all $t \geqslant 0$.

Having $V(0)>0$ by Lemma 3.1 and a comparison theorem (see, e.g., Refs. [6, p. 2; 7, p. 3]), it is concluded that

$$V(t) \leqslant V(0)\exp\left(-\frac{2\delta(1-v-v^2)}{K(1-v^2)}t\right), \tag{20}$$

for all $t \geqslant 0$. Thus, $V(t) \to 0$ exponentially as $t \to \infty$. □

Now, the stability of the EDM wire follows.

Theorem 3.5. *Consider system* (1)–(3) *and let inequality* (11) *hold. The solution of the system* $y(x,t) \to 0$ *as* $t \to \infty$ *for all* $x \in [0,1]$.

Proof. By Lemma 3.4, $V(t) \to 0$ as $t \to \infty$. Thus, by Eq. (6), $y(x,t) \to 0$ as $t \to \infty$ for all $x \in [0,1]$. □

By Theorem 3.5, the stability of the EDM wire represented by Eqs. (1)–(3) is guaranteed when the wire speed satisfies inequality (11), i.e., when $v < v_c \approx 0.6$. The critical speed v_c is a conservative bound on the wire speed since Theorem 3.5 furnishes only a sufficient condition for the stability. Thus, for wire axial speeds larger than v_c, but less than 1, the wire can possibly be stable.

4. Conclusions

In this note, a mathematical model of electro-discharge machining (EDM) is studied. The model is a non-linear partial differential equation which represents the transversal vibration of the moving EDM wire. It is rigorously shown that the transversal vibration of the wire decays to zero for wire axial speeds below a critical value. That is, the wire is stable. The wire stability is established by showing that an energy-like (Lyapunov) function corresponding to the wire decays to zero exponentially.

Appendix A. Various proofs

Proof of Lemma 3.1. The non-negativeness of $V(\cdot)$ is obvious from Eq. (6). Recall that at least one of the functions $f(\cdot)$ or $g(\cdot)$ in Eq. (3) is not identically equal to zero over [0, 1]. Furthermore, the function $f(\cdot)$, for which $f(0)=0$ by Eq. (2), cannot assume a non-zero constant value over [0, 1]. Thus, from Eq. (7a), it follows that $E(0)$ is positive, and so is $V(0)$ by Eq. (7b). □

Proof of Lemma 3.2. By Scheeffer's inequality, which is a Poincaré-type inequality (see, e.g., Ref. [8, p. 67]), it is concluded that

$$\int_0^1 y^2(x,t)\,\mathrm{d}x \leqslant \frac{1}{\pi^2}\int_0^1 y_x^2(x,t)\,\mathrm{d}x, \tag{A.1}$$

for all $t \geqslant 0$. Furthermore,

$$\int_0^1 y_t(x,t)y(x,t)\,\mathrm{d}x \leqslant \frac{1}{2\pi}\int_0^1 y_t^2(x,t)\,\mathrm{d}x + \frac{\pi}{2}\int_0^1 y^2(x,t)\,\mathrm{d}x, \tag{A.2}$$

for all $t \geqslant 0$. Using inequality (A.1) in inequality (A.2), it is concluded that

$$\int_0^1 y_t(x,t)y(x,t)\,\mathrm{d}x \leqslant \frac{1}{2\pi}\int_0^1 y_t^2(x,t)\,\mathrm{d}x + \frac{1}{2\pi}\int_0^1 y_x^2(x,t)\,\mathrm{d}x, \tag{A.3}$$

for all $t \geqslant 0$.

Substituting inequalities (A.3) and (A.1) into Eq. (4) and noting that $T(x) \geqslant 0$ for all $x \in [0,1]$, it follows that

$$V(t) \leqslant E(t) + \frac{\delta}{2\pi}\left(\int_0^1 y_t^2(x,t)\mathrm{d}x + \left(\frac{1+2\delta/\pi}{1-v^2}\right)\int_0^1 [1-v^2+kT(x)]y_x^2(x,t)\,\mathrm{d}x\right), \tag{A.4}$$

for all $t \geqslant 0$. Since $v<1$, it is concluded that

$$V(t) \leqslant E(t) + \frac{\delta(1+2\delta/\pi)}{\pi(1-v^2)}\left(\frac{1}{2}\int_0^1 (y_t^2(x,t) + [1-v^2+kT(x)]\; y_x^2(x,t))\mathrm{d}x\right), \tag{A.5}$$

and finally

$$V(t) \leqslant \left(1 + \frac{\delta(1+2\delta/\pi)}{\pi(1-v^2)}\right)E(t), \tag{A.6}$$

for all $t \geqslant 0$. Thus, inequality (8) holds with K given in Eq. (9). □

Proof of Lemma 3.3. From Eq. (2), it follows that

$$y_t(0,t) = y_t(1,t) = 0, \tag{A.7}$$

for all $t \geqslant 0$.

Having Eq. (A.7), proofs of Eqs. (10a) and (10d) are, respectively, as follows:

$$2\int_0^1 y_{xt}y_t\,\mathrm{d}x = \int_0^1 (y_t^2)_x\,\mathrm{d}x = y_t^2(1,t) - y_t^2(0,t) = 0, \tag{A.8}$$

for all $t \geqslant 0$, and

$$\int_0^1 (y_{xx}y_t + y_{xt}y_x)\,\mathrm{d}x = \int_0^1 (y_x y_t)_x\,\mathrm{d}x = 0. \tag{A.9}$$

Having Eq. (2), proofs of Eqs. (10b) and (10c) are, respectively, as follows:

$$\int_0^1 y_{xt} y \, \mathrm{d}x = \int_0^1 (y_t y)_x \, \mathrm{d}x - \int_0^1 y_t y_x \, \mathrm{d}x = -\int_0^1 y_t y_x \, \mathrm{d}x, \tag{A.10}$$

$$\int_0^1 y_{xx} y \, \mathrm{d}x = \int_0^1 (y_x y)_x \, \mathrm{d}x - \int_0^1 y_x^2 \, \mathrm{d}x = -\int_0^1 y_x^2 \, \mathrm{d}x. \tag{A.11}$$

Moreover, having Eq. (2), the truth of Eq. (10e) is established as follows:

$$\begin{aligned}\int_0^1 T(x) y_{xx} y \, \mathrm{d}x &= \int_0^1 (T(x) y_x y)_x \, \mathrm{d}x - \int_0^1 T_x(x) y_x y \, \mathrm{d}x - \int_0^1 T(x) y_x^2 \, \mathrm{d}x \\ &= -\int_0^1 T_x(x) y_x y \, \mathrm{d}x - \int_0^1 T(x) y_x^2 \, \mathrm{d}x. \end{aligned} \tag{A.12}$$

Finally, by Eq. (A.7), a proof of Eq. (10f) is as follows:

$$\begin{aligned}\int_0^1 T(x)(y_{xx} y_t + y_{xt} y_x) \, \mathrm{d}x &= \int_0^1 (T(x) y_x y_t)_x \, \mathrm{d}x - \int_0^1 T_x(x) y_x y_t \, \mathrm{d}x \\ &= -\int_0^1 T_x(x) y_x y_t \, \mathrm{d}x. \end{aligned} \tag{A.13}$$

Thus, Lemma 3.3 is proved. □

References

[1] E.B. Guitrau, The EDM Handbook, Hanser Gardner Publications, Cincinnati, OH, 1997.

[2] K.P. Rajurkar, Nontraditional manufacturing processes, in: R.C. Dorf, A. Kusiak (Eds.), Handbook of Design, Manufacturing and Automation, Wiley, New York, 1994, pp. 211–241.

[3] K.D. Murphy, Z. Lin, The influence of spatially nonuniform temperature fields on the vibration and stability characteristics of EDM wires, International Journal of Mechanical Sciences 42 (2000) 1369–1390.

[4] T.A. Lambert Jr., K.D. Murphy, Modal convections and its effect on the stability of EDM wires, International Journal of Mechanical Sciences 44 (2002) 207–216.

[5] S.M. Shahruz, Boundary control of the axially moving Kirchhoff string, Automatica 34 (1998) 1273–1277.

[6] D. Bainov, P. Simeonov, Integral Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[7] V. Lakshmikantham, S. Leela, A.A. Martynyuk, Stability Analysis of Nonlinear Systems, Marcel Dekker, New York, 1989.

[8] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.