Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 267 (2003) 105–166
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Review

Automotive disc brake squeal

N.M. Kinkaid, O.M. O'Reilly, P. Papadopoulos

Department of Mechanical Engineering, University of California, Berkeley, CA 94720-1740, USA

Received 7 October 2002; accepted 7 October 2002

Abstract

Disc brake squeal remains an elusive problem in the automotive industry. Since the early 20th century, many investigators have examined the problem with experimental, analytical, and computational techniques, but there is as yet no method to completely suppress disc brake squeal. This paper provides a comprehensive review and bibliography of works on disc brake squeal. In an effort to make this review accessible to a large audience, background sections on vibrations, contact and disc brake systems are also included.

Contents

E-mail address: oreilly@me.berkeley.edu (O.M. O'Reilly).

doi:10.1016/S0022-460X(02)01573-0

1. What is brake squeal?

Brakes are one of the most important safety and performance components in automobiles. Appropriately, ever since the advent of the automobile, development of brakes has focused on increasing braking power and reliability. However, the refinement of vehicle acoustics and comfort through improvement in other aspects of vehicle design has dramatically increased the relative contribution of brake noise to these aesthetic and environmental concerns. Brake noise is an irritant to consumers who may believe that it is symptomatic of a defective brake and file a warranty claim, even though the brake is functioning exactly as designed in all other aspects. Thus, noise generation and suppression have become prominent considerations in brake part design and manufacture. Indeed, as noted by Abendroth and Wernitz [1], many makers of materials for brake pads spend up to 50% of their engineering budgets on noise, vibration and harshness issues.

A wide array of brake noise and vibration phenomena are described by an even wider array of terminology. *Squeal*, *groan*, *chatter*, *judder*, *moan*, *hum*, and *squeak* are just a few of the names found in the literature. Of these phenomena, the one generally termed *squeal* is probably the most prevalent, disturbing to both vehicle passengers and the environment, and expensive to brake and automotive manufacturers in terms of warranty costs (see Ref. [2]). No precise definition of brake squeal has gained complete acceptance, but it is generally agreed that squeal is a sustained, high-frequency (> 1000 Hz) vibration of brake system components during a braking action resulting in noise audible to vehicle occupants or passers-by. As shall be seen later on, this squeal is often subdivided into high- and low-frequency regimes.

There exists no general means for completely eliminating brake squeal. Brakes that squeal do not, in general, squeal during every braking action. Rather, the occurrence of squeal is intermittent or perhaps even random. Many different factors on both the micro- and macroscopic levels appear to affect squeal, and some of these factors (especially on the microscopic level) are not well understood. As will become apparent later on in this review, different brake squeal experiments produce widely different and even conflicting results. Generally, a central difficulty in modelling brake squeal is one of scales. Effects on very small scales in length and time (i.e., microscopic contact phenomena and high-frequency vibrations) interact in important ways with effects on large scales (such as wear over the life of the brake and dynamics of large vehicle substructures).

A number of theories have been formulated to explain the mechanisms of brake squeal, and numerous studies have tried with varied success to apply them to the dynamics of disc brakes. There are many models for squealing disc brakes: we have counted well over 15 different models in the literature. However, none of these models have attempted to include the effects at all scales mentioned above. This has led to models which capture some features of brake squeal well and ignore many others. Experimental studies also tend to have limited applicability with their results only pertaining to one brake configuration or to one automobile type.

The contents of this review paper are as follows: first, a brief overview of other reviews of the brake squeal literature is presented. Next, the components of a disc brake system are briefly discussed. To facilitate understanding of the later sections of this paper, two sections are included which contain background on vibrations and contact and wear, respectively. Experimental studies of disc brake squeal dating from the 1930s to 2001 are discussed in some depth along with empirical remedies for squealing disc brakes. Major features of theories of disc brake squeal will then be explained and examined. The final sections of the paper inspect the various works that examine these theories using lumped parameter and finite element models. A discussion of some of the open issues in understanding squealing disc brakes closes the paper.

The motivation for writing this paper was to complement, update, and expand upon several earlier reviews on the subject. This has proved to be a difficult task in part because the literature on disc brake noise is growing rapidly every year, the subject has an interdisciplinary nature, and the vast majority of papers appear in conference proceedings. Indeed, many of the conference papers in the bibliography are from the *Annual SAE Brake Colloquium and Engineering Display*. With these issues in mind, we have attempted to cover conference proceedings up to the end of 2001.

1.1. Earlier reviews of brake squeal

The subject of brake squeal has generated a considerable volume of literature, including several review articles. Many of these reviews also discuss other types of brake noise, such as drum brake squeal and brake groan.

In his 1976 review, North [3] compared early lumped parameter squeal models that fell into two classes—pin–disc (or pin-on-disc) systems incorporating kinematic constraints and eight-degree-of-freedom systems. These models featured frictional forces which have some of the features of a follower force.[1] He tended to favour the latter approach, as he pointed out that parameters in the pin–disc systems are difficult to correlate to disc brake systems. Results from follower force models, he said, can be easily verified by experiments on brake systems. However, he stopped short of total condemnation of pin–disc models, arguing that both models are useful for demonstrating different mechanisms that lead to squeal and recommending that future effort be aimed at reconciling the differences between the two approaches.

In their review a decade and a half after North's, Crolla and Lang [2] noted that, in spite of the large increases in sophistication of analytical models for disc brake squeal, the results from these (including large degree-of-freedom finite element models) have not yielded many principles for the design of noise-free brakes. The most useful results for such design, they said, had been obtained through experimental and empirical techniques. They opined, however, that finite element methods would in the future provide the most useful design tools once the difficulty of modelling frictional interactions therein had been overcome.

Another review of the disc brake squeal literature appears in Refs. [6,7]. The interesting feature of this short review is that it firmly placed brake squeal in the context of other friction-induced vibration phenomena.[2]

Yang and Gibson [9] concurred with Crolla and Lang [2] that experiments have been the most successful method for investigating brake squeal. In fact, they deemed experiments to be the sole means for verifying any solution to brake squeal. They argued that the finite element method is as yet incapable of accurately modelling the complicated boundary conditions and friction mechanisms found in brake systems, and that the lumped parameter models used to model squealing brakes simplify the problem to a degree that many of the important parameters are neglected. However, Yang and Gibson saw significant promise for computer-aided design and engineering in developing quiet but effective brakes.

While completing the present review, we found a new review on disc brake squeal by Papinniemi et al. [10]. This short review can be considered as a complement to Yang and Gibson [9]. The authors were also informed of a new review by Ibrahim [11]. This review can be considered as a sequel to Refs. [6,7] and discusses thermal effects and noise in several types of automotive braking systems. Unique to all the aforementioned reviews, Ref. [11] also discusses the role of random vibrations in the generation of brake noise. Finally, we would also like to mention the recent review by Akay [12] which contains an illuminating discussion of the acoustics of friction-induced vibration.

2. Disc brake systems

Disc brakes and their actuation systems have been developed since the mid-1890s. In this section, we outline some of this historical development, present an overview of the components of

[1] A follower (or circulatory) force is a force which depends on the displacements of a system but which cannot be derived from a potential energy associated with these displacements (see, e.g., the texts of Leipholz [4] and Ziegler [5]).

[2] A friction-induced vibration is a particular example of a self-excited vibration where the excitation is caused by friction forces. An elegant definition of self-excited (or self-induced) vibration is provided by Den Hartog [8]: "*In a self-excited vibration the alternating force that sustains the motion is created or controlled by the motion itself; when the motion stops the alternating force disappears.*"

a disc brake system, and provide some representative numerical values for quantities associated with a disc brake.

2.1. Some historical background

A century ago, the British engineer Frederick William Lanchester (1868–1946) patented a disc brake (see Refs. [13–15]). In his 1902 patent [16], he described a disc brake consisting of a disc of sheet metal which is rigidly connected to one of the rear wheels of the vehicle. To slow the vehicle, the disc is pinched at its edge by a pair of jaws. This period also evidenced many of the early developments in brake technology. For instance, according to Newcomb and Spurr [15], Mercedes (Daimler Motor Gesellschaft) and Renault both introduced variants of the modern (internal expanding) drum brake in 1903. On the other side of the Atlantic in the 1890s, according to Hughes [17], the American inventor Elmer Ambrose Sperry (1860–1930) invented a brake featuring an electromagnetically actuated disc [18–20]. The disc, which was known as a brake magnet, was placed in contact with another disc (known as a brake disc) to achieve a braking torque. Interestingly, Sperry [19] noted that the braking torque he attained was partially due to friction between the discs and partially due to Foucault (eddy) currents.

Needless to say, the early brake designs of Lanchester and Sperry were substantially modified during the twentieth century. In particular, both the materials and actuation methods used have been improved. According to Harper [14], one of the main arenas for these developments was the aviation industry during the Second World War. Aircraft disc brakes are what are known as clutch-type. That is, the friction pads contact the disc in an annular region which extends over most of angular extent of the disc. In contrast, in spot-type disc brakes, which are now used in automobiles, the angular extent of the friction pads ranges from 30° to 50°. Lanchester's 1902 design is a spot-type disc brake, whereas Sperry's is a clutch-type disc brake.

The evolution of spot-type disc brakes in automobiles can be traced to developments by the Dunlop, Girdling, and Lockheed Corporations in the 1950s [14,21]. Their spot-type disc brakes are substantially similar to those present in automobiles today. The widespread use of disc brakes on the front wheels of passenger vehicles can be partially attributed to increasingly stringent regulations throughout the world on vehicle braking. For example, prior to the 1970s, most automobiles in the United States were equipped with front wheel drum brakes. This situation changed with the introduction of the Federal Motor Vehicle Safety Standard (FMVSS) No. 105. FMVSS No. 105, versions of which became effective for passenger vehicles on January 1, 1968 and 1976, imposed standards on stopping distance, brake fade and water resistance for automotive braking systems (see Refs. [22,23]).[3] As the brakes on the front wheels contribute 70–80% of the braking power, the braking systems for the front wheels were crucial to satisfying FMVSS No. 105. Compared to drum brakes, disc brakes have superior water resistance and fade performance [24]. These features, coupled with the oil crisis and the increased popularity of imported automobiles, contributed to the profusion of disc brake systems in the United States.

[3] According to the National Highway and Transportation Safety Association (NHTSA), FMVSS No. 105 was subsumed in September 2002 by FMVSS No. 135 for all light vehicles. Contrary to popular belief, front wheel drum brakes were not banned by FMVSS No. 105. They were replaced because it was easier for automobiles equipped with front wheel disc brakes to pass the safety standards (see Ref. [22]).

2.2. Components of a disc brake

There are several major components of a modern (spot-type) disc brake: the rotor, caliper, brake pad assemblies and a hydraulic actuation system. Although there is a considerable range of designs for these components, we will attempt to give a brief overview of their function and composition. Our main references for this subsection were Harper [14], Newcomb and Spurr [21] and numerous trade catalogues. Some readers might also find the textbook on automotive braking systems by Halderman and Mitchell [24] to be helpful.

The rotor (or disc) is rigidly mounted on the axle hub and therefore rotates with the automobile's wheel. The pair of brake pad assemblies, which consist of friction material, backing plates and other components, are pressed against the disc in order to generate a frictional torque to slow the disc (and wheel's) rotation (see Fig. 1). The caliper houses the hydraulic piston(s) which actuate the pad assemblies. It is attached by a caliper mounting bracket to the vehicle. The methods and points of attachment depend on the type of caliper.

When a driver depresses the brake pedal, it effects an increase in hydraulic pressure in the pistons housed inside the caliper. The device which converts the brake pedal's motion to hydraulic pressure is known as the master cylinder. It is connected by brake lines and hoses to the disc brake's caliper. The master cylinder may be supplemented by another servo-device to boost the hydraulic pressure in the brake lines. We shall shortly discuss how the pistons and the seals between the piston and the caliper move the brake pad assemblies.

In the majority of automotive disc brakes, the disc is made of grey cast iron. This material is wear resistant and relatively inexpensive. According to Newcomb and Spurr [21], in order to protect the wheel bearings from the high temperatures induced in a braking action at the rotor-pad interface, the rotor is shaped like a top-hat. The hat section increases

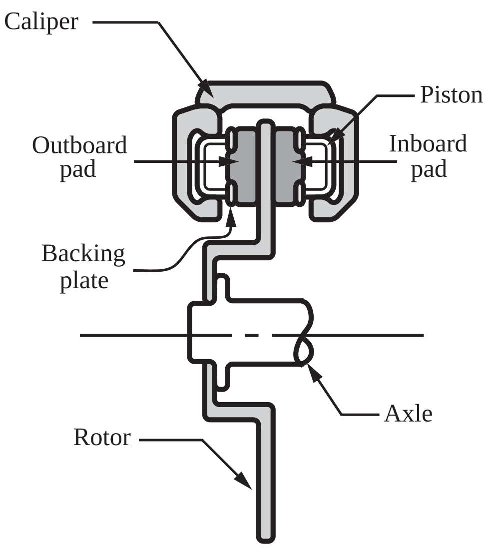

Fig. 1. Cross-section of a simplified disc brake. The wheel (not shown) is also attached to the rotor, typically at the axle flange. The disc brake shown in this Figure has a fixed caliper.

both the surface area (to improve cooling) and the length of the path that the heat must travel to affect the bearings. In vented discs, cooling is further enhanced by constructing the disc from two thinner discs connected by a series of thin fins (which are also known as ribs). A prime number (e.g., 31, 37 or 41) of fins is typically used in this case in order to inhibit symmetric modes of vibration in the disc. In high performance and customized disc brakes, the rotors are often slotted and/or drilled and are not necessarily composed of cast iron.

The brake pad assemblies consist of a friction material which is mounted to a rigid pad backing plate.[4] The mounting is achieved in a variety of manners: using rivets, using adhesives or integrally molding the friction material to the backing plate. The two brake pad assemblies in a single disc brake are usually distinguished as inboard and outboard. These assemblies are mounted to the caliper or the caliper mounting bracket in various ways. We also note that it is common to use squeal prevention measures, usually in the form of dampers (also known as shims) or damping substances, in the contact regions between each of the pad backing plate and the caliper, and the caliper mounting bracket and piston(s). These shall be discussed in further detail in a later Section of this paper.

Friction materials can be considered as composite materials which Anderson [25] classifies as organic, carbon-based, and metallic. The first of these are predominantly used in the automotive industry. According to Bergman et al. [26] and Jacko et al. [27], there can be up to 25 components of an organic brake pad which are collectively termed the friction material. These components can be divided into five categories:

(i) a matrix, which is composed of a binder and other materials,
(ii) fibres,
(iii) friction modifiers, which include metallic particulates,
(iv) mineral fillers, which serve to improve manufacturability,
(v) solid lubricants.

The binder is usually composed of a thermosetting polymer with some possible additions of rubber and cashew nut resin. Pertaining to the fibres, Anderson [25] notes that there are three types that are used: asbestos, non-asbestos organic, and resin bonded metallic (semimet or semi-metallic). Due to health issues and public awareness, substantial attention has been recently focused on friction materials with non-asbestos organic fibres (see, e.g., Loken [28]).[5] The metallic particulates serve to control the wear and thermal properties of the friction material and the solid lubricants serve as stabilizers for the friction coefficient (see Ref. [31]). The precise mixture of the five components employed in a friction material depends on the wear, thermal range of operation and friction coefficient required. The operating temperature range discussed in the literature is usually from 0°C to 500°C. At the upper extremes of this range, the wear rate of the friction material increases exponentially (see Refs. [25,27,31,32]). Although examples of various mixtures

[4] It is common to refer to the brake pad assemblies as brake pads, and (with some confusion) friction material also as brake pads.

[5] Some 80 years after Herbert Frood's ((1864–1931) and the founder of Ferodo Ltd.) pioneering use of asbestos fibres in friction materials, in the mid-1980s the EPA proposed to ban asbestos in, among other products, friction materials. However, a US Court of Appeals found this ban unwarranted [29]. In Europe, the EC's directive 98/12/EC prohibits asbestos in brake friction materials [30].

of the five components can be found, e.g., in Refs. [27,28,33,34], the precise constitutions of most commercial friction materials are trade secrets.

There are numerous designs of calipers found in disc brake systems: including the so-called sliding, fixed and floating. The caliper serves to house the hydraulically activated pistons. For the sliding and floating designs, the caliper is supported vertically by a caliper (mounting) bracket. This bracket is attached to either the steering knuckle or the axle housing. A fixed caliper is rigidly mounted to the vehicle so that it may transmit the braking torque and it contains a set of one or more pistons for both the inboard and outboard pad assemblies (see Fig. 1). The pistons are hydraulically linked so as to equalize the actuating force between the assemblies. On the other hand, sliding and floating calipers usually contain one piston (see Fig. 2), and their mounting bracket allows them to travel in the direction transverse to the disc while transmitting braking torque to the vehicle. Specifically, as the piston forces the inboard pad assembly against the disc, the reaction forces the caliper to move in the opposite direction. As a result, the attached outboard pad assembly is pressed into contact with the other side of the disc. In this way, the normal force on both sides of the disc in contact with the pads is equalized.

The amount of travel of the pad assemblies during a braking is small. In essence, the pad assemblies are always in glancing contact with the rotor. Several sets of seals are present between the caliper wall and the piston. Among other functions, the seals act as wear adjusters and guard against the leakage of brake fluid. When the brake pedal is depressed and, as a result, the

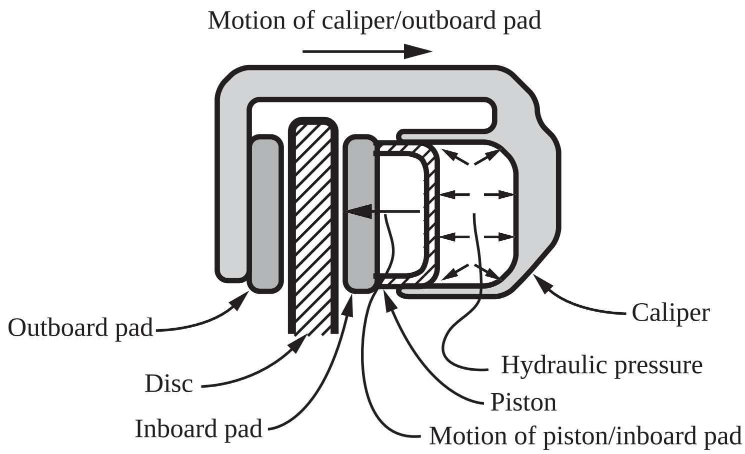

Fig. 2. Schematic of a simple floating caliper design for a disc brake.

Table 1
Dissipation rates in brakes for stops from 70 mph at impending brake lockup

Make & model	Initial kinetic energy (kJ)	Stopping distance (m)	Average dissipation (kW)	Maximum dissipation (kW)
Ford Explorer	1031.6	61.3	134.4	526.9
Porsche Boxster	656.2	48.8	139.6	434.9
Honda Civic LX	529.6	59.4	73.4	278.8

hydraulic pressure exerted on the piston(s) is increased, the action of the piston causes the friction material in the brake pads to press further against the rotor. Then, when the driver's foot is removed from the brake pedal and the pressure exerted on the piston is reduced to its nominal value, the brake pad assembly in contact with the piston is retracted using the seals just mentioned.

2.3. Energy dissipation

The purpose of the present section is to give the reader some perspective on the dissipation rates sustained in modern disc brake systems. To this end, Table 1 contains pertinent data for some representative production automobiles.[6] As approximately 70–80% of a vehicle's braking power is in the front brakes, it can easily be seen from data in Table 1 that, during a braking action, each front brake may be dissipating energy at a rate of over 50 kW at an interface the size of a brake pad. Nearly all of this mechanical dissipation takes the form of heat generation at the interface. However, sounds of over 100 dB (as measured at a reasonable distance from the source) represent orders of magnitude less mechanical power.[7] Thus, any mechanism which transforms even a tiny fraction of the dissipated mechanical energy into sound represents a serious potential squeal problem.

The technical difficulties that need to be overcome in order to develop disc brakes are also evident from Table 1. In contrast to drum brakes, the contact area where the friction forces are active is far smaller. As a result, the temperatures experienced in the contact area have the potential to be higher than in drum brakes. This issue has promoted vented brake rotors. It is also interesting to note that drum brakes, because they feature a moment arm, require a smaller actuation force than disc brakes. As a result, the development of hydraulic actuation systems and friction materials for disc brakes capable of withstanding higher pressures and stresses, respectively, are some of the reasons why it took so long for disc brakes to be incorporated in automobiles.

3. Background on vibration and waves

In subsequent sections of this review, considerable attention will be focused on the vibration of disc brake assemblies. Understanding some of the issues associated with this topic necessitates a certain amount of background material which we will now provide.

3.1. Modelling the brake rotor

Modelling a disc brake assembly with the aim of eventually understanding its vibrations and dynamics is complicated by the fact that the brake disc is rotating and the pads and caliper are fixed. A further complication, which has attracted much attention and a variety of models, is the

[6] The vehicle data for Table 1 was taken from Refs. [35–37] and, in our calculations, we assumed constant deceleration.

[7] For example, a sound pressure level of 100 dB measured a distance of 2 m from a source, has a sound power of approximately 0.25 W. Compared to the power being dissipated during braking this is a miniscule quantity.

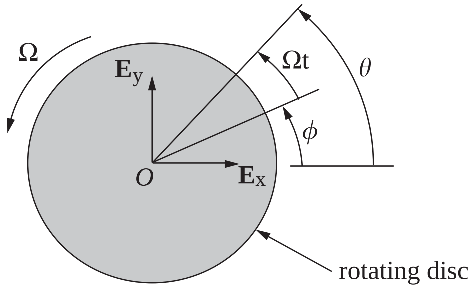

Fig. 3. Schematic of a rotating disc and the angular co-ordinates θ and ϕ. The disc is assumed to be rotating counterclockwise at a constant speed Ω. In this figure, the vectors $\mathbf{E}_x$, $\mathbf{E}_y$ and $\mathbf{E}_z = \mathbf{E}_x \times \mathbf{E}_y$ form an orthonormal, right-handed basis.

nature of the contact between the pads and the disc. Arguably the models for the rotor display the greatest diversity: they include the single-mode approximation of Ref. [38], the beam model of Hultén and Flint [39], the plate models of Ouyang et al. [40–42] and Tseng and Wickert [43] and the finite element models of Baba et al. [44], Bae and Wickert [45], and Kung et al. [46].

To understand some of the issues associated with the vibration of the brake rotor, it is convenient to start with a discussion of a particular plate model for this body. In this model, the rotor is modelled as a uniform circular plate of thickness h, inner radius r_i and outer radius r_o. The inner radius is assumed to be fixed (or clamped) to a rigid axle. For any point of the disc, the displacement vector $\mathbf{u}$ can be represented as

$$\mathbf{u} = u_R(R, \phi, z, t)\mathbf{e}_r + u_\phi(R, \phi, z, t)\mathbf{e}_\phi + u_z(R, \phi, z, t)\mathbf{E}_z, \tag{1}$$

where R, ϕ and z are a cylindrical polar co-ordinate system. Referring to Fig. 3, these co-ordinates are Lagrangian or material. In Eq. (1), we have used the unit vectors, $\mathbf{e}_r = \cos(\phi)\mathbf{E}_x + \sin(\phi)\mathbf{E}_x$, and $\mathbf{e}_\phi = -\sin(\phi)\mathbf{E}_x + \cos(\phi)\mathbf{E}_y$, and assumed that $\mathbf{E}_z$ points along the rigid axle. The displacements u_R and u_ϕ are known as the in-plane or longitudinal displacements of the disc.

The plate model assumes that the transverse (out-of-plane) vibration of the rotor is independent of z, $u_z = u_z(R, \phi, t)$, and is superposed on the steady state radial and tangential deformations that are induced by rotating the axle at a constant rotational speed Ω. The resulting equations of motion are (from Ref. [47])

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = \frac{h}{R}\frac{\partial}{\partial R}\left(\sigma_{RR} R \frac{\partial u_z}{\partial R}\right) + \frac{h\sigma_{\phi\phi}}{R^2}\frac{\partial^2 u_z}{\partial \phi^2} - \frac{Eh^3}{12(1-\nu^2)}\nabla^4 u_z + q. \tag{2}$$

Here, $q = q(R, \phi, t)$ represents the contribution of external forces and moments, E is Young's modulus, ρ is the mass density, ν is the Poisson ratio, and $\nabla^4 = (\nabla^2)^2$ with

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}. \tag{3}$$

The stresses σ_{RR} and $\sigma_{\phi\phi}$, which are sometimes known as membrane stresses, are the stresses induced in the plate by the rotation.[8] Although the rotation at angular speed Ω does induce

[8] The lengthy expressions for these stresses, which are functions of r_o, r_i, Ω, E and ν can be found in the literature on spinning discs (e.g., Ref. [47]).

stresses in the plane of the disc, the resulting strains are assumed to be sufficiently small that ρ, r_o, r_i and h are negligibly different from those of the stationary disc.[9]

The plate model (2) has two limiting cases. First, setting σ_{RR} and $\sigma_{\phi\phi}$ to zero in Eq. (2) is known as "ignoring the centrifugal effects". On the other hand, setting h to zero in Eq. (2) produces a model for the spinning disc which ignores bending effects. Clearly, such a model for a brake rotor would be poor to say the least.

In models of disc brake systems, the term q in Eq. (2) can be used to incorporate the effects of the brake pads. These may be modelled in a variety of manners. For instance, Ouyang et al. [40–42] used a finite element model for the brake pads, while Nishiwaki [48] modelled the brake pads as a deformable body with appended massless springs.

We have written the equations of motion of the plate in terms of the Lagrangian (or material) co-ordinates R and ϕ. Owing to the rotation of the disc, and because the radial and tangential displacements induced by this rotation are assumed negligible, it is common to use a set of Eulerian (or spatial) co-ordinates r and θ to describe the equations of motion of the disc (see Fig. 3). Based on the aforementioned assumptions, the spatial and material co-ordinates are related:

$$r \approx R, \quad \theta = \phi + \Omega t. \tag{4}$$

Any function of R, ϕ, and t can then be written as a different function of r, θ, and t:

$$f(R, \phi, t) = f(r, \theta - \Omega t, t) = \tilde{f}(r, \theta, t). \tag{5}$$

Well-known identities are then used to transform Eq. (2) to its Eulerian form. Most of the experimental results in the literature for the vibration of the brake rotor are expressed as functions of r and θ.

3.2. Standing waves and travelling waves

In the absence of the loadings due to brake pad assemblies, the brake rotor is usually considered to be axisymmetric. Returning to the plate model (2), setting $q = 0$ and ignoring centrifugal effects, we find that

$$\rho h \frac{\partial u_z}{\partial t^2} + \frac{Eh^3}{12(1-\nu^2)} \nabla^4 u_z = 0. \tag{6}$$

As is well known, the general solution of this equation is a doubly infinite sum of eigenmodes:

$$\begin{aligned} u_z(R, \phi, t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{mn}(R) \sin(\omega_{mn} t)(A_{mn} \sin(n\phi + \psi_n)) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{mn}(R) \sin(\omega_{mn} t)(B_{mn} \sin(n\phi) + C_{mn} \cos(n\phi)). \end{aligned} \tag{7}$$

[9] The plate model (2) does not account for any shear stress $\sigma_{R\phi}$ that might be induced by the presence of a sufficiently large external force system. This effect is incorporated into the model of Tseng and Wickert [43].

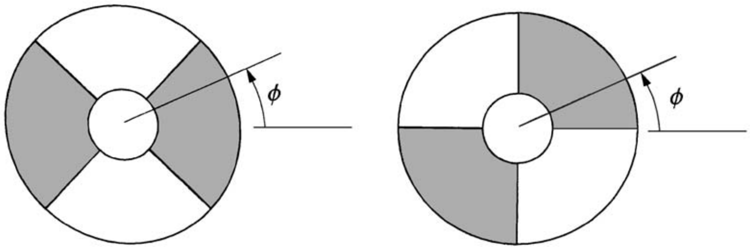

Fig. 4. Two doublet modes of u_z for an annular disc which have $m = 0$ nodal circles and $n = 2$ nodal diameters. The shaded parts of these figures have upward displacement: $u_z > 0$.

Here, A_{mn}, $B_{mn} = A_{mn}\sin(\psi_n)$, $C_{mn} = A_{mn}\cos(\psi_n)$, and ψ_n are constants, and ω_{mn} is the natural frequency of the mode which has n nodal diameters and m nodal circles.[10] We can also write the Eulerian representations of Eq. (7) as

$$\begin{aligned}\tilde{u}_z(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{mn}(r)\sin(\omega_{mn}t)(A_{mn}\sin(n\theta - n\Omega t + \psi_n)) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{mn}(r)\sin(\omega_{mn}t)(B_{mn}\sin(n\theta - n\Omega t) + C_{mn}\cos(n\theta - n\Omega t)). \end{aligned} \quad (8)$$

Both Eqs. (7) and (8) express the vibration of the plate as a linear superposition of standing waves.

The solutions involving B_{mn} and C_{mn} can also be interpreted as the infinite sum of singlet modes and doublet modes. A singlet mode is a mode which has no nodal diameters (i.e., $n = 0$), while a doublet mode is a member of a pair of modes having the same frequency ω_{mn}, and the same number of non-zero n nodal diameters and m nodal circles. For example, from Eq. (7), $C_{30}F_{30}(R)\sin(\omega_{30}t)$ is a singlet, while $B_{02}F_{02}(R)\sin(\omega_{02}t)\sin(2\phi)$ and $C_{02}F_{02}(R)\cos(\omega_{02}t)\cos(2\phi)$ constitute the elements of the doublet mode with two nodal diameters and zero nodal circles (see Fig. 4). As remarked in Ref. [49,50], the linear combination of the sin and cos doublet pairs serve to fix the nodal diameters in an axisymmetric structure. We shall return to these modes in a later section, Section 7.5, on theories of brake squeal.

For the plate model of a circular disc, where $q = 0$ and the centrifugal effects are ignored, the frequencies ω_{mn} are independent of Ω. Tabulations of ω_{mn} as functions of the dimensions of the plate and its elastic properties are easily found in the literature. When the centrifugal effects are included and q due to the brake pad assemblies are incorporated, an exact solution to Eq. (2) is not available. However, it is common to use modes (7) or (8) in conjunction with various approximation and perturbation methods to analyze the dynamics of the disc (see, e.g., Ref. [51]). For axisymmetric discs, a topic of considerable interest is how the natural frequencies of the doublet modes might split apart when the symmetry of the disc is broken (see, e.g., the papers [44,49,51–55]). This splitting can occur either when a $q \neq 0$ is applied to the disc, the disc is rotated

[10] A nodal circle occurs at points of the disc where the function $F_{mn}(R) = 0$. A mode has m modal circles if $F_{mn}(R)$ has m non-trivial zeros.

or when certain features, such as bolt assemblies or geometric imperfections, destroy the rotational symmetry of the disc.

In a seminal paper on spinning discs, Lamb and Southwell [56] pointed out that either of solutions (7) and (8) can also be considered as the superposition of travelling waves. To see this, one uses trigonometric identities to find that

$$2\sin(\omega_{mn}t)\sin(n\theta+\psi_n)=\sin(n\theta-\omega_{mn}t+\psi_n)+\sin(n\theta+\omega_{mn}t+\psi_n),$$

$$2\sin(\omega_{mn}t)\sin(n\theta-n\Omega t+\psi_n)=\sin(n\theta-(\omega_{mn}+n\Omega)t+\psi_n)+\sin(n\theta+(\omega_{mn}-n\Omega)t+\psi_n). \quad (9)$$

In other words, each of the eigenmodes in Eqs. (7) and (8) can be considered as the sum of two travelling waves.[11] Viewing the motion of the disc as the superposition of travelling waves has the advantage that one immediately perceives the symmetry breaking effect of the rotation of the disc. It should be noted from Eq. (9) that backward travelling waves when viewed by an observer fixed in space, will, for the same natural frequency ω_{mn}, travel slower than forward travelling waves.

3.3. Some issues associated with plate models for brake rotors

Experimental studies on brake squeal have primarily classified the modes of the brake rotor using ideas from the dynamics of spinning discs. In particular, the number of nodal diameters is used to distinguish different modes. Although this does not appear to have caused any controversy, this classification ignores the fact that the brake rotor is a composite structure containing a top-hat and fins, and the influences of the brake pad contact. These effects can produce deformations in the rotor which are not accounted for by the plate theory we have mentioned. Here, we discuss some works which explore these deficiencies.

Bae and Wickert [45] developed a finite element model of the brake rotor to partially examine the influence of the hat on the modes of the brake rotor. Their results, in Section 5 of Ref. [45], showed that it is possible in a stationary brake rotor to have two distinct modes with the same number of nodal diameters and nodal circles but different frequencies. The two modes are distinguished by the fact that their in-plane displacement fields will be different. They also found that by adjusting the dimensions of the hat, certain transverse (bending) modes in their model could be eliminated. For brake squeal analysis, these two results are interesting. The former suggests that the popular method of categorizing modes in brake assemblies using nodal diameters of the brake rotor may not be accurate, while the latter result gives a method to tailor the modes of the brake rotor.

Partially motivated by the need to have an analytical model for a disc brake rotor, another model was discussed by Meel [57]. In this work, the rotor was modelled as two annular discs which are attached by ribs. A rigid hub is then mounted on top of one of the annular discs. Although the annular discs and ribs are both modelled using the three-dimensional theory of linear isotropic elasticity, several assumptions are placed on the displacement fields of these

[11] Recall that a wave $f(r,\theta-ct)$ where $c>0$ is a forward travelling wave with respect to a fixed observer, while a wave $f(r,\theta+ct)$ where $c>0$ is a backward travelling wave with respect to a fixed observer. Similarly, $g(R,\phi-ct)$ and $g(R,\phi+ct)$ are, respectively, forward and backward travelling waves when viewed by an observer who is fixed on the disc.

bodies. After using the Rayleigh–Ritz technique to obtain an approximate solution to the equations of motion for each of the discs, the assumed modes method is used to find the equations of motion for the free vibration of the rotor. The resulting vibration analysis is compared to a finite element analysis for validation purposes.[12] One of the interesting results of this work is the difficulty in developing a tractible model which is useful for analyzing high-frequency modes of a disc brake rotor.

It is well known (see e.g., the paper [59]) that the presence of the brake pads contacting the rotor will, by the Poisson effect, couple the transverse and in-plane deformations of the rotor. Finite element models of rotors (see, e.g., Refs. [57,58]) can accommodate this effect. Another avenue, which has not yet been explored, is to use plate theories which have the ability to model both the transverse and in-plane deformations of the rotor. Such theories have been developed by Green and Naghdi and are discussed in Refs. [60,61].

3.4. Linear vibration analysis

In the development of models for squealing disc brakes, one goal is to examine the stability of small-amplitude vibrations of the brake assembly. As will be seen in later sections, many researchers believe that the onset of instability coincides with the squealing. We shall adopt this viewpoint in this Section.

The canonical form of the equations governing the linear vibrations of the disc brake assembly is

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{G} + \mathbf{D})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{10}$$

Here, $\mathbf{x} = \mathbf{x}(t)$ is an n-dimensional vector, $\mathbf{M}$ and $\mathbf{D}$ are the respective $(n \times n)$ mass and viscous damping matrices, $\mathbf{G}$ is the skew-symmetric gyroscopic matrix, $\mathbf{K}$ is an $(n \times n)$ matrix, and the dot denotes derivative with respect to time. The components of $\mathbf{x}$ consist of the individual amplitudes of the assumed modes of vibration of the disc brake assembly. An important caveat here is that neither $\mathbf{M}$ nor $\mathbf{K}$ are necessarily symmetric (see, e.g., Eq. (1.19) of Ref. [59]).

To examine the stability of the equilibrium $\mathbf{x} = \mathbf{0}$ of (10), the solutions λ of the characteristic equation,

$$\det(\mathbf{M}\lambda^2 + (\mathbf{G} + \mathbf{D})\lambda + \mathbf{K}) = 0, \tag{11}$$

need to be determined. As will be seen later, varying the parameters of the disc brake assembly to see when any of the $2n$ roots λ of (11) develop positive real parts is often used as a criteria for brake squeal. In particular, if we denote the real part of λ by σ and the imaginary part by ω, then some authors, such as Millner [62] define a "squeal propensity" to be the largest value of σ among the $2n$ roots λ. Other authors, such as Flint [63] and Yuan [64], use a different, but obviously related, definition based on the ratio $\sigma/(\sqrt{\sigma^2 + \omega^2})$.

Examples of equations of form (10) can be found in numerous works on disc brake squeal (see, e.g., Refs. [42,43,46,48]), and the matrices $\mathbf{M}$, $\mathbf{G}$, $\mathbf{D}$ and $\mathbf{K}$ are presumed to depend on the parameters of the brake system. The motivation for analyzing (10) is that squeal is assumed to coincide with the instability of the equilibrium $\mathbf{x} = \mathbf{0}$. The non-symmetry of $\mathbf{K}$ is attributed to the

[12] A related model is discussed in Ref. [58].

follower force nature of the dynamic Coulomb friction forces between the pads of friction material and the rotor. Discussions on the follower force nature of the friction forces can be found in the review by Mottershead [51]. In particular, Chen [65] and Ono et al. [47] point out that a follower force $F_\phi \mathbf{e}_\phi$ acting at a point $R = R^*,\ \phi = \phi^*$ on the surface of the plate would introduce a term

$$q = -\left(\frac{\delta(R - R^*)\delta(\phi - \phi^*)}{R^*}\right)\frac{F_\phi}{R^*}\frac{\partial u_z}{\partial \phi} \tag{12}$$

in Eq. (2). Here, $\delta(\cdot)$ is the Dirac delta function.

The structure of Eq. (10) is unusual because of the simultaneous presence of a non-zero $\mathbf{G}$, and a $\mathbf{K}$ and an $\mathbf{M}$ which are not necessarily symmetric. Systems such as Eq. (10) where $\mathbf{G} = \mathbf{0}$ and $\mathbf{M}$ is symmetric, have been extensively studied in the context of follower forces in mechanical systems (see Refs. [4,66]), and, where $\mathbf{K}$ and $\mathbf{M}$ are symmetric, in the context of gyroscopic systems (see, e.g., Ref. [67]). However, there are few general results known for linear systems of form (2). Several studies of brake squeal however ignore the rotation of the brake rotor and, consequently, set $\mathbf{G} = \mathbf{0}$. They also feature models where $\mathbf{M}$ is symmetric. In this case, as the friction force is a follower force and such forces can destabilize the equilibrium of Eq. (10), it is easy to anticipate that the friction force has the capability of inducing squeal.

4. Background on contact, temperature, and wear

The central purpose of a disc brake is to exploit the friction between the pads and rotor to dissipate the kinetic energy of the vehicle. Accurate modelling of the frictional forces is one of the crucial elements in the analysis of any disc brake system. Unfortunately, the choice of a correct or even adequate model is non-trivial. Some of the main reasons for this difficulty are the effects of temperature and wear. To appreciate these matters, some background is necessary, and in the present section we gather parts of this knowledge.

This section is organized into three parts as follows: First, we discuss some friction models that have appeared in the literature and establish some of our notations for later sections of this paper. Next, the issues of temperature and wear in the context of braking systems are examined. Finally, additional studies in the literature on temperature and pressure fields in disc brakes are discussed.

4.1. Friction and contact

This section introduces some terminology and models from the field of tribiology. Our purpose is two-fold: first, to provide some necessary background, and second to point out the state of knowledge in frictional contact which may serve to improve existing models of brakes. Given the limited space, we cannot do justice to all the work done on developing friction models. The interested reader is referred to the texts of Bowden and Tabor [68], Kragelskii [69], Rabinowitz [70], and Suh [71] for further details and references. We also mention the historical treatise of Dawson [72]. Comprehensive reviews of friction models from the perspective of dynamics and controls are provided by Armstrong–Hélouvry [73], and from the perspective of computational mechanics by Oden and Martins [74].

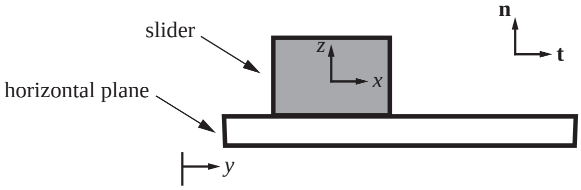

Fig. 5. Schematic of a slider on a horizontal foundation. The displacement of the slider is assumed planar and equal to $x\mathbf{t}$, while the displacement of the foundation is $y\mathbf{t}$.

The mathematical modelling of the frictional forces acting at the contact interface has a long and rich history. Although, several models have been, and continue to be, proposed and developed, the classical model for friction is the one most commonly adopted in studies on disc brake squeal.

To discuss the classical model for friction, it is convenient to consider the motion of a slider on a flat, rigid foundation (see Fig. 5). The unit normal to the surface of the slider at the contact interface is denoted by $\mathbf{n}$ and the resultant normal force acting on the slider is denoted by $\mathbf{N} = N\mathbf{n}$. It is assumed that the motion of the system occurs in the direction of the unit vector $\mathbf{t}$. Consequently, the motion of the slider has the representation $x\mathbf{t}$, where $x = x(t)$ is the displacement of the slider. Similarly, the motion of the foundation is assumed to be $y\mathbf{t}$. The sliding velocity of the slider is $v_s = \dot{x} - \dot{y}$, and the resultant friction force on the slider is $\mathbf{F}_f = F_f\mathbf{t}$. Due to the presence of asperities on the upper surface of the foundation and lower surface of the slider, the actual area of contact A_r and the apparent area of contact A will not be equal.

Amontons' two laws of friction [72], which date from 1699, are (i) F_f is independent of A and (ii) F_f is directly proportional to N. These laws pertain to the case where there is relative motion between the slider and the plane. As a consequence of his second law, the coefficient of kinetic (or dynamic) friction μ_k is independent of N. These laws are supplemented by the observations that the friction force opposes the relative motion:

$$\mathbf{F}_f = F_f\mathbf{t} = -\mu_k N \frac{\mathbf{v}_s}{||\mathbf{v}_s||}, \tag{13}$$

where the sliding velocity $\mathbf{v}_s = v_s\mathbf{t}$. We also note that, according to Kragelskii (p. 178 of the book [69]), in 1785 Coulomb showed that μ_k could be a function of v_s. Supplementing Amontons' laws, (13), and Coulomb's observations that μ_k could depend on v_s, is the belief that the static coefficient of friction μ_s is greater than μ_k.[13] We shall collectively refer to these laws, this observation and this assumption as the classical friction model.

Several generalizations of the classical friction model are available in the literature, e.g., Bowden and Tabor's adhesion and ploughing theory [68], state variable friction laws [75,76], and Suh and Sin's theory [71,77]. But, probably because existing models for brake systems are so complex and exhibit a rich array of dynamical behaviours, these friction laws have not (yet) been adopted by the community researching brake noise.

It is known that μ_k can be non-linearly dependent on v_s (see, e.g., Figs. 6.18–6.22 in the book [69]). This functional dependency has attracted considerable attention both in studies of

[13] Dawson [72] credits the conclusion that $\mu_s > \mu_k$ to Euler's work in 1748.

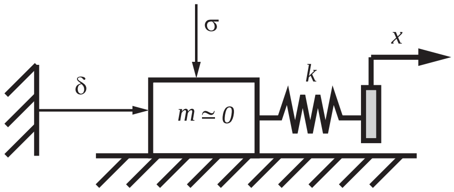

Fig. 6. Ruina's idealized friction testing machine with finite stiffness that is discussed in Ref. [76]. A displacement x is supplied to the machine by a screw or hydraulic ram and induces a displacement δ in the slider. The spring constant k represents the stiffness of the slider and the machine, σ is the normal force acting on the slider and $\dot{\delta} = v_s$.

non-linear dynamical systems featuring frictional forces and in theories of brake noise. However, it has also been the subject of some pointed discussions, see e.g., Refs. [74,76,78,79]. As pointed out by Tolstoi [79], the dependency of the friction force on the normal displacement between the contacting bodies can, in many cases, be the source of an apparent dependency of μ_k on v_s. This may be the reason why most modern theories of brake squeal, which feature multi-degree-of-freedom models for the brake system and incorporate (to varying extents) the flexibility of the pads of friction material and displacements of these pads relative to the rotor, assume that μ_k is independent of v_s.

A further issue concerning the dependency of μ_k on v_s was raised by Ruina [75]. He pointed out that the inherent instability of a friction characteristic where $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$ makes it impossible to observe this phenomenon in a friction experiment. According to Ruina, the frictional behaviour of material pairs is typically investigated via steady state sliding experiments in friction testing machines, of which an idealized version appears in Fig. 6. Such a machine, he says, has an elastic rather than viscous response to sudden displacements. Therefore, a coefficient of friction μ_k, where $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$, rules out the possibility of steady state sliding as the dynamics of this machine leads to a negative damping and unstable oscillations.[14] Thus, the appearance of a μ_k where $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$ in a steady state sliding experiment must indicate the presence of a more complicated constitutive relationship for friction. In his work [76], Ruina suggested (more general) state variable friction laws as a possible explanation.

The friction coefficient μ_k in the contact region between the pads of friction material and the brake rotor is found, using brake dynamometer tests, to range from 0.2 to above 0.6 in the literature. It should be clear that these values of μ_k are not valid pointwise throughout the contact region between the pads of friction material and the brake rotor.[15] Rather, they are average values for both pads. Sources of the variation in μ_k could be the brake line pressure p_B which ranges from 0 to 4 MPa, and the temperature of the rotor which ranges from 0°C to 500°C in the literature (see, e.g., Refs. [83–85]). Because of this range of dependencies and the fact that the rotor temperature will change during a braking, Eriksson et al. [86] have aptly commented that it is difficult to determine if μ_k depends on v_s for a particular braking system.

[14] We shall discuss this matter later—see Eq. (20).

[15] Finite element-based predictions of the interface pressure and friction forces in disc brakes can be found in the papers [80–82]. It is clear from these works that neither of these fields are uniform.

Of course, μ_k also depends on the wear properties of the brake—a subject to which we now turn.

4.2. Temperature and wear

Most researchers realize the tremendous influence temperature plays on the design and performance of brakes. However, accurately modelling the thermomechanical properties of braking systems has proven to be very difficult. This is partially due to the fact that the brake pads are complex anisotropic materials. Prior to discussing some of the researches performed on temperature and wear in disc brake systems, we pause to present some background.

When two surfaces slide over each other, the heat Q produced is proportional to the sliding velocity and frictional traction t_f:

$$Q = \int_{\mathscr{A}_r} t_f v_s \,\mathrm{d}\mathscr{A}. \tag{14}$$

Here, $\mathscr{A}_r$ is the actual area of contact and t_f is the frictional force per unit area. When Coulomb's theory of dynamic friction is employed, then $t_f = \mu_k p$ and Q becomes a linear function of the normal traction p. It should be obvious that the area of contact $\mathscr{A}_r$, the tractions p and t_f, and the sliding velocity field v_s will depend on the thermomechanical properties of the two contacting bodies.

Pertaining to wear, most arguments in the literature on disc brakes use a version of Archard's classical wear model. This model states that the rate of wear $\dot{W}$ is proportional to the pressure:

$$\dot{W} = \int_{\mathscr{A}_r} K\left(\frac{p v_s}{3H}\right) \mathrm{d}\mathscr{A}, \tag{15}$$

where K is the wear coefficient and H is the hardness of the softer material. A critique of Archard's model is contained in Ref. [71]. Studies of wear in disc brake systems have been the subject of several investigations, among them, Eriksson et al. [87], Österle et al. [88], and Rhee et al. [85]. In addition, Day et al. [80] have presented some numerical simulations of wear in the pads of friction material in a disc brake.

4.3. Hot spots and thermoelastic instability

Of particular interest in braking systems are situations where the contact between the brake pads and the rotor results in the formation of hot spots. Hot spots can lead to thermal distortion of the brake pads and rotor and (solid–solid) phase transformation of the brake rotor. The thermal distortion may result in a low-frequency instability known as *judder* (cf. Refs. [2,89,90]) and may be a key component in the Rhee et al. [91] hammering theory of brake squeal. The formation of hot spots is generally attributed to a thermoelastic instability (TEI).

As originally proposed by Barber [92,93], TEI may arise when two sliding surfaces are in contact. In this case, the actual contact between the two surfaces initially occurs in some localized regions. As these regions transmit the contact forces between the bodies, large pressures can develop. According to Barber [93], these localized regions distort due to the presence of temperature gradients which are an inherent part of the frictional heating process. In particular, the distortion results in increasing the localized regions of high contact pressures. However, these areas of high contact pressure are also more susceptible to wear, and the wear produces a *smoothing* effect whereby the contact area becomes progressively less localized. In a stable state, the rate of wear and the rate of thermal expansion are in equilibrium and localization of the actual contact is not promoted. However, if the rate of wear cannot overcome the rate of thermal expansion of the contact area, then an instability will arise. Barber argued that this instability, which he called TEI, leads to the formation of localized regions of contact which will have to sustain high pressures and, consequently, high temperatures.

Following Barber's work, Dow and Burton [94] showed that TEI could be achieved in the absence of wear. In particular, they established a necessary criterion for the growth of a disturbance to the temperature or pressure fields of a semi-infinite blade sliding on a rigid surface. Their work showed that beyond a critical sliding speed, a TEI would occur. The resulting instability is often referred to as *friction-induced* TEI. Another notable extension to Barber's work was presented by Kennedy and Lin [95]. In this work, a finite element analysis of the axisymmetric thermoelastic deformation of an annular disc brake was presented. Kennedy and Lin also pointed out that disc brakes have the unusual feature that the sliding velocity of the pad relative to the rotor is a linear function of the radius. Consequently, the heat generated Q will be non-uniform through the contact interface, and this will influence the pressure distribution in the contact interface.

Pressure distributions on the disc-pad of friction material interface under static conditions have been computed by Samie and Sheridan [96] using a compliance-based method. They also validated the results by experimental measurement. Tirovic and Day [82] also used finite elements to investigate the dependence of pressure on parameters such as the compressibility of the friction material, the stiffness of the backing plate and the flexure of the caliper. In a companion paper [80], an uncoupled thermomechanical analysis is conducted to study the effect of disc coning (due to friction-induced steady heat flux) on the pressure distribution.

The geometries analyzed by Dow and Burton [94] and Kennedy and Lin [95] were not representative of actual automotive disc brakes. Since their work, there has been much progress in establishing more realistic models for a TEI. These models can be found in the works of Barber and his co-workers [90,97–101]. With the exception of Ref. [101], all of these works assume a steady state process. We also remark that although considerable attention has been placed on accurately representing the geometry of the brake pads and rotor, to date there has been little attention paid to the role of anisotropy of the brake pads in the analysis of TEI.

We have only discussed a small portion of the published work on TEI. For further details on this instability and its importance in automotive brakes, we refer the interested reader to Anderson and Knapp [102], Barber et al. [100,103,104], Day et al. [80], and Kennedy [105].

5. Experimental studies on brake squeal

To discuss experimental studies on brake squeal it is convenient to first consider work prior to the mid-1970s. We refer to this period as the classical era. The 1978 work of Felske et al. [106]

where dual pulsed holographic interferometry (DPHI) was used to examine the modes of squealing brakes appears to have accelerated interest in publishing experimental results on brake squeal. Works based on DPHI dominated the literature until the use of another optical technique, electronic speckle pattern interferometry (ESPI), became popular in the late 1990s (cf. the papers [107,108]).[16] We shall categorize works whose primary focus is vibration measurement as vibration-based studies. Finally, starting in the late 1980s, an increased number of researchers have refocused interest in the role played by tribiology in squeal generation. We shall categorize works of this type as tribiology-based studies.

In reviewing the experimental literature on brake squeal, few details are provided by the vast majority of authors on the precise disc brake assembly they are using. As a result, given the wide range of designs and material selections that are possible for a brake system, it is almost impossible to compare results and conclusions from different research groups. In the forthcoming sections, the fact that most researchers analyzed different disc brake assemblies should be continually borne in mind.

A short summary of conclusions reached in the experimental literature and discussed in the next three sections is as follows:

(a) A braking system can squeal at a number of distinct frequencies.
(b) The amplitude of vibration of squealing brakes is on the order of microns.[17]
(c) The vibration of a squealing disc brake assembly where the rotor is spinning at a constant speed may, but does not have to, be a standing wave or a travelling wave.
(d) It is not clear if knowledge of the modes of the components of a disc brake system or of the modes of a stationary disc brake assembly can be used to infer a squeal propensity.
(e) There appears to be a consensus that the squealing frequencies are slightly lower than the natural frequencies of the stationary rotor.
(f) Increased μ_k between the rotor and pads of friction material increases the propensity for squeal.
(g) There is no consensus as to which component of the disc brake system contributes the most to squeal. Indeed, it is likely that the contributions of each of the components depend on the squealing frequency of interest.

Several other interesting results are also present in the literature to which we now turn.

5.1. Classical experiments

In the late 1930s, H.R. Mills [111] conducted one of the first sets of investigations on brake squeal. For drum brakes, he attempted to correlate friction pairs showing a decreasing coefficient of friction μ_k with increasing sliding velocity v_s to the occurrence of squeal. Unfortunately, he did not arrive at any definitive conclusions.

Nearly two decades later, Fosberry and Holubecki [112,113] tested disc brake systems extensively and reported that most instances of brake squeal coincided with a decreasing $\mu_k(v_s)$

[16] Further details on DPHI and ESPI can be found in Refs. [109,110].

[17] By way of contrast, the pads of friction material move approximately 25–75 μm during braking [29].

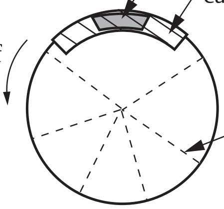

Fig. 7. Schematic of Fosberry and Holubecki's [113] modal results for a squealing rotor of a disc brake system. This Figure is adapted from Figs. 5 and 6 of Fosberry and Holubecki [113]. The asymmetry of the mode, which is partially due to the frictional contact between the pads and the rotor, should be noted.

function. However, on page 12 of Ref. [110] they cautioned that the friction characteristics of the brake pads did not show a complete correlation to brake squeal.[18]

Fosberry and Holubecki's experimental work on a disc brake system also indicated that the rotor is the resonant member, vibrating in transverse modes with diametral nodes while the centres of the brake pads are located near anti-nodal points of this vibration.[19] One of their reported mode shapes is reproduced in Fig. 7. As expected, the pads and caliper also vibrate during squeal. Indeed, Fosberry and Holubecki reported that (from p. 14 of Ref. [113])

> "During squeal, the friction pads and backing plate generally move with the disc, often with an amplitude of the samemagnitude; unsupported parts of the backing plate and the parts of the caliper vibrate in a more complex manner, however, with the caliper vibrations always very much smaller than the disc vibrations."

It is interesting to note that the same authors noted that the amplitude of disc vibration is on the order of 10^{-4} in. in some cases when the squeal is loud.

In later experiments by Spurr [114], brake pads were machined so as to only contact the rotor along a strip less than 1/8 in. wide. Spurr found that squeal was only generated when this contact strip was sufficiently close to the leading edge of the pad. This contact configuration caused the vector of the resultant contact force to pass through the pivot point supporting the pad (see Fig. 8). Squeal was ameliorated when the contact strip was moved toward the trailing edge of the pad. This phenomenon was subsequently correlated in cantilever-on-disc systems by Jarvis and B. Mills [38] and in several pin–disc systems by Earles and various co-workers [115–119].

In 1972, North [120] published an ambitious paper on squealing disc brakes. This paper marks the first one we found where experimental results on an actual squealing brake system and a model were correlated. North's experimental apparatus centred on a Lockheed disc brake system

[18] Paralleling their earlier work on squealing drum brakes, Fosberry and Holubecki coated the rotor with oils of known frictional characteristic (i.e., known $\mu_k(v_s)$) in order to ascertain whether the μ_k–v_s characteristic effected brake squeal. However, they noted that these experiments were not as conclusive as their earlier related experiments on drum brakes.

[19] As we noted earlier, a diametral node is a series of nodal points which lie along a ray emanating from the center of the disc. The interpretation of the disc vibration in Ref. [113] corrects some errors in the earlier paper [112].

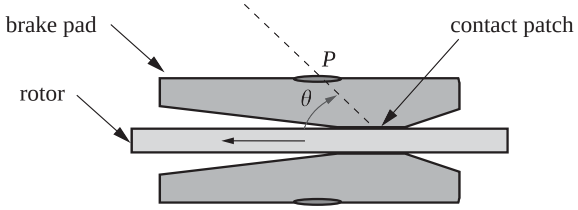

Fig. 8. Schematic of two brake pads contacting a rotor which is used to explain Spurr's sprag-slip theory of brake squeal. The angle θ defines the slope of the line connecting the pivot point P to the mid-point of a pad's contact area. Notice that the pads have been modified so that the contact area is displaced from the leading edge of the brake pad. This figure is adapted from Ref. [114].

which featured a single cylinder swinging caliper.[20] North instrumented his experiment using accelerometers to measure the vibration of the pads of friction material and the caliper, and a vibration meter was used to determine the rotor vibration. We shall discuss North's model later in Section 8, but it is interesting to note that the agreement between the disc vibration and the model's prediction was, according to North, "encouraging."

In Refs. [112,113], as in all other experimental works that we found, there was a discrete number of squeal frequencies in the audible range for a given braking system. As will shortly become evident, the nature of the vibrations associated with these frequencies has been the subject of much investigation and some controversy. Further, it has not been conclusively demonstrated that knowledge of the uncoupled modes of the stationary rotor, pads, and calipers is sufficient to determine the squeal frequencies.

5.2. Vibration-based works

To examine the vibration of a squealing disc brake assembly, Felske et al. [106], who were working for Volkswagen AG, began using holographic interferometry and, in particular, DPHI. Measurements of the vibrations of squealing brake systems were possible using this technique and several surprising results were found by various researchers in the 1980s and 1990s. Many of these results were confirmed and clarified by other researchers using another optical measurement technique, ESPI, in the late 1990s. As ESPI and DPHI can both be used to obtain three-dimensional displacement fields of the imaged object, these techniques can, in principle, isolate various squeal mechanisms.

In 1978, Felske et al. assumed that the mechanism for brake squeal was the coupled vibration of the disc brake assembly. To this end, their experimental work established that the entire disc brake assembly vibrated during squeal. One of the interesting findings in Ref. [106] is that

> The greater the coefficient of friction of the rubbing surface, the more the likelihood of squeal.

In other words, the larger μ_k is between the pads of friction material and the rotor, the higher the likelihood of brake squeal. Related comments are contained in the earlier paper by Spurr [114], and the later works by Murakami et al. [121], Nishiwaki et al. [122], and Ichiba and Nagasawa

[20] A good discussion of this early type of caliper can be found in Ref. [14].

[123]. If one considers brake squeal to be a friction-induced instability, then, among others, Millner [62] and Kung et al.'s [46] models for squealing disc brakes exhibit behavior consistent with Felske et al.'s remark.

During squeal, Felske et al. [106] were also able to establish that the pads of friction material vibrate in various bending modes, that no nodal circles and an even number of radial nodes were seen in the vibration of the disc, and that the main contribution to audible noise was the vibration of the pads of friction material and the brake calipers. These conclusions about the source of the noise seem to contradict the aforementioned conclusions of Fosberry and Holubecki [112,113]. By considering two types of calipers, what they called yoke- and fist-type, Felske et al. also pointed out that for some (but not all) braking systems, the squeal frequencies correspond to modes of the system's components.[21] Unfortunately, the amount of detail presented by Felske et al. on the precise meaning of "artificially excited disc brakes"—which we understand to mean the frequency response of the disc brake assembly when the disc is not rotating and the hydraulic pistons are not being actuated—makes it difficult to interpret some of their measurements and conclusions.

Felske et al. reported that the maximum amplitude of vibration of a pad of friction material in a squealing brake was 3 μm. This is consistent with the earlier cited amplitude reported by Fosberry and Holubecki. Other researchers have found similar results (see the papers [124,125]). Unfortunately, amplitudes of this magnitude make the development of tractable models for brake squeal very difficult.

Six years after Felske et al.'s work, Murakami et al. [121], working for the Nissan Motor Company, also examined squealing disc brakes using DPHI. They too believed that squeal was generated by coupled vibrations of the disc brake system. Supplementing the optical measurement technique, they used piezoelectric accelerometers which were mounted on several components of the braking system. Murakami et al.'s experimental results agreed with Felske et al.'s concerning the vibration modes of the disc and pads. They did not comment on the issue of whether or not the pads and calipers are the primary acoustic source of squeal. Murakami et al. noted that the amount of squeal increased when the natural frequencies of the pads, caliper, and brake rotor were close to each other. They cautioned that closeness of these frequencies is not a necessary condition for squeal generation. A further interesting conclusion that Murakami et al. reached was that squeal was more likely to occur when μ_k between the rotor and pads of friction material was a decreasing function of v_s than when it was a constant function of v_s.

Also in 1984, Ohta et al. [126] examined a squealing disc brake. Their experimental observations were performed using holographic interferometry, and were used to develop a lumped parameter model for a squealing disc brake. Ohta et al. showed that the maximum amplitude of vibration of a rotor in a squealing disc brake assembly is approximately 20 μm. Other interesting results in the paper [126] include very clear time traces of the vibrations of the disc, pads of friction material and supporting plate of a squealing disc brake assembly.

In their subsequent work, Nishiwaki et al. [122], working for the Toyota Motor Corporation, investigated the vibrational characteristics of squealing brake systems using DPHI. They also examined the individual modal responses of the rotor, brake pads and caliper using an

[21] Fist-type calipers are known as pin-slider or sliding calipers. We have been unable to ascertain the meaning of yoke-type caliper.

electromagnetic vibration exciter, found the modes and frequencies of the components to be quite similar under both conditions, and agreed with some of the above researchers that the disc vibrated in a diametral bending mode which was stationary with respect to a fixed observer even when the disc was rotating (see page 982 of Ref. [122]). During squeal, they found that the pads of friction material underwent bending vibrations. They concluded that brake squeal could be eliminated by modifying the brake rotor. The modification they considered eliminated a number of cooling fins in the vented brake rotor.

Nishiwaki et al. [122] also concluded that the vibration mode and frequency of a squealing disc brake's rotor are heavily influenced by the natural frequencies and modes of the stationary rotor. They based this conclusion on their observation that the squealing frequency and mode shape of the brake's rotor were close to a natural frequency and (corresponding) mode shape of a stationary rotor.[22]

Further holographic interferometry studies were recently performed on squealing disc and drum brakes by Fieldhouse and Newcomb [127,128] at the Universities of Huddersfield and Loughborough. They concurred with Fosberry and Holubecki [112,113], Felske et al. [106], and Nishiwaki et al. [122] that the disc exhibited purely diametral nodes, but found instances where the mode also travelled around the disc.[23] When the rate of rotation of the mode around the disc was uniform, then the speed of rotation depended on the frequency and mode order. In this case, the mode was a travelling wave. For instance, Fieldhouse and Newcomb [127,128] present an example of a squealing brake where the main frequency is 10750 Hz while the speed that the mode travels around the disc is 1344 Hz. As shown in Fig. 9, the mode shape has eight nodal diameters and the disc is rotating at 10 r.p.m.[24] This mode is a forward travelling wave:

$$w_z(r, \theta, t) = G(r)f(n\theta - \omega t), \tag{16}$$

where w_z is the amplitude of the transverse vibration of the upper surface of the rotor, r is the radial co-ordinate, θ is the Eulerian angular co-ordinate, n is the wave nodal number, and the phase speed is ω/n.[25] In addition, $G(r)$ is a function of r which describes the radial behavior of the vibration and $f(n\theta - \omega t)$ is a function which describes the angular and temporal behavior of the vibration. For the example just discussed, $\omega = 10750$ Hz, $n = 8$ and, as expected, $\omega/n \approx 1344$. It is important to note here that the wave form discussed by Fieldhouse and Newcomb is travelling both with respect to the disc and the ground. These authors also explicitly state that the speed of rotation of the mode around the disc is not always uniform (see, e.g., Fig. 14 of Ref. [127]).

[22] It is unfortunate that insufficient detail is presented in Ref. [122] to ascertain the boundary conditions on the stationary rotor that are used to determine its modal response. This criticism also pertains to several other papers we cite in this Section.

[23] A related mode in drum brakes is discussed, in chronological order, by Lang and Newcomb [129], Hultén et al. [130], Fieldhouse and Rennison [131] and Servis [132]. Servis developed a model for a drum brake assembly and showed how a travelling wave occurred after an equilibrium of this model became unstable because of a flutter instability (see Figs. 4.14–4.17 in Ref. [132]). We shall shortly discuss this instability in the context of disc brake squeal.

[24] Animations of the travelling motion of the mode are discussed by Talbot and Fieldhouse [125] and can be found at http://scom.hud.ac.uk/external/research/brakenoise/index.html

[25] The co-ordinates r and θ were defined in the earlier section on vibration and waves.

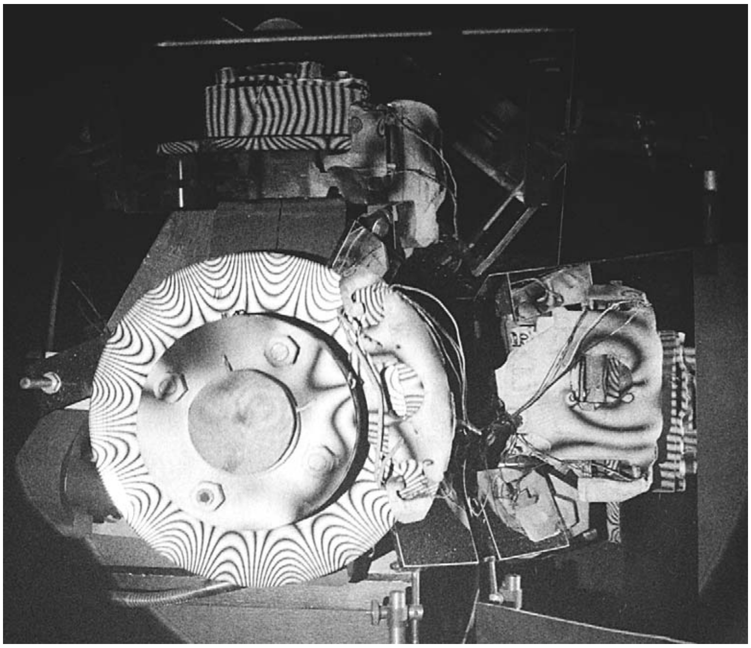

Fig. 9. Fieldhouse and Newcomb's reconstructed hologram of a squealing disc brake system. In this figure, the rotor is rotating counterclockwise at 10 rpm, the vibration of the upper surface of the rotor has eight nodal diameters, the frequency of the vibration is 10750 Hz, and the nodal diameters are rotating about the center of the rotor at $\approx$1344 Hz. This figure was generously supplied to the authors by, and is reproduced here with the permission of, J. D. Fieldhouse.

The DPHI results for the case when the rotational speed of the mode around the disc was not uniform were examined in further detail by Talbot and Fieldhouse [125]. By using a Fourier series, they showed in this case that a mode they observed with n modal diameters could be approximated as the sum of forward and backward travelling waves:

$$w_z(r, \theta, t) = G_1(r)\sin(8\theta - \omega t + \alpha) + G_2(r)\cos(8\theta + \omega t + \beta). \tag{17}$$

Here, α and β are constants and $G_1(r)$ and $G_2(r)$ are functions of r. These authors also found a similar representation for the tangential displacement w_θ of the rotor.

Fieldhouse and Newcomb's experiments in Refs. [127,128] also showed a more complicated mode shape for the pads of friction material than previously known. They reported that the vibration of some of these pads consisted of a bending vibration of the entire pad which was superposed with a torsional vibration of its trailing edge. This localized torsional vibration was found to account for a squeal frequency which was not close to a natural frequency of the brake rotor.

At the same time as Fieldhouse and Newcomb's early work was being published, Ichiba and Nagasawa [123], working for the Toyota Motor Corporation, used accelerometers to examine a squealing disc brake. They viewed squeal as a self-excited vibration caused by the friction forces between the friction material and rotor. After measuring the vibrations of brake rotors and pad backing plates, they concluded, with the assistance of a simplified model for the friction material vibration, that the variation of μ_k with pressure played a more important role in causing squeal than dependence of μ_k on v_s.[26] Their results also indicated that squeal usually occurs at a frequency that is 200–400 Hz lower than a (stationary) rotor resonance.[27] They also suggest that the transverse vibration w_z of the upper surface of the brake rotor during squeal consists of the sum of two standing waves:

$$w_z(r,\theta,t) = G(r)\left(A_{s1}\sin(n\theta)\sin(\omega t) + A_{c1}\sin\left(n\theta - \frac{\pi}{2}\right)\sin\left(\omega t - \frac{\pi}{2}\right)\right), \quad (18)$$

where A_{s1} and A_{c1} are constants. This can be loosely interpreted as a waveform where the nodal diameters rotate at a non-uniform rate about the centre of the disc. As shall shortly become apparent, it is remarkably similar to some of Fieldhouse and Newcomb's results. Finally, Ichiba and Nagasawa [123] also found that the backing plates of the brake pad assemblies vibrate in a rigid mode when squeal is at a low frequency (<7 kHz) and in bending modes during high frequency (> 7 kHz) squeal.[28]

Concurrent with the papers [123,127], Matsuzaki and Izumihara [136], working at the Akebono Research and Development Center, presented evidence that audible squeal in disc brakes is the result of longitudinal (or in-plane) vibrations rather than transverse vibrations of the disc, as asserted in some of the studies mentioned above. Their experiments showed that the audible squeal frequencies showed much better agreement with the frequencies of even-order longitudinal modes in the disc than with any of its bending (transverse) mode frequencies. To further demonstrate their point, Matsuzaki and Izumihara cut slits through the rotors of several brakes.[29] These slits altered the frequency response of the rotor: in particular, the frequency of in-plane vibrations of the rotor increased. They then correlated the decrease of squeal to the presence of the slits. Agreement was found in most cases.[30]

In 1999, Dunlap et al. [138], working at Delphi Chassis Systems, concurred with the work of Matsuzaki and Izumihara [136] that what they termed *high-frequency squeal* corresponded to in-plane modes of the brake rotor.[31] Subsequently, Chen et al. [139,140], working at Ford Motor Company, presented experimental evidence that coupling between the in-plane and out-of-plane modes was a "key to produce squeal". For completeness, we note that Baba et al. [44] also discussed the role of in-plane modes.

The works we have just discussed on the vibration of squealing brakes demonstrate that when the rotor is spinning at a constant speed, the transverse vibration $w_z(r,\theta,t)$ of a surface of the rotor might either be a standing wave or a travelling wave, or neither. Using ESPI to measure w_z and a set of microphones to measure the acoustic radiation from the squealing disc brake assembly, Reeves et al. [124], working at Rover Group, recently revisited this issue. For their

[26] This dependency is reminiscent of the emphasis Tolstoi [79] placed on the dependency of μ_k on the normal displacements.

[27] This is reminiscent of Nishiwaki et al.'s [132] results. We note that a related result was also discussed by Fieldhouse [133].

[28] Related results are discussed by Lang and Smales [134] and Chowdhary et al. [135].

[29] A similar method was used by Tarter [137].

[30] It would be interesting to attempt to reconcile the disagreements between Ref. [136] and Refs. [123,127] using a vibration analysis similar to Bae and Wickert's [45] but incorporating the brake pad dynamics and friction, and the rotation of the rotor.

[31] The paper [138] also discusses several case studies of brake squeal and how, by varying the components of the braking system, one can eliminate or exasperate brake squeal.

particular disc brake system, they found a standing wave vibration of the rotor in a squealing brake assembly. Their main purpose however was to re-examine Fieldhouse and Newcomb's experimental evidence of a mode which rotated at a non-uniform rate relative to the disc. In the course of their investigation, they showed that an example of such a mode, which Reeves et al. called *a complex mode*, could be considered as the superposition of two standing waves:

$$w_z(r,\theta,t) = G(r)(\sin(\omega t)\sin(n\theta) + \sin(\omega t + \tau)\sin(n\theta + \alpha)), \tag{19}$$

where τ and α are constants: $|\alpha| \in (0, \pi/2)$ and $|\tau| \in (0, \pi/2)$. We emphasize that the vibration (19) is not the same as the travelling wave (16) but is similar to Ichiba and Nagasawa [123] suggestion (18) and Talbot and Fieldhouse's [125] result (17).

Recent work at Boston University by McDaniel et al. [141] argues that measurements of the acoustic radiation of the stationary modes of the brake system can be used to infer information about squealing brakes. Specifically, a laser-Doppler vibrometer was used to measure the modes of a disc brake assembly where the brake pad assemblies were hydraulically actuated to press on to the rotor. The modal measurements were then imported into a boundary element method numerical package in order to compute the acoustic radiation intensities and efficiencies. Using this methodology, McDaniel et al. concluded that the rotor was the primary source of acoustic radiation. These authors also argued that, for low rotational speeds of the rotor, the stationary modes of the brake system are qualitatively similar to those observed in [122] and Refs. [127,128] for squealing brakes.

Saad et al. [142,143] considered the effects of pad vibration and the μ_k–v_s relationship in their experiments. Their results emphasized the importance of brake pad resonance in squeal as they assumed that the audible squeal frequencies often corresponded to the measured vibration frequency of the pad in isolation. These authors also used their experimental results to calibrate a three-degree-of-freedom model for brake squeal that was developed by Matsui et al. [144].

All of the experiments we have discussed tacitly presume that the explanation for squeal lies in the vibrations of the brake system, and some of them have also studied the noise emitted by a squealing brake. Those works which have attempted to explicitly correlate the vibrations to the acoustic radiation include Cunefare and Rye [145], Felske et al. [106], Fieldhouse and Newcomb [127,128], Mahajan et al. [146], Matsuzaki and Izumihara [136], McDaniel et al. [141], and Reeves et al. [124].[32]

5.3. Tribology-based works

As friction and wear plays an important role in brake squeal, many tribiological studies have been performed in this area. The effects discussed in these studies are undoubtedly important. However, it is not immediately clear how to incorporate them into predictive models for disc brake squeal. In fact, few analyses found in the brake squeal literature have incorporated complex tribiological processes in their models.

[32] In contrast to these papers, Hald [147] recently used time domain acoustical holography to analyze a squealing brake's spectral emissions. This technique may prove useful in subsequent works on brake squeal.

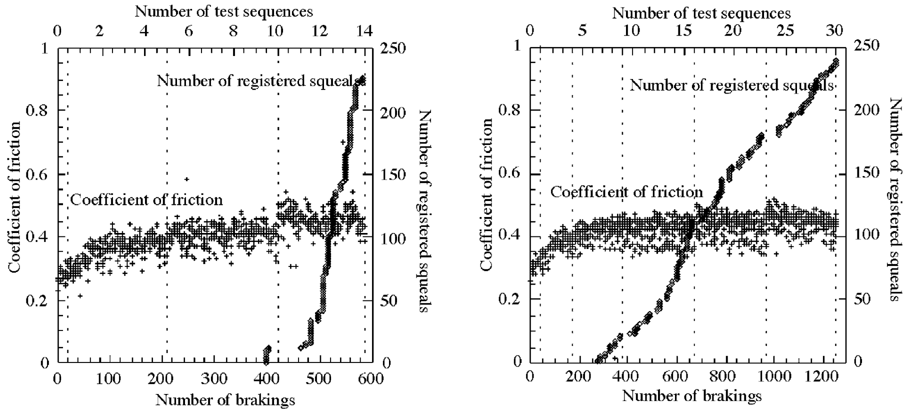

Fig. 10. Two sets of the experimental results of Bergman et al. for the accumulated incidents of squeal and coefficient of friction μ_k as a function of the number of brakings. This figure is identical to Fig. 2(a) and (b) from Ref. [26] and was generously supplied to the authors by S. Jacobson.

In one study, Rhee et al. [85], working at Allied-Signal Inc., examined the formation and destruction of friction films[33] on the rotor and subsequent effects on brake squeal in a series of dynamometer tests. These authors believed that friction films form on the contact surfaces of brake rotors and pads of friction material as a result of the compaction of wear debris removed from these components during braking actions. Rhee et al. reported that, for a semi-metallic friction material and a grey cast iron disc brake rotor, the formation of the friction film increases the coefficient of friction and that brake squeal did not occur until the films were well formed and the coefficient of friction μ_k had reached a steady level. This usually occurred in the temperature range of 100–300°C. Higher temperatures destroyed the films, and squeal was eliminated. However, this process often led to severe thermal damage to both the rotor and the pads.

In a series of works started in the mid-1990s, Jacobson and his colleagues investigated brake squeal and focused on the role played by the tribiology of the pad–disc interfaces. Their work was conducted at Uppsala University in Sweden, and much of this work, as well as the relevant experimental equipment, is summarized in the Ph.D. Theses of Bergman [148] and Eriksson [149].

One of the main results from Jacobson's group was to correlate changes in μ_k to the number of brakings and squeal generation. For instance, in Ref. [26], they carefully prepared the disc brake rotors and examined the change in μ_k as the braking system was run-in (see Fig. 10). They noted that μ_k gradually increased from a value just above 0.2 and, after over one thousand brakings began to plateau at a value below 0.6.[34] As μ_k increased beyond a critical value, Bergman et al. observed that squealing behaviour sharply increased. These experimental results are reminiscent of the remark quoted earlier from Felske et al. as well as Rhee et al.'s remarks on increased squeal generation coinciding with increased brake use.[35]

[33] These friction films are also known as *glazes* or *transfer films*.

[34] Related results are discussed in a later paper [83].

[35] It is interesting to note that Vola et al. [150] found a similar threshold for the coefficient of friction phenomenon in the squealing of a rubber–glass contact.

Further contributions of the Jacobson group include, in Ref. [151], studies of the brake pad's contact area on the generation of brake squeal, in Refs. [33,34], the effects of various solid lubricants on the generation of brake squeal, in Ref. [152], characterizing the contact plateaus on the brake pads, in Ref. [153], examining the creation and wear of contact spots in the interface between a piece of a brake pad and a glass disc, and in Ref. [87], the influence of humidity on brake squeal. It is interesting to note that Eriksson et al. [150] showed that brake pads which had many small contact plateaus were more likely to generate squeal than brake pads with fewer larger contact plateaus. In later papers [83,86], they also report hysteresis in the relationship between the brake line pressure and μ_k during a braking procedure.

Ibrahim et al. [154] measured the average normal and friction forces acting on a friction element (in the form of a dowel) which was placed in contact with a rotating disc. Both the rotation speed and direction of the disc were variable, and their tests were performed at constant rotational speeds. They noted several interesting features. Most notably, neither the normal force nor μ_k were constant. In fact, these authors reported that the friction and normal forces acting on the friction element are random, non-Gaussian processes. They also develop a single-degree-of-freedom model for the vibration of the friction element. This model was subsequently analyzed in further detail by Qiao and Ibrahim [155]. Recently, Ibrahim [11] has discussed the relationships between the results of Refs. [154,155] and brake noise.

6. Methods to eliminate brake squeal

Various empirical methods have been developed to reduce brake squeal in disc brakes. Fortunately, several methods are discussed in the open literature.

The earliest discussion of methods to eliminate disc brake squeal we found was in the papers of Fosberry and Holubecki [112,113]. In particular, they examined the effects of several design changes such as increasing the damping between the brake pad and backing plate and the backing plate and the brake piston, changing the caliper geometry and stiffness, and modifying the backing plate.

Fosberry and Holubecki's advocation of increasing the viscous damping in the brake system in order to suppress brake squeal is reflected in some of the commonly used remedies in the automotive maintenance community:[36]

(a) Use of an anti-squeal product, such as *disc brake quiet*[37] between the backing plates and calipers.
(b) Application of a grease, which can be an anti-seize compound, to the piston-backing plate contact areas.[38]

---

[36] Our references here include the papers and articles [24,106,113,114,133,138,156–160].

[37] This product, which is manufactured by several companies, contains water, ethylene glycol, and an acrylic polymer.

[38] An anti-seize compound, in addition to often serving as a lubricant, may prevent the backing pads and pistons from fusing or welding together at the high temperatures found in a braking system.

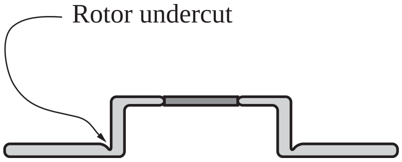

Fig. 11. Schematic of one of the brake squeal counter-measures used in the disc brakes of a Mercedes-Benz M-Class vehicle. This figure is adapted from Fig. 13 of [163].

(c) Use of vibration shims between the backing pads and calipers. These shims often consist of constrained layer dampers.[39] Many off-the-shelf brake pads come equipped with these shims.
(d) Chamfering and/or slotting of the pads of friction material.
(e) Sanding the surfaces of brake rotors.
(f) Lubrication of the pins that connect the caliper to its mounting bracket.

We note that the chamfering remedy (d) is clearly reminiscent of Spurr's observations in [114]. A recent paper by Fieldhouse [133] presents an explanation for the effects of chamfering on squeal. His arguments center on the interplay between the angular extent of contact between the pads of friction material and the modes of vibration of the stationary rotor. This matter is also analyzed, with the help of a model, in Chapter 4 of Chowdhary's recent thesis [58], and discussed from an experimental viewpoint in Ref. [161]. Remedy (e) is probably related to the friction film formation that we mentioned earlier. As noted by Rhee et al. [85], the roughness of the rotor is intimately related to the friction film formation, but the precise correlation remains to be established.

Apart from (d) and (e), the remedies presented above are based on reducing or damping vibrations of components of the disc brake system. They can all be traced to the diagnosis that brake noise is the result of vibrations of the braking system.[40] The use of shims in braking systems is also discussed by Fosberry and Holubecki [113], North [3], and Brooks et al. [162]. The latter authors comment that shims can be used to alter the centre of pressure between the pistons and the pad backing plate, and advocate that having the center of pressure lie towards the leading edge of the brake pad assembly reduces squeal. In Ref. [113] this is achieved simply by moving the backing plate relative to the piston in the direction of rotation of the rotor.

One of the most interesting papers on methods to eliminate disc brake squeal that we found was by DiLisio et al. [163]. These authors discussed various measures that were used with the Mercedes-Benz M-Class. In particular, a constrained layer noise insulator[41] was added to front disc brakes, the manufacturing tolerance on the compressibility of the brake pads was decreased, and the undercut of the rotor was deepened (see Fig. 11). In work which is reminiscent of Matsuzaki and Izumihara [136] and Spurr [114], DiLisio et al. also

[39] A constrained layer damper is a layer of viscoelastic polymer which is constrained between two layers of a stiffer material, such as a metal.

[40] In a related context, Tolstoi [79] notes that one method to eliminate self-excited vibrations is to increase the "normal damping." That is, increase the damping of motions perpendicular to the sliding direction. This is precisely what several of the remedies (a)–(f) achieve.

[41] This insulator might be what is commonly referred to as a constrained layer damper.

investigated the effects of placing slits in the brake pads and chamfering the ends of the brake pads.

As mentioned in a previous section, several authors have discussed the influence of the natural frequencies of the individual components of a braking system. For example, Murakami et al. [121] state that

> "It was found that a great deal of squeal occurred when the natural frequency of these components were in the same vicinity."

It appears to be a commonly held belief that designing the brake pads, rotor, and calipers such that their natural frequencies in the audible range are as isolated as possible suppresses squeal (see also Refs. [162,164]). However, this is not sufficient (see, e.g., Refs. [138,136]). For instance, the experimental evidence of Nishiwaki et al. [122] indicates that the squealing frequencies are related to the frequencies of vibration of the (stationary) brake rotor only.

Several other novel methods for the suppression of brake squeal are discussed in the literature.[42] One such method which was recently proposed by Cunefare and Graf [165,166] involves using dither to eliminate squeal. Specifically, dither is a high-frequency disturbance which is used to "smoothen" the effects of frictional forces. This is achieved in Refs. [165,166] by placing a piezoelectric transducer in contact with the backing plate of the inboard pad of friction material in a floating caliper disc brake. The transducer can be conveniently placed in the piston. A high frequency voltage is applied to the transducer and this causes the backing plate (and pad of friction material) to vibrate. For example, to quench squeal at 5.6 KHz, Cunefare and Graf [165] excite the transducer at frequencies up to 20 KHz and voltages of up to 177 V (root mean squared). Another novel method is discussed by Baba et al. [44] and consists of reshaping the hat of the brake rotor so as to destroy the symmetry of the brake rotor.

7. Central features of some theories for brake squeal

There is a large body of literature devoted to developing models that explain brake squeal and determine its dependence on the parameters of the brake system. To understand this literature, it is convenient to examine the distinct central features of some of the proposed theories of brake squeal. Here, we have categorized six such features. Later in this review paper, we shall discuss the models used to explore theories of brake squeal which are based on these features.

7.1. Decreasing μ_k with increasing v_s

As mentioned earlier, Mills [111] in 1938 examined various drum brake and brake lining combinations where μ_k was a decreasing function of v_s. His work lead to a school of thought which advocated that a necessary condition for a brake to squeal was the presence of a $\mu_k(v_s)$ relationship where $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$. We shall refer to the theory of brake squeal which is founded on this premise as the $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$ theory of brake squeal.

[42] Several of these suppression methods are detailed in the patent literature—which, in the interests of brevity, we have not reviewed here.

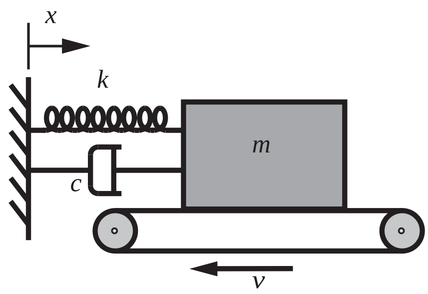

Fig. 12. A spring–mass–damper system mounted on a moving surface. For this oscillator, the spring constant is k, the mass of the block is m and the viscous damping coefficient is c.

It is known that systems where μ_k is a decreasing function of v_s can have effectively negative damping and consequently exhibit unstable oscillations. For example, consider the oscillator shown in Fig. 12. The governing equation for this oscillator is

$$m\ddot{x} + (c - mg\mu_2)\dot{x} + kx = 0, \tag{20}$$

where x measures the horizontal displacement of the block of mass m from equilibrium. The dependency of μ_k on v_s for this system is taken to be

$$\mu_k = \mu_k(v_s) = \mu_1 - \mu_2 v_s, \tag{21}$$

where μ_1 and μ_2 are constants. Having $\mu_2 > c/mg$ then gives rise to negative damping and self-excited vibrations in the system. Following this line of reasoning, Fosberry and Holubecki [113] concluded in 1961 that

> "Disc-brake squeal has the characteristic of a frictional vibration of the type which can be induced by a frictional pair having either a static coefficient of friction higher than the dynamic coefficient, or a dynamic coefficient which decreases with increase of speed."

A similar comment, pertaining to all automotive brakes, can be found in a 1955 paper by Sinclair [167].

It is interesting to note that Mills' experiments coincided with Bowden and Leben's [168] work on *stick–slip* vibrations in friction oscillators. Indeed, Mills comments in Ref. [111] that

> "The way in which squeak is produced has not yet been definitely established. It seems most probable that it is set up by a type of "stick-and-slip" action between the rubbing surfaces, as with a violin string and its bow, in which the source of energy is the variation of friction force with speed."

To elaborate further on this matter, consider the schematic of a friction oscillator shown in Fig. 12. Briefly, for a constant v and for $\mu_s > \mu_k$, the slider is initially stuck to the plane, but eventually the spring force overcomes the frictional force and sliding commences. After a certain time, the slider comes to rest relative to the plane, i.e., $\dot{x} = v$, and the process repeats itself. The resulting motions of the slider are also known as relaxation or self-excited oscillations. Paralleling Mills' work, it was shown that a necessary condition for the existence of these oscillations in a friction oscillator was a coefficient of friction μ_k which was a decreasing function of v_s (cf. Refs. [169,170]). Complementing these results, Blok [171] pointed out that these oscillations could also be excited provided the assumed constant μ_k was less than μ_s.

The $d\mu_k/dv_s < 0$ theory of brake squeal has not received much attention in recent years. Indeed, Lang and Smales [134] in 1983 commented that they only considered it to be important for brake noise under 100 Hz. The most recent work on this topic is Eriksson and Jacobson [86]. Among other matters, they experimentally examined the correlation between squealing brakes and friction material brake rotor combinations with $d\mu_k/dv_s < 0$ and found none.

7.2. Sprag-slip

Spurr [114] coined the term *sprag-slip* to describe his theory of brake squeal.[43] This theory of squeal applies even when μ_k is independent of v_s and it associates this phenomenon with unstable oscillations of the system. Quoting directly from Spurr [114]:

> "Brake squeal similarly is, in general, due to contact occurring in such a position on the friction material that because of the geometry of the brake assembly the frictional force is much increased above the value it would have in a perfectly rigid system. The assembly then deflects elastically, reducing the frictional force and returns to its first state to repeat the cycle."

Notice that the variation of friction force in this theory of brake squeal is achieved (even when μ_k is constant) by varying the normal force. We now turn to elaborating on Spurr's theory and discussing some extensions to his work.

In sprag-slip, the oscillations occur due to the constrained interaction of various degrees-of-freedom in the system. Specifically, one of the sliding components is oriented with respect to another such that the frictional torque applied by the latter to the former causes the normal force (and hence the friction force and torque) to increase, creating a positive, possibly unstable, feedback. In systems with no flexible members, the contact forces can become unbounded. To illustrate this point, Spurr considered the system shown in Fig. 13. Assuming $F_f = \mu_k N$, Spurr showed that equilibrium for the system implies

$$N = \frac{L}{(1 - \mu_k \tan(\theta))}, \quad F_f = \frac{\mu_k L}{(1 - \mu_k \tan(\theta))}. \tag{22}$$

He then used this expression to show that if $\theta \to \tan^{-1}(1/\mu_k)$, then $F_f \to \infty$. This critical case is what he termed *spragging*.

To relate spragging to disc brake squeal, Spurr went on to examine disc brake pads whose contact patch with the brake rotor leads the pivot point on which the pads are mounted (see Fig. 8). As in the rigid case examined above, this scenario can lead to high contact forces. However, the flexibility of these components allows them to free themselves from the spragging situation by slipping loose after they have been sufficiently deformed by the large normal and friction forces at the contact interface. Once the spragging has been relieved and the original contact situation reestablished, the contact forces again start to build up. In this manner, spragging in an elastic system can lead to a sprag-slip limit cycle.

[43] Spurr's paper also contains a set of very interesting communications on brake squeal by Fosberry, Jarvis, H.R. Mills and Rabinowitz, among others, in addition to Spurr's reply to these communications.

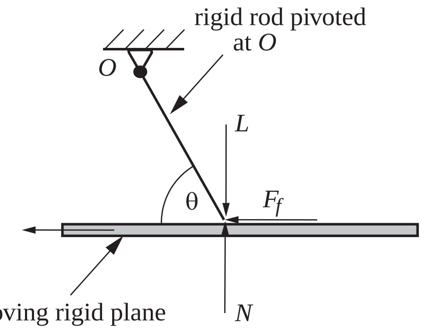

Fig. 13. A rigid, massless rod, which is pivoted at O (whose position is fixed) and loaded by an external force L at its free end, contacting a rigid moving plane. At the point contact between the plane and the rod, a friction force F_f is present. This figure is adapted from Ref. [114].

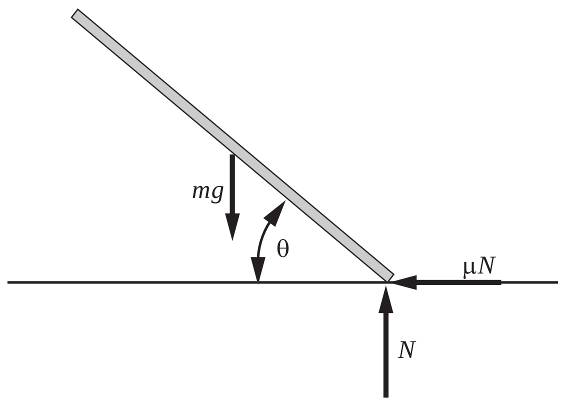

Fig. 14. Schematic of Painlevé's paradox adapted from Ref. [173]. The rod has mass m, length l, and moment of inertia J, and its end contacting the surface is assumed to be moving to the right.

Spurr cited the variable contact both between the piston and the friction material's backing plate (the effective pivot point of the pad) and between the friction material and rotor as a cause of the intermittent nature of squeal. The surfaces of all of these elements are subject to wear, a process that changes contact geometry over long time periods; and their bulk is subject to thermal deformations which distort the contact geometry over short time periods.

Spurr's discussion of the spragging of the rigid rod mentioned above bears some striking parallels with Painlevé's [172] famous example of a rigid rod of length l, mass m and inertia J which is moving on a horizontal plane. One point of the rod is in contact with the plane and the rod is free to rotate. In Stewart's [173] treatment of Painlevé's example (cf. Fig. 14), he shows that when

$$(1/m) - (l^2/4J)\cos(\theta)(\mu_k \sin(\theta) - \cos(\theta)) < 0 \tag{23}$$

the rigid rod may rotate into the rigid table (an impossibility), unless unbounded forces act at the interface. Note that Eq. (23) may be rewritten as

$$(1/m) - (l^2/4J)\cos^2(\theta)(\mu_k \tan(\theta) - 1) < 0 \tag{24}$$

and that this term may only be less than zero if $\mu_k > \cot(\theta)$. (Stewart's other necessary condition for unbounded interfacial forces,

$$(l/2)\cos(\theta)\dot{\theta}^2 - g < 0 \tag{25}$$

is always satisfied if applied to Spurr's case as θ is fixed.) To summarize, in Painlevé's example, unbounded contact forces may result when $\mu_k > \cot(\theta)$; and in Spurr's spragging problem, unbounded contact forces result when $\mu_k = \cot(\theta)$.

Spurr's theory of brake squeal has been championed by Earles and co-workers [115–117,119], Felske et al. [106], Jarvis and Mills [38], Millner [62], and North [3,120], among others. In a discussion of Jarvis and Mills' paper, Crisp in Ref. [174] coined the terms "geometrically induced" or "kinematic constraint" for the type of instability arising in the sprag-slip theory of brake squeal.[44]

7.3. Decreasing μ_k with increasing v_s combined with sprag-slip

Earlier, we discussed the experimental work of Murakami et al. [121]. They hypothesized that brake squeal was generated by a mechanism which combined the $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$ theory and Spurr's sprag-slip theory. Their approach to brake squeal was to consider squeal as a self-excited vibration of the components of the disc brake system. To induce squeal with their mechanism, one needs the correct combination of friction response and geometric and material properties of the disc brake assembly. Murakami et al.'s experiments showed that the disc, calipers and brake pad assemblies were the chief vibrating elements when squeal occurred, while the corresponding vibration of the suspension system and axle were negligible.

7.4. Self-excited vibration with constant μ_k

It appears that the first researcher to consider brake squeal as a self-excited vibration induced by friction forces with constant μ_k was North [120] in 1972. North presents an eight-degree-of-freedom model for a particular disc brake system. This (linear) model is novel in the manner in which the friction forces between the brake rotor and pads of friction material are incorporated as follower forces. The criterion for brake squeal North adopted is the onset of instability of the equilibrium of his model.

To discuss North's theory of brake squeal further, it is sufficient to consider the two-degree-of-freedom model considered by North [3] in 1976. Here, the disc is modelled as a rigid body of mass M, thickness $2h$, and moment of inertia J. As in the more complex model, it has two-degrees-of-freedom y and θ. The rationale for this discretization is discussed in Section 8. The disc is assumed to be sandwiched between two layers of friction material, each of length L, of total stiffness k_1. The friction forces F_1 and F_2 acting on the disc are

$$F_1 = \mu(k_1 y + N_0), \quad F_2 = \mu(-k_1 y + N_0). \tag{26}$$

[44] An example of a kinematic constraint can be seen in our discussion of Jarvis and Mills' paper below: see in particular Eqs. (31) and (33).

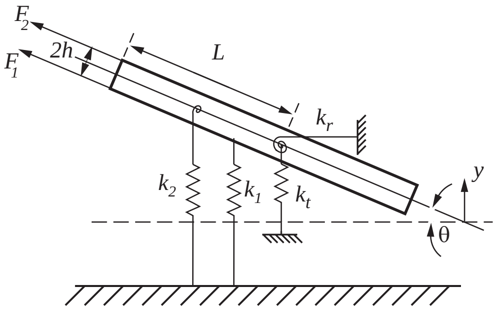

Fig. 15. North's two degree-of-freedom model of a disc brake system [3]. This model features a rigid body model of the brake rotor and two friction forces F_1 and F_2 for the friction forces between the pads of friction material and the rotor.

As can be seen from Fig. 15, these forces are follower forces and N_0 is the (static) preload between the pads and the disc.[45] The equations of motion are of form (10),

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \tag{27}$$

with

$$\mathbf{x} = \begin{bmatrix} y \\ \theta \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_t + k' & 2\mu N_0 \\ -2\mu h k' & k_r + k'L^2/3 \end{bmatrix}. \tag{28}$$

The stiffnesses k_t and k_r pertain to the disc and $k_2 = k_1 L^2/3$. When the equilibrium ($y = 0, \theta = 0$) of Eq. (27) is stable, then the eigenvalues λ consist of complex conjugate purely imaginary pairs, $\lambda_{1,2} = \pm i\omega_1$ and $\lambda_{3,4} = \pm i\omega_2$, which lie on the imaginary axis. As the parameter values of the system are varied, it is possible for the pairs to coalesce ($\omega_1 \to \omega_2$) and split into a quartet $\lambda_{1,2} = -\sigma \pm \omega_2$, $\lambda_{3,4} = \sigma \pm \omega_2$ where $\rho > 0$. The equilibrium is then said to have become unstable in a flutter instability.[46] The criterion for the flutter instability to occur can be established in a standard manner from Eq. (27):[47] First, a criterion for instability of the equilibrium is established:

$$\frac{1}{MJ}\left((k_t + k_1)J - \left(k_r + \frac{k_1 L^2}{3}\right)M\right)^2 \leqslant 16\mu_k^2 h k_1 N_0. \tag{29}$$

The flutter instability occurs when the $\leqslant$ is replaced by an equality in this equation. It is not too difficult to see from Eq. (29) that increasing μ_k can induce instability—a result which is in remarkable agreement with the observations that squeal propensity increases with increasing μ_k that we quoted earlier.

Several other researchers have followed North's lead, albeit with different models. For instance, Chowdhary et al. [58,135], Hulten and Flint [39],[48] Millner [62], and Nishiwaki et al. [48,177]

[45] The prescription of N_0 as a known force rather than a constraint force in North's models is an interesting simplification.

[46] This instability is sometimes called *binary flutter* in earlier texts, and a *reversible Hopf bifurcation* in modern dynamics terminology.

[47] The motivated reader will find that Eq. (29) does not agree with Eq. (43) of North [3]. This discrepancy does not effect North's conclusions.

[48] This paper is based on earlier works by Hultén et al. [130,175,176] on squealing drum brakes. One of the principal components in the drum brake model in these papers is the assumption of a constant μ_k.

develop multi-degree-of-freedom models for disc brakes whose equations of motion are of form (27) often with the inclusion of linear viscous damping, $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$, and adopt the same criterion for squeal as North.

Millner's paper champions the sprag-slip theory, or, more precisely, the kinematic constraint theory, of brake squeal. However, the development of the model in Millner features six-degrees-of-freedom and one constraint. After the constraint has been imposed, Millner ends up with a five-degrees-of-freedom model of form (27) with a non-symmetric $\mathbf{K}$ matrix, and an instability theory for squeal that is identical to North's.

Another notable work in the vein of assuming constant μ_k and self-excited vibrations is presented by (Ouyang, Mottershead et al. [40–42]). These authors developed a finite element model for the brake pad assemblies, calipers and piston of a floating caliper disc brake and a plate model for the brake rotor. In contrast to many other works, these authors considered the effects of the rotation of the disc rotor. A statement of a theory for squeal based on self-excited vibrations was also presented by Ouyang et al. [42] in 1999:

> "Squeal is supposed to be the consequence of the unstable parametric resonances induced by the rotating dry friction between the disc and the pads."

In this statement, the authors tacitly acknowledge the fact that friction forces are follower forces.

It is important to note that the models used to develop an understanding of brake squeal which employ a constant μ_k are sufficiently complex that a variation in the friction force is achieved. In North [3], for instance, this is achieved by modelling the contact between the brake pads and rotor by a discrete contact element (see Eq. (26) above). A similar approach is adopted in more recent models, see e.g., Chowdhary et al. [58,135] and Nishiwaki [48]. Other works, such as Ouyang et al. [40–42], which use a finite element model, consider the flexibilities of the rotor and the brake pad assemblies. As these bodies deform during the vibrations of the disc brake system, a variation in the normal forces between them occurs. This in turn causes a variation in the friction forces (even if μ_k is constant).

7.5. Splitting the doublet modes

At an ASME conference in 1992, Mottershead and Chan [49] presented an interesting theory of brake squeal.[49] Two years later, they commented in Ref. [178] that their "result was already understood in the vehicle brake industry." The premise of this theory was based on the splitting of the frequency of the doublet modes in the symmetric disc when a friction force was applied. The splitting could lead to flutter which was equated to brake squeal.

To describe the theory, Mottershead and Chan [49], considered a clamped elastic annular disc which was loaded (at a discrete number of points) by a tangential follower force traction. This traction was related to a normal pressure p by the coefficient of friction μ_k. Using a finite element model for the disc, Mottershead and Chan developed a model for the linear vibrations of the stationary disc subject to the tangential follower forces. This model was of the familiar form $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$ where $\mathbf{K}$ was not symmetric. Mottershead and Chan then showed that by increasing p from 0, an eventual threshold was reached where the eigenvalues of some of the doublet modes

[49] A revised version of this paper was later published as Ref. [50].

of the unloaded disc split.[50] This signified a flutter instability of the disc and was equated by Mottershead and Chan to the occurrence of brake squeal. They concluded:

> "Flutter instabilities in brake systems occur primarily as a result of symmetry. The frictional mechanism which has been the subject of much research over the past forty years is of secondary importance."

It is interesting to note that Mottershead and Chan [49] also equated their theory to Lang and Newcomb's [129] then newly discovered travelling modes in squealing drum brakes.

Although we shall discuss models which are intimately related to Mottershead and Chan's theory later, we note here the splitting phenomenon in a model of a disc brake system is also analyzed in Chowdhary et al. [135] (see in particular their Fig. 5 or, equivalently, Fig. 2.6 in Ref. [58]).

7.6. Hammering

Rhee et al. [91] found both the $d\mu_k/dv_s < 0$ and sprag-slip theories lacking in that they only described parameter spaces in which squeal might occur but not the actual mechanism exciting the squeal. They also pointed out several examples of common brake behaviour which are contrary to predictions of the $d\mu_k/dv_s < 0$ theory.

Noting that the frequencies of brake squeal often correspond to natural frequencies of brake system components, the authors proposed a mechanical impact model they called *hammering* to explain the noise excitation.[51] Several hammering mechanisms were proposed in Rhee et al. [91], among them, disc imperfections and spragging.

Hammering due to disc imperfections refers to the rocking of the pads of friction material over macroscopic waviness in the disc's surface (as may be caused by the local hot spots evidenced in Abendroth [179]). This causes repeated impacts of the disc and can excite it into a state of vibration.

At first glance, the hammering theory appears to contrast the sprag-slip theory, where persistent contact between all components is assumed. However, Rhee et al. remarked that hammering was, in fact, compatible with sprag-slip instabilities. They argued that unstable interactions between various system elements (as in spragging) may cause hammering which leads to squeal. Referring to Schallamach's work [180] on sliding rubber, they also argue that stick–slip interaction associated with waves of detachment (which are also known as Schallamach waves and moving creases), can also be viewed as a series of impulses acting at the frictional interface. These impulses can, in principle, excite natural frequencies of the brake system components.

The sanding of the rotor discussed in an earlier section can be easily understood using the hammering theory of brake squeal. After all, the sanding removes the hot spots that may have formed and thus eliminates a source of hammering. It would be interesting to relate the hammering theory of brake squeal to the work we discussed earlier by Bergman, Eriksson and Jacobson. Here, we are referring to their observation that brake pads which had many small

[50] For instance, if the eigenvalue for the mode featuring $\sin(n\theta)$ developed a positive real part x, then the eigenvalue for the mode featuring $\cos(n\theta)$ developed a negative real part $-x$.

[51] Their choice of the term hammering was meant to evoke vibration tests which feature an impact hammer.

contact plateaus were more likely to generate squeal than brake pads with fewer larger contact plateaus in Ref. [152], and their work in Ref. [26] on brake squeal and disc topography.

8. Models of disc brake squeal and analyses thereof

The first detailed analysis of a model for a squealing disc brake was presented by Jarvis and Mills in 1963. Their analysis, which used a three-degree-of-freedom model subject to a single holonomic constraint, championed Spurr's sprag-slip theory. It also has had a considerable influence on later works.

Since 1963, the models for squealing disc brakes have featured an increased number of degrees-of-freedom, increased complexity, and, often, more complex friction models. As we shall show, there are also some researchers who have bucked the trend towards complexity and developed simple models. To review most of the models that have appeared in the literature, it is convenient to summarize a partial list of their chronological order:

The model of Jarvis and Mills (1963),
The pin-on-disc system of Earles and co-workers (1971–1987),
North's models (1972,1976),
Millner's model (1978),
Murakami, Tsunada and Kitamura's seven-degree-of-freedom model (1984),
Liles' finite element model (1989),
Matsui, Murakami et al.'s three-degree-of-freedom model (1992),
Brooks, Crolla, Lang and Schafer's 12 degree-of-freedom model (1993),
Nishiwaki's models for brake noise (1993),
Chargin, Dunne and Herting's finite element model (1997),
Hultén and Flint's model (1999),
Nack's finite element model (1999–2000),
Ouyang, Mottershead et al.'s finite element model (1999–2000),
Chowdhary, Bajaj and Krousgrill's model (2001),
McDaniel, Li, Moore and Chen's three-degree-of-freedom model (2001),
Rudolph and Popp's 14 degree-of-freedom model (2001).

Even though this list is not exhaustive, it is a vivid demonstration of the large body of models devoted to brake squeal. A detailed review of all of the models listed above is not possible. Rather, we will review some of these models and point out inter-relationships between them.

In the sequel, it is useful to follow Lang and Smales [134] and distinguish two types of squeal: low-frequency *squeal* (noise between 1 and 5 kHz where the rotor typically vibrates with 2–4 nodal diameters) and high-frequency *squeak* (5–10 nodal diameters). In low-frequency squeal, the nodal spacings are greater than the length of the pad, hence suggesting that treatment of the pads as rigid bodies is acceptable. This is one of the philosophies behind the models of North [3,120], Millner [62], Brooks et al. [162], and Rudolph and Popp [181,182]. In high-frequency squeak, Lang and Smales pointed out that the disc nodes are much closer together and pad bending

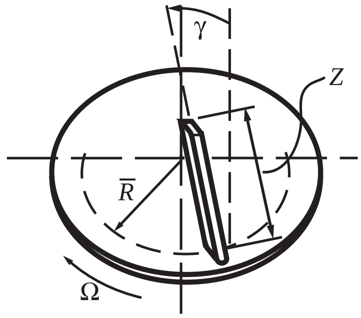

Fig. 16. Schematic of Jarvis and Mills' cantilevered beam on disc arrangement [38].

vibration becomes important. They recommend that these problems be treated with the sprag-slip models such as Earles' pin–disc systems.[52]

8.1. The model of Jarvis and Mills and its legacy

The first theoretical treatment of sprag-slip instabilities in brakes was given by Jarvis and B. Mills [38].[53] To develop insight into brake system noise, they considerably simplified the earlier experiments of Fosberry and Holubecki and examined the vibrations of a rotating disc that was in contact with the tip of a cantilever (see Fig. 16). Their experimental and analytical results have significance to the present day.

From their experimental work, they issued the following statement concerning the vibration of the disc as it contacted the cantilever:

> "In practice, there were no true nodes and if a vibration pick–up was scanned round the disc the amplitude was seen to vary about a mean level and the phase changed continuously. This behaviour is explained if it is assumed that at least two vibrations of the same order of mode exist simultaneously with different phases in time and space."

This statement foreshadows the works on travelling waves in squealing brakes that we discussed earlier in Section 5 (cf. Eqs. (17)–(19)). Jarvis and Mills statement was loosely interpreted by several researchers, for instance, Brooks et al. [162] and Millner [62], to imply that the vibration of the rotor of a squealing brake could be considered as the superposition of two standing waves relative to a fixed observer. Furthermore, if one assumes that the number of nodal diameters of these waves was sufficiently small (2 or 3) so that the pads of friction material were deformed rigidly, then these pads and the rotor could be modelled using lumped masses. As illustrated some 35 years later by Hultén and Flint [39], one of these standing waves is responsible for the rotation of the lumped mass modelling the rotor, while the other standing wave is responsible for the

[52] Lang and Smales comments about the pad modes appear to have been verified experimentally by Ichiba and Nagasawa [123].

[53] It is insightful to also read the accompanying communications [174] and authors' reply [183].

translation of this lumped mass (see Fig. 2 in Ref. [39]). As mentioned earlier, this is part of the philosophy behind many lumped parameter models for low-frequency brake squeal.

To model their experiment, Jarvis and Mills considered a cantilevered beam of length Z whose free end contacted the face of a rotating disc at a radius $r = \bar{R}$. Applying Lagrange's equations and using the appropriate normal functions for cantilevers and discs as generalized co-ordinates, their non-linear analysis using the method of slowly varying parameters showed that a decreasing friction velocity relationship, $\mathrm{d}\mu_k/\mathrm{d}v_s < 0$, had negligible effect in generating unstable vibrations in such a system. Once this effect had been discounted and the terms relating μ_k to the sliding velocity v_s removed, the equations of motion simplified considerably. This allowed Jarvis and Mills to consider two modes of the disc (whose amplitudes were q_1 and q_2) and one mode for the cantilever (whose amplitude was q_c). In particular the (transverse) vibration of the disc is assumed to be

$$w_z(r, \theta, t) = f(r)(-q_1(t)\cos(n\theta) + q_2(t)\sin(n\theta)), \tag{30}$$

where $f(r) = \mathrm{J}_n(kr) + \lambda \mathrm{i}^n \mathrm{I}_n(kr)$, J_n and I_n are Bessel functions, and r and θ are co-ordinates which we defined in Section 3.[54] The resulting equations of motion are

$$\begin{aligned} a_c\ddot{q}_c + b_c\dot{q}_c + c_c q_c &= -M_1 P + \mu_k M_2 P, \\ a_d\ddot{q}_1 + b_d\dot{q}_1 + c_d q_1 &= ({}_1N_1)P, \\ a_d\ddot{q}_2 + b_d\dot{q}_2 + c_d q_2 &= \mu_k({}_2N_2)P. \end{aligned} \tag{31}$$

Here, $a_{c,d}$, $b_{c,d}$, $c_{c,d}$, $M_{1,2}$, ${}_1N_1$, and ${}_2N_2$ are constants which are functions of the modes of the disc:

$$\begin{aligned} {}_1N_1 &= \left(\frac{nh}{\bar{R}}\right) f(\bar{R}), \\ {}_2N_2 &= -\left(\frac{nh}{\bar{R}}\right), \\ M_1 &= \left(\frac{2\sinh(mZ)\sin(mZ)}{\sinh(mZ) - \sin(mZ)}\right)\sin(\gamma), \\ M_2 &= \left(\frac{2\sinh(mZ)\sin(mZ)}{\sinh(mZ) - \sin(mZ)}\right)\cos(\gamma), \end{aligned} \tag{32}$$

where h is half the thickness of the disc, and m depends on the experimentally determined vibrational frequency of the cantilever. The force P in Eq. (31) is the normal force between the disc and the cantilever. Also, these equations are supplemented by the kinematical constraint:

$$M_1 q_c = {}_1N_1 q_1. \tag{33}$$

In total, Eqs. (31) and (33) constitute four equations for four unknowns.

Crisp's comments in Ref. [173] that Jarvis and Mills are looking at a holonomically constrained mechanical system should be apparent from Eqs. (31) and (33). What Crisp pointed out explicitly was that imposing a constraint on a dynamical system can induce instability.

[54] The constants k and λ are calculated from known results on the bending vibrations of discs (see, e.g., Ref. [51]). Jarvis and Mills are treating the vibration of the disc as if it were stationary (cf. Ref. [183]).

Jarvis and Mills performed a stability analysis for the equilibrium ($q_c = q_1 = q_2 = 0$) of Eqs. (31) and (33). This analysis showed that instability arose when

$$({}_1N_1)^2 b_c > M_1(M_2\mu_k - M_1)b_d. \tag{34}$$

With the help of Eq. (33), one finds that this inequality is equivalent to

$$f^2(\bar{R})b_c > \left(\frac{2\sinh(mZ)\sin(mZ)}{\sinh(mZ) - \sin(mZ)}\right)^2 (\mu_k \cot(\gamma) - 1)\sin^2(\gamma). \tag{35}$$

It should be clear that stability is intimately related to the angle γ and the coefficient of friction μ_k. Indeed, as Spurr noted in Ref. [174], Eq. (35) is very similar to his spragging criterion $\mu_k = \tan(\theta)$ (cf. Eq. (22)).[55] Because of the simplicity of Jarvis and Mills' model, the rotational speed of the disc does not explicitly appear in Eq. (35).

Although we agree with Fosberry and Holubecki [174] that it is possible to over-emphasize the relevance of Jarvis and Mills' work to actual brake squeal, their paper nonetheless make a significant contribution to the modelling of brake squeal. It is also important to note that experimental comparisons with Eq. (35) performed by Jarvis and Mills in Ref. [38] showed significant disagreement. Bringing model and experiment into closer agreement is a theme that has since dominated the literature on brake squeal. With this in mind, numerous researchers, among them, Earles, North and Millner, have developed other models and we now turn to discussing them.

8.2. The pin-on-disc models of Earles and co-workers

Starting in the 1970s, Earles and various co-workers [115–117], who were working at King's College in London, used pin on disc (or pin–disc) systems to investigate and quantify the sprag-slip mechanism for squeal. The investigations performed consisted of examining how the damping and orientation influenced squeal. A good review of some of the early papers by Earles and co-workers is presented by North [3]. Here, we supplement some of North's comments.

Some of the works of Earles and co-workers featured experiments where a disc is spun at a constant speed (ranging up to 1000 r.p.m.). The disc is contacted by a pin which is supported by a flexible cantilever [118]. In the early papers of this group, this experimental system was correlated to the model shown in Fig. 17. Later works by Badi, Earles and Chambers [116,117,184] considered systems where two pins were acting on the disc.

In these works, a linear stability analysis was performed on lumped parameter models of pin–disc systems in order to find the flutter boundaries in parameter space (i.e., the manifolds in parameter space on which Hopf bifurcations occur). After the constraints had been incorporated, these models were generally linear three- or five-degree-of-freedom systems of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\mathbf{x} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{36}$$

Results from these analyses indicated that at least one pin in the system must have a configuration that causes it to dig into the disc for squeal to occur. For example, Earles and Lee [118] were able

[55] In making this comparison, we noted that Spurr's θ and Jarvis and Mill's γ are related by $\gamma + \theta = \pi/2$.

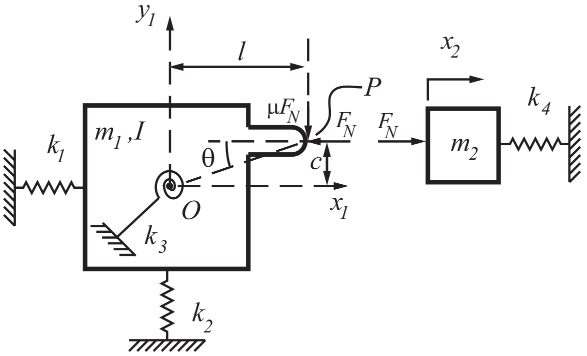

Fig. 17. Earles and Lee's model of a pin–disc oscillatory system. The pin (m_1) and disc (m_2) are meant to represent the pad and rotor in a disc brake. The kinematic constraint is provided by the assumed persistent contact between the pin and disc at point P. Notice that varying the angle θ (or the distance c) is equivalent to varying the location of the pin relative to the point O.

to find an equation for the eigenvalues λ of an equilibrium for the system in Fig. 17 in the form

$$a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{37}$$

where a_i are constants which depend on the system's parameters. It is easy to see that the equilibrium state is unstable if any of the coefficients of (37) are negative. Earles and Lee's analysis shows, however, that these coefficients can only be negative if the term $c(c - \mu_k l)$ is negative (i.e., $\mu_k l > c$). Taking into account the relationship between l and c, a necessary condition for instability (squeal) in the system is that θ be sufficiently small so that

$$\mu_k > \tan(\theta) > 0, \tag{38}$$

where a negative value of θ would indicate that the direction of disc rotation is the opposite of that shown in Fig. 17. Once again, the stability of an equilibrium in a model for a disc brake depends on μ_k and an angle characteristic of the contact geometry.

In the final work on pin-on-disc systems by this group, Earles and Chambers [117] found that damping in the pin assembly (corresponding to damping of the brake pad assembly in a disc brake) could enlarge the unstable regions under certain circumstances, while disc damping always reduced these regions.

The models used by Earles and co-workers feature follower forces for friction. This is transparent if one considers the expression for the matrix **K** on page 550 of Ref. [117]. However, we could not find any explicit mention of follower forces in the papers that we have cited by Earles and co-workers.

8.3. The lumped parameter models of North and Millner

In contrast to Earles and his co-workers, North [120] developed a more complicated model for a disc brake assembly in 1972. As noted earlier, North was interested in correlating the squeal of a

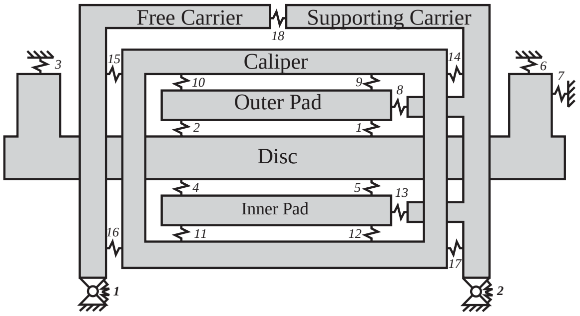

Fig. 18. The 14 degree-of-freedom model of a disc brake system that is used in the work of Rudolph and Popp [181,182]. The numerals in this Figure label the 18 coupling elements between the rigid bodies. The forces and moments in these couplings depend on the relative displacement and relative velocity of the two connected rigid bodies. Rudolph and Popp's model is unique in the manner in which μ_k is assumed to depend on the brake line pressure and the initial temperature of the brake rotor.

particular brake system with an eight-degree-of-freedom model he developed.[56] The direction of rotation of the rotor in North, and several other researchers', models can be inferred from the assumed direction of the friction forces.

North's eight-degree-of-freedom model featured four rigid bodies each with two degrees-of-freedom (in parenthesis): the inboard (y_1, θ_1) and outboard (y_2, θ_2) brake pads, the swinging caliper (y_3, θ_3) and the disc (y_0, θ_0). The model is similar in appearance to that shown schematically in Fig. 18, and features well over 20 parameters that must be specified. The normal forces, N_1 and N_2, acting on the disc due to the pads are linearly dependent on the static preload N_0, and the displacements y_0, y_1, and y_2 and their velocities (see Eqs. (3.1)–(3.2) of Ref. [120]). In other words, there are no holonomic constraints on the eight-degree-of-freedom model. The friction forces on the disc are $\mu_k N_1$ and $\mu_k N_2$ where μ_k is a constant.

The equations of motion formulated by North for his eight-degree-of-freedom model are of the form (10):

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \qquad (39)$$

Here, for the first time in a model of a squealing brake, we find that $\boldsymbol{K}$ is not symmetric. In other words, and as we discussed earlier, North's model incorporated friction as a follower force. Apart from the direction of the friction forces, North's model did not include the effects of the rotation of the disc (see p. 32 of Ref. [120]). Indeed, North followed Jarvis and Mills in modelling the vibration of the disc as Eq. (30). By varying the parameters in $\mathbf{M}$ and $\mathbf{K}$, North examined when stability of the equilibrium of (39) is lost. The onset of instability was taken to be the criterion for squeal to occur and was correlated to North's experiments.

Although we noted earlier that the agreement achieved by North's model to the disc vibration was "encouraging", the novelty and scope of North's work was substantial for its time. One of its

[56] The reader may wish to recall that North was examining a Lockheed disc brake which featured a single cylinder (piston) swinging caliper.

successes was the correlation of increased damping to decreasing squeal in an analytical model for the brake.

In 1978, Millner [62], who was working for Ferodo Ltd., developed a model for a fixed caliper disc brake. Using symmetry, only the rotor, a single brake pad and half of the caliper were modelled. Each of these components were given two-degrees-of-freedom (one translational and one rotational), and a single (holonomic) constraint was imposed on the relative motion of the brake pad and caliper. The contact between the rotor and the pad was similar to North's model but with no static preload N_0.[57] The model Millner developed had a substantial number of parameters, which were obtained from comparison to an experimental set-up. Millner found from the model that while the geometry was important, instabilities could be excited in almost any contact configuration if the coefficient of friction μ_k had a high enough value and the caliper's stiffness was in a certain range. The mass of the brake caliper was also determined to be of critical importance.

An interesting feature that Millner examined was the effect of varying the centre of pressure between the piston and the brake pad. It was found that moving this centre towards the leading edge of the brake pad reduced the propensity for squeal (as defined by Millner's squeal propensity). This conclusion is apparently in agreement with a subsequent 12 degree-of-freedom model of Brooks et al. [162], but at odds with a recent study, using a seven degree-of-freedom model, by El-Butch and Ibrahim [185]. It would be interesting to correlate these results to the sprag-slip theory.

Both Millner and North realized that the pads of friction material were non-linearly elastic. Indeed, North [120] noted that the brake pad assemblies could be considered as a bi-metallic strip. However, to this date these material non-linearities have yet to be fully incorporated into lumped parameter models of disc brakes.

8.4. Some subsequent lumped parameter models

Since North and Millner's development of lumped parameter models, several other researchers have also constructed lumped parameter models to understand brake squeal. In contrast to the earlier models, most of them included the effects of linear viscous damping. All of these models are based on the premise that vibration is a friction-induced instability caused by a friction force which is a follower force. Although the equations of motion for these models are of the form (10) with

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \tag{40}$$

it is important in comparing results to note that many of these authors are modelling different disc brake assemblies. Furthermore, comparison to experimental results is a key component of many of their analyses.

We have already mentioned some of the lumped parameter models which succeeded those of North and Millner. These models include, Brooks et al.'s 12 degree-of-freedom model developed

[57] That is, the normal force R acting on the pad due to the rotor was, in Millner's notation, $R = KPL(y_d - y_p)$ and the friction force $F = \mu KPL(y_d - y_p)$. Here, KPL is a spring constant, y_d and y_p are the rotor and brake pad translational displacements, respectively.

in 1993, Nishiwaki's [48,177] four-degree-of-freedom model developed in 1993, Hultén and Flint's variable degree-of-freedom model developed in 1999, El-Butch and Ibrahim's seven-degree-of-freedom model developed in 1999, Chowdhary et al.'s multi-degree-of-freedom model and Rudolph and Popp's 14 degree-of-freedom model which were both published in 2001. There are several others and space precludes us from outlining all of them here. However, we will attempt to give a representative selection of the important achievements found with these models.

One common issue that all of the studies of the lumped parameter models feature is the issue of parameter prescription. Given the complexity of a brake assembly and its wide range of operating brake line pressures and temperatures, this is not a trivial matter. To assist in this, several authors use the assumed modes methods (see Refs. [39,177]), while others correlate experimental modal analysis and/or finite element analysis to their model (see Refs. [59,121,135,142,182]).

Because the squeal mechanism for lumped parameter models centres on an instability criterion, varying the system parameters is a crucial component of all papers featuring these models. In particular, for squeal to arise according to the theory proposed by North, two sets of purely imaginary eigenvalues, each of which corresponds to a different eigenmode of Eq. (40), must merge as a system parameter is varied. In other words, merging is synonymous with the onset of squeal. The merging phenomenon is emphasized in the work of Chowdhary et al. [135] which we shall shortly discuss. Several other interesting contributions are contained in the literature. For instance, Brooks et al. [162] use an eigenvalue sensitivity analysis, which was developed in Ref. [186], to help them analyze the effects of small variations in the parameters of their 12 degree-of-freedom model on the onset of squeal. Another example lies in the paper of Murakami et al. [121]. These authors notably considered a lumped parameter model containing both a decreasing μ_k with increasing v_s friction relationship and a kinematic constraint instability. They were able to divide most system parameters into two categories. The first category contains parameters for which squeal is always a strictly increasing or decreasing function of the change in a parameter value. This group includes the absolute level of μ_k and the negative slope of the μ_k–v_s relationship. The second category consists of parameters for which there is a single value which produces a minimum or maximum squeal level. All component masses and stiffnesses belonged to this second category. These authors cautioned that the critical values for parameters in the second category were highly dependent on other system parameters, and therefore the values for these parameters must be chosen with care. It is significant to note that damping parameters could not be successfully categorized in this manner, and had to be considered on a case-by-case basis.

To elaborate further on merging, it is appropriate to discuss the work of Chowdhary et al. [135] in further detail. In the model used by these authors, the rotor is modelled as a thin (clamped-free) annular disc (see Eq. (6)), the pad backing plates are modelled as thin annular sectors and the friction material is represented by a distributed system of elastic springs of linear stiffness k_{lin} interconnecting the plates and the rotor.[58] Inertial effects due to the rotor's rotation are ignored in Ref. [135], and a constant coefficient of friction μ_k is assumed. The upper surface of the backing plates are assumed to be fixed by a set of four linear springs of stiffness k_n. The friction force is incorporated into this model in a manner identical to that used by North (see Eq. (26)). The modes for the rotor are assumed to be of form (7), while those for the backing plates are generated

[58] This model, which can be considered as a more elaborate version of Nishiwaki's model [48], is also discussed in further detail in Chapter 2 of Chowdhary's thesis [58].

using the Rayleigh–Ritz method. These two sets of modes, along with the contact elements and expressions for the work done by friction on the rotor and pad backing plates, are used with the assumed modes method to generate a lumped parameter model for the disc brake system. The equations of motion for the resulting model are of form (10):

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{41}$$

When $\mu_k \neq 0$, $\mathbf{K}$ is not symmetric. The formulation of the model in Ref. [135] is such that an arbitrary number of modes N and N_P can be used for the rotor and each backing plate, respectively. Based on a comparison with a finite element analysis of the stationary brake system when $\mu_k = 0$, the authors choose $N = 117$ and $N_P = 20$.

To analyze the onset of squeal, Chowdhary et al. examined the influences of varying the friction coefficient μ_k and the stiffnesses k_{lin} and k_n. Some of their results are shown in Fig. 19. From this figure, we see that when $k_{lin0} = k_{lin}$, the eigenmodes 43 and 44 merge when $\mu_k = 0.0276$ and squeal is said to occur.[59] Although one can also see from the figure that, for other discrete values of k_{lin}/k_{lin0} and μ_k, different sets of eigenmodes merge, Chowdhary et al. show that those modes which are the closest spaced when $\mu_k = 0$ are the ones which will merge first. This important result suggests the possibility of characterizing the propensity of a brake system to squeal by simply measuring the stationary modal response. In particular, their work emphasizes that if one can tailor the eigenmodes of the system so that merging is avoided, then squeal can be eliminated. The analysis of the eigenvalues and eigenmodes of Eq. (41) in Ref. [135] also showed that the backing plates exhibit rigid body modes at low frequencies (<1500 Hz), whereas they bend and twist at higher frequencies.[60] In the context of earlier work on the vibrations of squealing disc brake systems, the eigenmode analysis in Ref. [135] shows that although the rotor modes dominate, the brake pads (even when $\mu_k = 0$) have significant effects (see Fig. 5 of Ref. [135]). Although Servis [132] has shown that a travelling wave was generated in a model for a drum brake after the onset of the flutter instability, this phenomenon has not yet been established for Chowhardy et al.'s model. We strongly suspect that a travelling wave will be present and it would be of great interest to compare it to the experimental results on this wave.

We close this section by noting that most lumped parameter models that we found in the literature featured constant μ_k. The exceptions were, in order of appearance, Murakami et al. [121], Sherif [187], Yuan [64] and Rudolph and Popp [182]. The latter paper considered the variation of μ_k with the initial rotor temperature and the brake line pressure, while the other three papers featured the variation of μ_k with v_s.

8.5. Models featuring the vibration of plates

One of the experimental results we focused upon earlier was the presence of both standing waves and travelling waves in squealing disc brakes that were observed by Fieldhouse and Newcomb and Ichiba and Nagasawa in the early 1990s. Some progress towards explaining these

[59] The reader may have noticed that the coefficient of friction μ_k needed for squeal to occur in Fig. 19 is very small compared to the experimental results we quoted earlier. This partially motivated improvements to model (41), which are discussed in the Chowdhary's thesis [58], which we shall presently review.

[60] This result is similar to the experimental results of Ichiba and Nagasawa [123] and the comments by Lang and Smales [134] that we mentioned earlier.

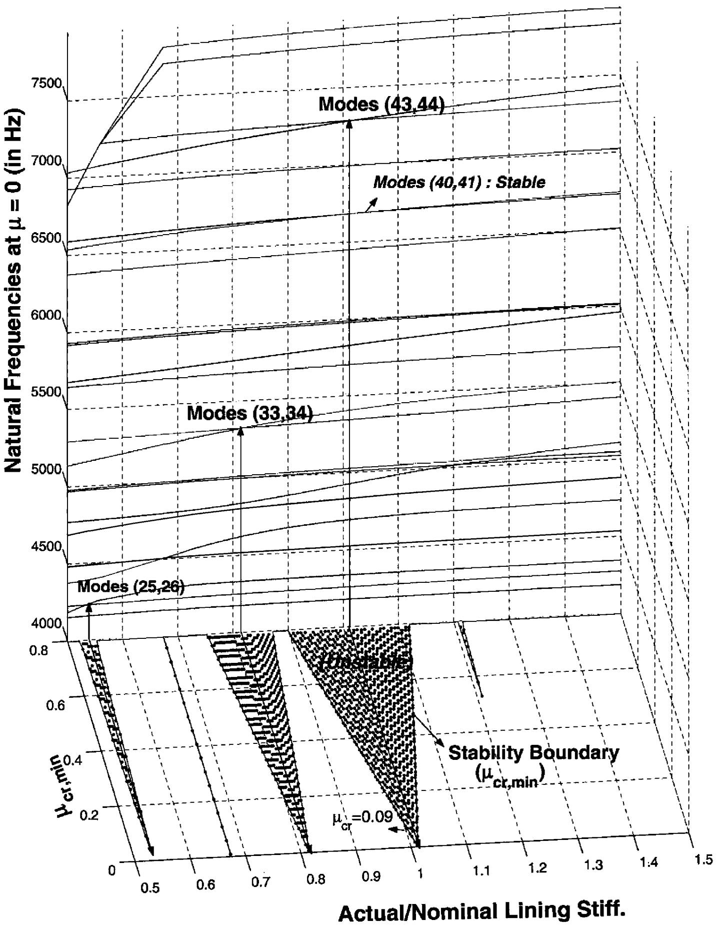

Fig. 19. The variation of the eigenfrequencies in the model of Chowdhary et al. with varying μ_k and the lining stiffness k_{lin}. In the caption, $k_{lin0} = 17.2\ \mathrm{MN/m^2}$ is the nominal lining stiffness. Notice that when $k_{lin0} = k_{lin}$, modes 43 and 44 merge when $\mu_k = 0.0276$ and squeal is said to occur. In Chowdhary et al.'s notation, $\mu_{cr,min}$ is the minimum critical value of μ_k and $\mu = \mu_k$. This figure, which is identical to Fig. 7 in reference [135], was generously supplied to the authors by A.K. Bajaj, and is reproduced here with the permission of the American Society of Mechanical Engineers.

waves in brake systems, albeit using different models, has been achieved by several researchers. This Section is concerned with these works. We caution the reader that the authors of most of the papers we discuss in this Section do not claim to have solved the brake squeal problem, nor do many of them proclaim to be pertinent to brake noise. However, we believe that they do provide valuable insight into the dynamics of brake systems.

Earlier, we discussed Mottershead and Chan's theory in some detail. Concurrently, Ono, Chen et al. [47,188,189], motivated by hard drive developments in the computer industry, were examining the dynamics of spinning discs interacting with slider heads. Ono et al. used a plate model for the disc (see Eq. (2) above and Figs. 9a and b in Ref. [47]) and showed that regardless of the rotational speed Ω of the disc, the frictional follower force always destabilized the forward

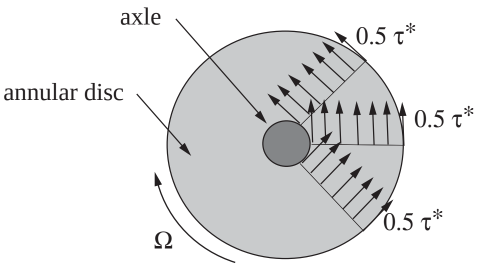

Fig. 20. Schematic of the system considered by Tseng and Wickert [43].

travelling waves (relative to an observer fixed in space) in the disc—again in a flutter instability which occurred when Ω differed from 0.[61] In addition, by considering more elaborate loading effects, Ono et al. were able to suppress the instability of the forward travelling waves.

In a work published in 1998 by Tseng and Wickert [43], another model for a disc which was loaded by a frictional follower force was examined. In contrast to Mottershead and Chan [49,50], the disc was rotated at a constant speed Ω. Mimicking a disk brake system, two sectors on the top and bottom surfaces of the disc were subjected to a traction field τ^* consisting of a constant amplitude frictional follower force (see Fig. 20). Using a plane stress analysis, Tseng and Wickert analyzed the stress fields σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ induced in the disc by the rotation. They were then able to establish the (linearized) equations governing the small amplitude transverse vibrations of the disc which are superposed on the steady deformed state of the disc. In short, they established a set of equations of the form (10):

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{42}$$

Again, **K** is not symmetric. From their numerical analyses of these equations, Tseng and Wickert found that a flutter instability can occur when $\tau^* > 0$.

Over the past decade, Mottershead and his co-workers have published a series of articles [50,178,190–192] on instabilities in annular discs to which a rotating system of a discrete mass-dashpot-spring and an accompanying frictional follower force was applied.[62] Much of this work is summarized in the review article by Mottershead [51] and its relationship to a floating caliper brake system is outlined in Ref. [191]. In theme, these papers focus on the parametric resonances induced in the disc by the rotating system. Our discussion of these papers is to record that they provide a benchmark for the information that can be lost when one ignores the rotation of the disc in a model for brake squeal—specifically instabilities and bifurcations due to the parametric resonances are ignored. Pertaining to travelling waves, it is interesting to note that the most recent

[61] As Mottershead [51] would later remark, Ono et al.'s results are at odds with Mottershead and Chan concerning the threshold friction forces needed to destabilize the disc. However, his remark must be viewed from the perspective that there are significant differences in the models used in the papers [49,50] compared to those used in references [47,188,189].

[62] This model is related, by a co-ordinate change, to the dual model of a rotating plate acted upon by a fixed load system, provided the simple plate model (6) is used. In other words, the deformation induced in the plate by the rigid rotation is not considered in the dual model.

of these papers, Ref. [171], relates the parametric resonances to the instability of travelling waves in the disc.

The previous system considered by Mottershead and co-workers considered the effects of a discrete rotating system. In 1997, this group presented a model where the disc is loaded by a rotating distributed system. This system is a more realistic model for the disc brake assembly. In Ref. [193], finite elements were used to formulate mass, damping, and stiffness matrices for the disc and pad. The disc damping was assumed to be of the proportional type so that the system could be described in terms of the disc's modal co-ordinates. As in the earlier works by this group, the effects of the rotating system were considered to be small perturbations to the dynamics of the undamped disc, and the method of multiple time scales[63] was used to analyze the stability of parametric resonances in this system. The model presented in Ref. [193] can be considered as a precursor to another model which we shall shortly discuss.

At this point in our discussion, it should be becoming apparent that the models for the brake pad assemblies are beginning to show that they can contribute significantly to any conclusions about the potential instabilities of a brake system. We now turn to reviewing some more elaborate models which capture the complex geometry of these systems.

8.6. Finite element models

In recent years, the finite element method has become an indispensable tool for modelling disc brake systems and providing new insights into the problem of brake squeal. This method provides a natural and straightforward means for generating finite dimensional approximations to the governing equations of motion for the components of the brake system. This is accomplished by admitting polynomial interpolations of the dependent variables (e.g., displacements, temperature) within each element subdomain.

One view of the finite element method is that it has the capability of generating high-resolution finite dimensional models of form (10) for a solid continuum. However, contrary to traditional lumped parameter techniques, the finite element method allows for accurate representation of complex geometries and boundary/loading conditions. Also, spatially resolved kinematic and kinetic quantities, such as strains and stresses, are readily computed as part of the finite element solution. Furthermore, the accuracy of a finite element model is typically controlled by the analyst, who may choose to refine the approximation in order to simulate the response of the brake system with a higher degree of fidelity.

The finite element method has been employed by researchers in brake squeal studies to several ends. One of its earlier uses was to investigate the modes and natural frequencies of the brake rotor (see, e.g., Refs. [122,136]). The most common use is to compute the $\mathbf{M}$ and $\mathbf{K}$ matrices in models of disc brakes. Subsequently, a linear eigenvalue analysis is conducted to determine the system's frequencies, modes, and stability.[64] As in the other analyses we have cited, lack of (linear) stability, evidenced in the form of one or more non-dissipating eigenmodes is associated with squeal propensity. The reader may also wish to recall that earlier we discussed the use of the finite

[63] The interested reader is referred to Nayfeh and Mook [194] for further details on this perturbation method.

[64] Because the eigenvalues of the equilibrium can be complex numbers with real and imaginary parts, such an analysis is often termed ''complex eigenvalue analysis'' in some of the papers we cite.

element analyses to calculate traction distributions at the disc-brake pad interface under static, ''pseudo-dynamic'' (i.e., steady) or dynamic conditions.

Some of the representative research efforts that make use of the finite element method are now discussed in more detail. One of the earliest such efforts is due to Liles [156], who generated finite element models of the brake components and validated their accuracy using an experimental modal analysis. Subsequently, he employed modal synthesis to reduce the dimension of each model and proceeded to construct a *system* model which he used in sensitivity studies for several brake design parameters. This analysis produced a series of interesting observations, such as that squeal propensity increases with higher coefficient of friction and thinner (or worn) linings.

A series of recent contributions have proposed refinements to the complex eigenvalue approach to squeal analysis. In particular, Nack [195,196], working at General Motors, constructed large-scale finite element models under steady sliding conditions with constant μ_k. The impenetrability constraint in this work is enforced by way of a penalty method. A similar approach is adopted by Kung et al. [46], where attention is focused on the modal coupling between rotor and caliper. Blaschke et al. [197] pursue a complex eigenvalue approach by including stick–slip and abandoning the penalty regularization in favor of an exact bilateral satisfaction of the contact constraint. A related methodology is proposed by Moirot et al. [198].

A full dynamic analysis of a coarse finite element model of a brake system was presented by Chargin et al. in [199]. This study assumed persistent contact between disc and pads and sliding velocity-dependent friction laws. A differential-algebraic integrator was employed to advance the solution in time. This paper also includes a parametric study of the effect of μ_k on the response of the brake system, and simulation results for the transient dynamics of the brake system.

In an extension to earlier works on squealing brakes which featured fluttering discs, Ouyang, Mottershead et al. [41,42], recently developed a model for a disc brake system. In this model, the brake rotor was modelled as a disc using thin plate theory (6) and the pads of friction material, caliper and caliper mounting are modelled using finite elements (cf. Fig. 21).[65] Ouyang et al.'s [41,42] criterion for brake squeal is instability of the system. Using a selection of parameters, they found that the region of instability was highly dependent on the rotating speed of the disc, even though gyroscopic and centrifugal effects in the disc were neglected in their model. Furthermore, they investigated the effects of varying the stiffnesses and damping of the pads of friction material. In agreement with many other works, they found that damping does reduce the squeal propensity.

One of the results in Refs. [41,42] that caused the authors of these papers concern was that an increase in the rotation speed Ω would result in increased instability. As Ouyang and Mottershead [40] pointed out, this was contrary to experience. To remedy this prediction, they included a more complex friction model in the model of Refs. [41,42]. Instead of assuming a constant μ_k, they now postulated that, in our notation,

$$\mu_k = \mu_k(v_s) = \mu_0\left(1 - \frac{v_s}{15}\right) + 0.0002v_s^2. \tag{43}$$

Using this model, they found that a bounded region of instability for a disc rotating at a constant Ω. That is, for a given $\mu_0 = \mu_k(0)$, they were able to vary Ω to induce instability, and upon further increasing Ω were eventually able to regain stability. It is interesting that their model also predicts

[65] Their model is for a floating caliper disc brake. It should also be noted that centrifugal and Coriolis accelerations of the disc were ignored.

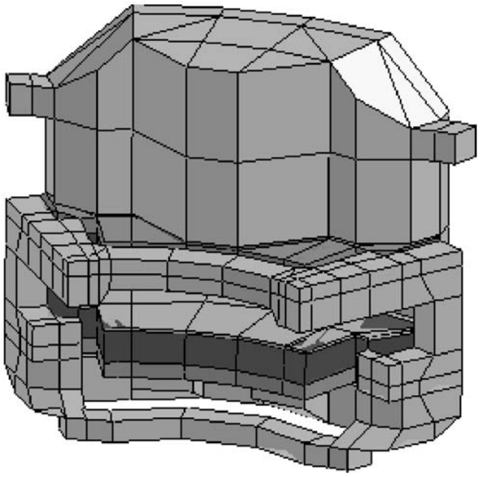

Fig. 21. The mesh for the finite element model of the caliper and brake pad assemblies that is used in Ref. [200]. The finite element model is similar to that used in Refs. [40–42]. This figure was generously supplied to the authors by, and is reproduced here with the permission of, H. Ouyang and J.E. Mottershead.

a threshold value of $\mu_k(0)$ below which instability was not observed. This threshold value depended on Ω.

In an effort to incorporate the in-plane vibrations of the rotor and brake pad backing plates, the effects of a floating caliper, and the influence of constrained layer damping, Chowdhary [58] extended the model used in Ref. [135] using the finite element method.[66] Specifically, the rotor, caliper and brake pad assemblies were individually modelled using finite elements and the resulting stiffness and mass matrices were imported into *Mathlab*.[67] Discrete contact elements were then used to interconnect the rotor, caliper and brake pad assemblies. Where appropriate, these contact elements featured a constant μ_k. The resulting equations governing the vibrations of the brake assembly are of form (10) and an eigenvalue analysis can be performed to determine the onset of squeal. Chowdhary used the model to examine the influence of the caliper on squeal, the effects of chamfering the brake pads on supression of squeal, and the influence of damping. He found that the damping significantly increased the value of μ_k needed for the onset of squeal, thereby correcting a deficiency in the earlier model of Chowdhary et al. [135]. In addition, he was able to explain that chamfering could modify the merging behaviour of the modes and thereby reduce squeal. These results are clearly significant because they provide an explanation for several squeal reduction techniques.

Research by a group at Ford Motor Company on finite element models for disc brake squeal is summarized in Ref. [146]. Of particular interest is the use of a finite element model which features

[66] The reader may wish to recall that in Ref. [135] the rotor was modelled using a plate model, and the backing plates were modelled as sectors of plates.

[67] It is interesting to note that the pads of friction material in the brake pad assemblies were modelled as orthotropic elastic bodies.

a coefficient of friction μ_k which depends on the velocity v_s, the interface temperature field T, and the interface pressure field p: $\mu_k(v_s, p, T)$. Another interesting aspect of some of the works by this group is their simulations, discussed in Refs. [201–203], of transient non-linear dynamics of a brake system where the rotor is initially rotating and the piston experiences a sharp increase in brake line pressure. These simulations are also correlated to experimental results. Perhaps because of the proprietary nature of this research, many of the details in the papers [201,202,146] are lacking.

Although it pertains to the analysis of aircraft brakes, the work of Hamzeh et al. [204] merits special mention because of its two significant novelties when compared to the previously cited works: first, it uses an experimentally fitted law for the contact between the pads of friction material and the rotor which is far more complex than the classical friction model that is normally used. Second, it departs from the logic of the linear stability analysis by attempting to identify instabilities by perturbing the steady state and tracing the growth or decay of the perturbation in time.

9. Conclusions

Despite a century of developing disc brake systems, disc brake squeal remains a largely unresolved problem. This is not to say, however, that no progress has been made. Many experimental and analytical studies have led to insight on the factors contributing to brake squeal or to the amelioration of squeal in disc brakes of a specific type or in a particular make and model of automobile. Experimental studies have accumulated a wealth of information about the nature of squeal, the vibration modes therein, the wear of brake components, and frictional interactions in brakes. Analytical studies have provided useful insights into how friction laws, geometry, and the dynamics of brake components can lead to squeal or instability in simple models of disc brakes. Finite elements have been used to try to extend these insights to more accurate brake models.

The development of a comprehensive predictive model of disc brake squeal is the ultimate goal of much of literature surveyed in this paper. This goal has been elusive, and whether or not such a model is possible remains to be seen. Indeed, given the number of factors involved and the range of designs, such a model, if developed, may be too complex to be useful. On the other hand, it is not too difficult to envisage improvements in existing models for disc brake assemblies. These improvements might include the influence of the variable decreasing speed of rotation of the rotor, the inclusion of thermal and wear effects, and the incorporation of more complex friction models and constitutive models for the pads of friction material.

Pertaining to analysis, most of the studies we have discussed centre on a linear analysis to predict the onset of instability. As noted earlier, several researchers have correlated this instability to the occurrence of squeal. However, the real issue is what happens after instability has occurred. It is well known that to address this issue one needs a non-linear model. This issue has not been significantly pursued in the disc brake squeal literature. One of the reasons for this is that a non-linear analysis of a large degree-of-freedom model is a formidable task.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. CMS0084808. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

During the course of researching this review article, the authors had several fruitful discussions with Prof. Arvind Raman (Purdue University) on the dynamics of spinning discs, with Louie Loufas on braking systems, and with Profs. Anil Bajaj and Chuck Krousgrill (Purdue University) on squealing brakes.

Several researchers kindly supplied figures: Prof. Anil Bajaj provided Fig. 19, Dr. John Fieldhouse (University of Huddersfield) provided Fig. 9, Prof. Staffan Jacobson (University of Uppsala) provided Fig. 10, and Prof. John Mottershead and Dr. Huajiang Ouyang (University of Liverpool) provided Fig. 21.

The authors are grateful to several researchers who sent papers and dissertations on request: Profs. Anil Bajaj and Chuck Krousgrill, Dr. Frank Chen (Ford Motor Company), Prof. Ken Cunefare (Georgia Institute of Technology), Prof. Staffan Jacobson, Dr. Johan Hultén (Volvo), Prof. Raouf Ibrahim (Wayne State University), Prof. John Mottershead and Dr. Huajiang Ouyang, Prof. Greg McDaniel (Boston University), and Herr Martin Rudolph (University of Hannover).

Several people, including Michael Drew, Patrick Kessler, Todd Lauderdale and Peter Varadi provided valuable comments and corrections on an earlier draft of this paper. Finally, the authors are grateful to a reviewer for several illuminating comments.

References

[1] H. Abendroth, B. Wernitz, The integrated test concept: Dyno—vehicle, performance—noise, Technical Report 2000-01-2774, SAE, Warrendale, PA, 2000.

[2] D.A. Crolla, A.M. Lang, Brake noise and vibration—the state of the art, in: D. Dowson, C.M. Taylor, M. Godet, (Eds.), Vehicle Tribology, number 18 in Tribology Series, Leeds, England, 1991, pp. 165–174.

[3] M.R. North, Disc brake squeal, in: Braking of Road Vehicles, Automobile Division of the Institution of Mechanical Engineers, Mechanical Engineering Publications Limited, London, England, 1976, pp. 169–176.

[4] H.H. Leipholz, Stability of Elastic Systems, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

[5] H. Ziegler, Principles of Structural Stability, Blaisdell, Waltham, MA, 1968.

[6] R.A. Ibrahim, Friction-induced vibration, chatter, squeal, and chaos: part II—dynamics and modeling, in: R.A. Ibrahim, A. Soom (Eds.), Friction-Induced Vibration, Chatter, Squeal, and Chaos, Vol. DE-49, ASME, New York, 1992 pp. 123–138.

[7] R.A. Ibrahim, Friction-induced vibration, chatter, squeal, and chaos: part II: dynamics and modeling, American Society of Mechanical Engineers Applied Mechanics Reviews 47 (7) (1994) 227–253.

[8] J.P. Den Hartog, Mechanical Vibrations, 4th Edition, McGraw-Hill, New York, 1956.

[9] S. Yang, R.F. Gibson, Brake vibration and noise: reviews, comments, and proposals, International Journal of Materials and Product Technology 12 (1997) 496–513.

[10] A. Papinniemi, J.C.S. Lai, J. Zhao, L. Loader, Brake squeal: a literature review, Applied Acoustics 63 (4) (2002) 391–400.

[11] R.A. Ibrahim, Friction-induced noise and related problems in automotive brakes, in: S.G. Pandalai (Ed.), Recent Research Developments in Sound and Vibration, Vol. 1, Transworld Research Network, Kerala, India, 2002.
[12] A. Akay, Acoustics of friction, Journal of Acoustical Society of America 111 (4) (2002) 1525–1548.
[13] C.S. Clark, The Lanchester Legacy, a Trilogy of Lanchester Works, Vol. 1, Butler & Tanner, Frome and London, England, 1995, pp. 1895–1931.
[14] G.A. Harper, Brakes and Friction Materials: The History and Development of the Technologies, Mechanical Engineering Publications Limited, London, England, 1998.
[15] T.P. Newcomb, R.T. Spurr, A Technical History of the Motor Car, Adam Hilger, New York, 1989.
[16] F.W. Lanchester, Improvements in the brake mechanism of power-propelled road vehicles, G.B. Patent No. 26407, 1902.
[17] T.P. Hughes, Elmer Sperry: Inventor and Engineer, John Hopkins Press, Baltimore, MD, 1971.
[18] E.A. Sperry, The electric brake in practice, Transactions AIEE 11 (1894) 682–728.
[19] E.A. Sperry, Electric brake, US Patent Nos. 534974, 534977, 1895.
[20] E.A. Sperry, Electric brake, US Patent No. 565937, 1896.
[21] T.P. Newcomb, R.T. Spurr, Automobile Brakes and Braking Systems, R. Bentley Inc., Cambridge, MA, 1969.
[22] C.J. Kahane, A preliminary evaluation of two braking improvements for passenger cars—dual master cylinders and front disc brakes, Technical Report DOT HS 806 359, NHTSA, Washington, DC, 1983.
[23] P. Oppenheimer, Braking regulations for passenger cars, Technical Report 770182, SAE, Warrendale, PA, 1977.
[24] J.D. Halderman, C.D. Mitchell Jr., Automotive Brake Systems, 2nd Edition, Prentice-Hall, Upper Saddle River, NJ, 2000.
[25] A.E. Anderson, Friction and wear of automotive brakes, in: ASM Handbook, Vol. 18, ASM International, Materials Park, OH, 1992, pp. 569–577.
[26] F. Bergman, M. Eriksson, S. Jacobson, Influence of disk topography on generation of brake squeal, Wear 225–229 (1999) 621–628.
[27] M.G. Jacko, P.H.S. Tsang, S.K. Rhee, Automotive friction materials evolution during the past decade, Wear 100 (1984) 503–515.
[28] H.Y. Loken, Asbestos free brakes and dry clutches reinforced with kevlar aramid fiber, Technical Report 800667, SAE, Warrendale, PA, 1980.
[29] Asbestos Institute, http://www.chrystolite.com/regulat/us.htm, 1999–2000.
[30] UK Department of the Environment, Transport and the Regions, http://www.roads.dtlr.gov.uk/vehicle/consult/asbestos, 1998.
[31] T.P. Newcomb, R.T. Spurr, Friction materials for brakes, Tribology 4 (2) (1971) 75–81.
[32] S.K. Rhee, Wear mechanisms for asbestos-reinforced automotive friction materials, Wear 29 (3) (1974) 391–393.
[33] F. Bergman, M. Eriksson, S. Jacobson, The effect of Cu_2S solid lubricant addition, and varying density, on the occurrence of brakes squeals for one low-metal, organic type brake pad, in: Proceedings of the 17th Annual SAE Brake Colloquium and Engineering Display, SAE, Warrendale, PA, 1999, pp. 137–143.
[34] F. Bergman, L. Gudmand-Høyer, M. Eriksson, S. Jacobson, The effect of Cu_2S, PbS, Sb_2S_3 solid lubricants on the occurrence of brakes squeals for three automotive brake pad matrix types, in: S.S. Eskildsen, D.S. Larsen, H. Reitz, E.J. Bienk, C.A. Straede (Eds.), Nordtrib '98: Proceedings of the Eighth International Conference on Tribology, DTI Tribology Center, Aarhus, Denmark, 1998, pp. 665–672.
[35] P. Bedard, Comparison test—little cars 6.1, Car and Driver 45 (12) (2000) 54–81.
[36] F. Markus, Road test–2002 Ford Explorer Eddie Bauer 4×4, Car and Driver 46 (7) (2001) 74–78.
[37] T. Swan, Comparison test—fanzy panzers, Car and Driver 46 (2) (2000) 46–59.
[38] R.P. Jarvis, B. Mills, Vibrations induced by friction, Proceedings of the Institution of Mechanical Engineers 178 (32) (1963) 847–857.
[39] J.O. Hultén, J. Flint, An assumed modes method approach to disc brake squeal analysis, Technical Report 1999-01-1335, SAE, Warrendale, PA, SAE, 1999.
[40] H. Ouyang, J.E. Mottershead, A bounded region of disc-brake vibration instability, Transactions of the American Society of Mechanical Engineers Journal of Vibration and Acoustics 123 (4) (2001) 543–545.
[41] H. Ouyang, J.E. Mottershead, D.J. Brookfield, S. James, M.P. Cartmell, A methodology for the determination of dynamic instabilities in a car disc brake, International Journal of Vehicle Design 23 (3/4) (2000) 241–262.

[42] H. Ouyang, J.E. Mottershead, D.J. Brookfield, S. James, M.P. Cartmell, T. Kaster, T. Treyde, B. Hirst, R. Allen, Dynamic instabilities in a simple model of a car disc brake, Technical Report 1999-01-3409, SAE, Warrendale, PA, 1999.
[43] J.-G. Tseng, J.A. Wickert, Nonconservative stability of a friction loaded disk, Transactions of the American Society for Mechanical Engineers Journal of Vibration and Acoustics 120 (4) (1998) 922–929.
[44] H. Baba, T. Wada, T. Takagi, Study on reduction of brake squeal caused by in-plane vibration on rotor, Technical Report 2001-01-3158, SAE, Warrendale, PA, 2001.
[45] J.C. Bae, J.A. Wickert, Free vibration of coupled disk-hat structures, Journal of Sound and Vibration 235 (1) (2000) 117–132.
[46] S.-W. Kung, K.B. Dunlap, R.S. Ballinger, Complex eigenvalue analysis for reducing low frequency brake squeal, Technical Report 2000-01-0444, SAE, Warrendale, PA, 2000.
[47] K. Ono, J.-S. Chen, D.B. Bogy, Stability analysis for the head-disk interface in a flexible disk drive, Transactions of the American Society of Mechanical Engineers Journal of Applied Mechanics 58 (4) (1991) 1005–1014.
[48] M. Nishiwaki, Generalized theory of brake noise, Proceedings of the Institution of Mechanical Engineers D 207 (1993) 195–202.
[49] J.E. Mottershead, S.N. Chan, Brake squeal—an analysis of symmetry and flutter instability, in: R.A. Ibrahim, A. Som (Eds.), Friction-Induced Vibration, Chatter, Squeal, and Chaos, Vol. DE-49, American Society of Mechanical Engineers, New York, 1992, pp. 87–97.
[50] J.E. Mottershead, S.N. Chan, Flutter instability of circular discs with frictional follower forces, Transactions of the American Society of Mechanical Engineers Journal of Vibration and Acoustics 117 (1) (1995) 161–163.
[51] J.E. Mottershead, Vibration- and friction-induced instability in disks, Shock and Vibration Digest 30 (1) (1998) 14–31.
[52] J.Y. Chang, J.A. Wickert, Response of modulated doublet modes to travelling wave excitation, Journal of Sound and Vibration 242 (1) (2001) 69–83.
[53] M. Kim, J. Moon, J.A. Wickert, Spatial modulation of repeated vibration modes in rotationally periodic structures, Transactions of the American Society of Mechanical Engineers Journal of Vibration and Acoustics 122 (1) (2000) 62–68.
[54] A. Raman, C.D. Mote, Jr., Non-linear oscillations of circular plates near a critical speed resonance, International Journal of Nonlinear Mechanics 34 (1) (1999) 139–157.
[55] A. Raman, C.D. Mote, Jr., Effects of imperfection on the non-linear oscillations of circular plates spinning near critical speed, International Journal of Nonlinear Mechanics 36 (2) (2001) 261–289.
[56] H. Lamb, R.V. Southwell, The vibrations of a spinning disk, Royal Society of London Proceedings Series A Mathematics Physics and Engineering Science 99 (699) (1921) 272–280.
[57] V.K. Meel, Modal characteristics of annular elastic solids and a disc brake rotor model, Master's Thesis, Purdue University, West Lafayette, Indiana, USA, 2000.
[58] H.V. Chowdhary, Modeling of Disc Brake Systems for Squeal Prediction, Master's Thesis, Purdue University, West Lafayette, Indiana, USA, 2001.
[59] J.G. McDaniel, X. Li, J. Moore, S.-E. Chen, Analysis of instabilities and power flow in brake systems with coupled rotor modes, Technical Report 01NVC-108, SAE, Warrendale, PA, 2001.
[60] P.M. Naghdi, Finite deformation of elastic rods and shells, in: D.E. Carlson, R.T. Shield (Eds.), Finite Elasticity: Proceedings of the IUTAM Symposium on Finite Elasticity, Bethlehem, PA, 1980, Martinus Nijhoff, The Hague, Netherlands, 1982, pp. 47–103.
[61] M.B. Rubin, Cosserat Theories: Shells, Rods and Points, Kluwer Academic Publishers, Dordrecht, Netherlands, 2000.
[62] N. Millner, An analysis of disc brake squeal, Technical Report 780332, SAE, Warrendale, PA, 1978.
[63] J. Flint, The effect of distributed parameters examined in a model for simulation of disc brake squeal, Technical Report 2000-01-2766, SAE, Warrendale, PA, 2000.
[64] Y. Yuan, A study of the effects of negative friction-speed slope on brake squeal, in: Proceedings of DETC'95, DE-Vol. 84-1, ASME, New York, 1995, pp. 1153–1162.
[65] J.-S. Chen, Stability Analysis of a Spinning Flexible Disk-stationary Load System: Application to Floppy Disk Drives, Ph.D. Thesis, University of California at Berkeley, Berkeley, USA, 1992.

[66] O.M. O'Reilly, N.K. Malhotra, N.S. Namachchivaya, Some aspects of destabilization in reversible dynamical systems with application to follower forces, Nonlinear Dynamics 10 (1) (1996) 63–87.
[67] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, T.S. Ratiu, Dissipation induced instabilities, Annales de L'Institut H. Poincaré: Analyse Non Linéaire 11 (1) (1994) 37–90.
[68] F.P. Bowden, D. Tabor, The Friction and Lubrication of Solids, Oxford University Press, Clarendon Press, Oxford, (second corrected edition, 2001. Oxford Classic Text in the Physical Sciences).
[69] I.V. Kragelskii, Friction and Wear, Butterworth, London, 1965.
[70] E. Rabinowicz, Friction and Wear of Materials, Wiley, New York, 1965.
[71] N.P. Suh, Tribophysics, Prentice–Hall, Englewood Cliffs, NJ, 1986.
[72] D. Dawson, History of Tribology, 2nd Edition, Professional Engineering Publishing, London and Bury St. Edmunds, England, 1998.
[73] B. Armstrong-Hélouvry, P. DuPont, C. Canudas de Wit, A survey of models, analysis tools and compensation methods for the control of machines with friction, Automatica 30 (7) (1994) 1093–1138.
[74] J.T. Oden, J.A.C. Martins, Models and computational methods for dynamic friction phenomena, Computer Methods in Applied Mechanics and Engineering 52 (1985) 527–634.
[75] H. Dankowicz, On the modeling of dynamic friction pheonomena, Zeitschrift für Angewandte Mathematik und Mechanik 79 (6) (1999) 399–409.
[76] A.L. Ruina, Constitutive relations for frictional slip, in: Z.P. Bažant (Ed.), Mechanics of Geomaterials: Rocks, Concrete, Soils, Wiley, New York, 1985, pp. 169–188.
[77] N.P. Suh, H.-C. Sin, The genesis of friction, Wear 69 (1981) 91–114.
[78] J.A.C. Martins, J.T. Oden, F.M.F. Simões, A study of static and kinetic friction, International Journal of Engineering Science 28 (1) (1990) 29–92.
[79] D.M. Tolstoi, Significance of the normal degree of freedom and natural normal vibrations in contact friction, Wear 10 (3) (1967) 199–213.
[80] A.J. Day, M. Tirovic, T.P. Newcomb, Thermal effects and pressure distributions in brakes, Proceedings of the Institution of Mechanical Engineers D 205 (1991) 199–205.
[81] C. Hohmann, K. Schiffner, K. Oerter, H. Reese, Contact analysis for drum brakes and disk brakes using ADINA, Computers & Structures 72 (1–3) (1999) 185–198.
[82] M. Tirovic, A.J. Day, Disc brake interface pressure distributions, Proceedings of the Institution of Mechanical Engineers D 205 (1991) 137–146.
[83] M. Eriksson, F. Bergman, S. Jacobson, On the nature of tribiological contact in automotive brakes, Wear 252 (1/2) (2002) 26–36.
[84] A. Ilm, P.S. Leung, P.K. Datta, Experimental investigations of disc brake friction, Technical Report 2000-01-2778, SAE, Warrendale, PA, 2000.
[85] S.K. Rhee, M.G. Jacko, P.H.S. Tsang, The role of friction film in friction, wear, and noise of automotive brakes, Technical Report 900004, SAE, Warrendale, PA, 1990.
[86] M. Eriksson, S. Jacobson, Friction behaviour and squeal generation of disc brakes at low speeds, Proceedings of the Institution of Mechanical Engineers D 215 (D12) (2001) 1245–1256.
[87] M. Eriksson, J. Lord, S. Jacobson, Wear and contact conditions of brake pads: dynamical in situ studies of pad on glass, Wear 249 (3–4) (2001) 272–278.
[88] W. Österle, M. Griepentrog, Th. Gross, I. Urban, Chemical and microstructural changes induced in friction and wear of brakes, Wear 251 (1–12) (2001) 1469–1476.
[89] W. Kreitlow, F. Schrödter, H. Matthäi, Vibration and hum of disc brakes under load, Technical Report 850079, SAE, Warrendale, PA, 1985.
[90] Y.-B. Yi, S. Du, J.R. Barber, J.W. Fash, Effect of geometry on thermoelastic instability in disk brakes and clutches, Transactions of the American Society for Mechanical Engineers Journal of Tribology 121 (4) (1999) 661–666.
[91] S.K. Rhee, P.H.S. Tsang, Y.S. Wang, Friction-induced noise and vibration of disc brakes, Wear 133 (1989) 39–45.
[92] J.R. Barber, The influence of thermal expansion on the friction and wear process, Wear 10 (2) (1967) 155–159.

[93] J.R. Barber, Thermoelastic instabilities in the sliding of conforming bodies, Royal Society of London Proceedings Series A Mathematics Physics and Engineering Science 312 (1510) (1969) 381–394.
[94] Th.A. Dow, R.A. Burton, Thermoelastic instability of sliding contact in the absence of wear, Wear 19 (3) (1972) 315–328.
[95] F.E. Kennedy, F.F. Ling, A thermal, thermoelastic, and wear simulation of a high-energy sliding contact problem, Transactions of the American Society of Mechanical Engineers Journal of Lubrication Technology 96 (3) (1974) 497–507.
[96] F. Samie, D.C. Sheridan, Contact analysis for a passenger car disc brake, Technical Report 900005, SAE, Warrendale, PA, 1990.
[97] S. Du, J.W. Fash, Finite element analysis of frictionally-excited thermoelastic instability in a 3D annular disk, International Journal of Vehicle Design 23 (3/4) (2000) 203–217.
[98] S. Du, P. Zagrodzki, J. Barber, G. Hulbert, Finite element analysis of frictionally excited thermoelastic instability, Journal of Thermal Stresses 20 (2) (1997) 185–201.
[99] K. Lee, J.R. Barber, Frictionally excited thermoelastic instability in automotive brakes, Transactions of the American Society of Mechanical Engineers Journal of Tribology 115 (4) (1993) 607–614.
[100] Y.-B. Yi, J.R. Barber, D.L. Hartsock, Thermoelastic instabilities in automotive disk brakes—finite element analysis and experimental verification, in: J.A.C. Martius, M.D.P. Monteiro Marques (Eds.), Proceedings of the Third Contact Mechanics International Symposium held in Peniche, Portugal, June 17–21, 2001, Kluwer, Dordrecht, Netherlands, 2002.
[101] P. Zagrodzki, K.B. Lam, E. Al Bahkali, J.R. Barber, Nonlinear transient behavior of a sliding system with frictionally excited thermoelastic instability, Transactions of the American Society of Mechanical Engineers Journal of Tribology 123 (4) (2001) 699–708.
[102] A.E. Anderson, R.A. Knapp, Hot spotting in automotive friction systems, Wear 135 (2) (1990) 319–337.
[103] J.R. Barber, Thermoelasticity and contact, Journal of Thermal Stresses 22 (4/5) (1999) 513–525.
[104] J.R. Barber, T.W. Beamond, J.R. Waring, C. Pritchard, Implications of thermoelastic instability for the design of brakes, Transactions of the American Society of Mechanical Engineers Journal of Tribology 107 (2) (1985) 206–210.
[105] F.E. Kennedy, Thermal and thermomechanical effects in dry sliding, Wear 100 (1984) 453–476.
[106] A. Felske, G. Hoppe, H. Matthäi, Oscillations in squealing disc brakes—analysis of vibration modes by holographic interferometry, Technical Report 780333, SAE, Warrendale, PA, 1978.
[107] H. Abendroth, Advances in brake NVH test equipment, Automotive Engineering International 107 (2) (1999) 60–63.
[108] R. Krupka, T. Walz, A. Ettemeyer, New techniques and applications for 3D-brake vibration analysis, Technical Report 2000-01-2772, SAE, Warrendale, PA, 2000.
[109] R. Jones, C. Wykes, Holographic and Speckle Interferometry: A Discussion of the Theory, Practise and Application of the Techniques, 2nd Edition, Cambridge University Press, Cambridge, 1989.
[110] Yu.I. Ostrovsky, V.P. Shchepinov, V.V. Yakovlev, Holographic Interferometry in Experimental Mechanics, Springer, Berlin, 1991.
[111] H.R. Mills, Brake squeak, Technical Report 9000 B, Institution of Automobile Engineers, 1938.
[112] R.A.C. Fosberry, Z. Holubecki, Interim report on disc brake squeal, Technical Report 1959/4, Motor Industry Research Association, Warwickshire, England, 1959.
[113] R.A.C. Fosberry, Z. Holubecki, Disc brake squeal: its mechanism and suppression, Technical Report 1961/1, Motor Industry Research Association, Warwickshire, England, 1961.
[114] R.T. Spurr, A theory of brake squeal, Proceedings of the Automobile Division, Institution of Mechanical Engineers 1961–1962 (1) (1961) 33–52.
[115] S.W.E. Earles, A mechanism of disc-brake squeal, Technical Report 770181, SAE, Warrendale, PA, 1977.
[116] S.W.E. Earles, M. Badi, Oscillatory instabilites generated in a double-pin and disc undamped system: a mechanism of disc-brake squeal, Proceedings of the Institution of Mechanical Engineers C 198 (1984) 43–49.
[117] S.W.E. Earles, P.W. Chambers, Disc brake squeal noise generation: predicting its dependency on system parameters including damping, International Journal of Vehicle Design 8 (1987) 538–552.

[118] S.W.E. Earles, C. Lee, Instabilities arising from the frictional interaction of a pin-disk system resulting in noise generation, Transactions of the American Society of Mechanical Engineers Journal of Engineering for Industry 98 (1) (1976) 81–86.
[119] S.W.E. Earles, G.B. Soar, Squeal noise in disc brakes, in: Vibration and Noise in Motor Vehicles, Institution of Mechanical Engineers, London, England, 1971, pp. 61–69, Paper number C 101/71.
[120] M.R. North, Disc brake squeal—a theoretical model, Technical Report 1972/5, Motor Industry Research Association, Warwickshire, England, 1972.
[121] H. Murakami, N. Tsunada, T. Kitamura, A study concerned with a mechanism of disc-brake squeal, Technical Report 841233, SAE, Warrendale, PA, 1984.
[122] M. Nishiwaki, H. Harada, H. Okamura, T. Ikeuchi, Study on disc brake squeal, Technical Report 890864, SAE, Warrendale, PA, 1989.
[123] Y. Ichiba, Y. Nagasawa, Experimental study on brake squeal, Technical Report 930802, SAE, Warrendale, PA, 1993.
[124] M. Reeves, N. Taylor, C. Edwards, D. Williams, C.H. Buckberry, A study of disc modal behavior during squeal generation using high-speed electronic speckle pattern interferometry and near-field sound pressure measurements, Proceedings of the Institution of Mechanical Engineers D 214 (D3) (2000) 285–296.
[125] C. Talbot, J.D. Fieldhouse, Animations of a disc brake generating noise, Technical Report 2001-01-3126, SAE, Warrendale, PA, 2001.
[126] K. Ohta, K. Kagawa, T. Eto, S. Nishikawa, Study on the disc brake squeal (simulation using modal analysis technique and experiments), Transactions of the Japanese Society of Mechanical Engineers, Series C 50 (457) (1984) 1585–1593 (in Japanese).
[127] J.D. Fieldhouse, P. Newcomb, The application of holographic interferometry to the study of disc brake noise, Technical Report 930805, SAE, Warrendale, PA, 1993.
[128] J.D. Fieldhouse, T.P. Newcomb, Double pulsed holography used to investigate noisy brakes, Optics and Lasers in Engineering 25 (6) (1996) 455–494.
[129] A.M. Lang, T.P. Newcomb, An experimental investigation into drum brake squeal, in: EAEC Conference on New Developments in Powertrain and Chassis Engineering, Institution of Mechanical Engineers Paper No. C382/051, 1989, pp. 431–444.
[130] J.O. Hultén, Friction phenomena related to drum brake squeal instabilities, in: CD-ROM Proceedings of DETC'97, DETC97/VIB-4161, ASME, New York, 1997, pp. 1–12.
[131] J.D. Fieldhouse, M. Rennison, An investigation of low frequency drum brake noise, Technical Report 982250, SAE, Warrendale, PA, 1998.
[132] B.K. Servis The Onset of Squeal Vibrations Resulting from a Coupled Mode Instability, Ph.D. Thesis, Purdue University, West Lafayette, Indiana, USA, 2000.
[133] J.D. Fieldhouse, A proposal to predict the noise frequency of a disc brake based on the friction pair interface geometry, Technical Report 1999-01-3403, SAE, Warrendale, PA, 1999.
[134] A.M. Lang, H. Smales, An approach to the solution of disc brake vibration problems, in: Braking of Road Vehicles. Automobile Division of the Institution of Mechanical Engineers, Mechanical Engineering Publications Limited, Suffolk, England, 1993, pp. 223–231.
[135] H.V. Chowdhary, A.K. Bajaj, C.M. Krousgill, An analytical approach to model disc brake system for squeal prediction, in: CD-ROM Proceedings of DETC'01, DETC2001/VIB-21560, ASME, Pittsburgh, PA, 2001, pp. 1–10.
[136] M. Matsuzaki, T. Izumihara, Brake noise caused by longitudinal vibration of the disc rotor, Technical Report 930804, SAE, Warrendale, PA, 1993.
[137] J.H. Tarter, Disc brake squeal, Technical Report 830530, SAE, Warrendale, PA, 1983.
[138] K.B. Dunlap, M.A. Riehle, R.E. Longhouse, An investigative overview of automobile disc brake noise, Technical Report 1999-01-0142, SAE, Warrendale, PA, 1999.
[139] F. Chen, S.-E. Chen, P. Harwood, In-plane mode/friction process and their contribution to disc brake squeal at high frequency, Technical Report 2000-01-2773, SAE, Warrendale, PA, 2000.
[140] F. Chen, J. Chern, J. Swayze, Modal coupling and its effect on brake squeal, Technical Report 2002-01-0922, SAE, Warrendale, PA, 2002.

[141] J.G. McDaniel, J. Moore, S.-E. Chen, C.L. Clarke, Acoustic radiation models of brake systems from stationary LDV measurements, in: Proceedings of IMEC' 99, American Society of Mechanical Engineers, New York, 1999.
[142] A. Saad, M. Watany, S. Abouel-Seoud, I. Abdel-Gawad, Passenger car disk brake squeal, Proceedings of the International SAMPE Symposium and Exhibition 43 (2) (1998) 1158–1169.
[143] M. Watany, S. Abouel-Seoud, A. Saad, I. Abdel-Gawad, Brake squeal generation, Technical Report 1999-01-1735, SAE, Warrendale, PA, 1999.
[144] H. Matsui, M. Murakami, H. Nakanishi, Y. Tsuda, Analysis of disc brake squeal, Technical Report 920553, SAE, Warrendale, PA, 1992.
[145] K.A. Cunefare, R. Rye, Investigation of disc brake squeal via sound intensity and laser vibrometry, Technical Report 2001-01-1604, SAE, Warrendale, PA, SAE, 2001.
[146] S. Mahajan, Y.-K. Hu, K. Zhang, Vehicle disc brake squeal simulations and experiences, Technical Report 1999-01-1738, SAE, Warrendale, PA, 1999.
[147] J. Hald, Time domain acoustical holography and its applications, Sound and Vibration 35 (2) (2001) 16–25.
[148] F. Bergman, Tribological nature of squealing disc brakes, Ph.D. Thesis, Uppsala University, Uppasala, Sweden, 1999.
[149] M. Eriksson, Friction and Contact Phenomena of Disc Brakes Related to Squeal, Ph.D. Thesis, Uppsala University, Uppsala, Sweden, 2000.
[150] D. Vola, M. Raous, J.A.C. Martins, Friction and instability of steady sliding: squeal of a rubber/glass contact, International Journal for Numerical Methods in Engineering 46 (10) (1999) 1699–1720.
[151] F. Bergman, M. Eriksson, S. Jacobson, The effect of reduced contact area on the occurrence of disc brake squeals for an automotive brake pad, Proceedings of the Institution of Mechanical Engineers D 214 (D5) (2000) 561–568.
[152] M. Eriksson, F. Bergman, S. Jacobson, Surface characterization of brake pads after running under silent and squealing conditions, Wear 232 (2) (1999) 163–167.
[153] M. Eriksson, A. Lundquist, S. Jacobson, A study of the influence of humidity on the friction and squeal generation of automotive brake pads, Proceedings of the Institution of Mechanical Engineers D 215 (D3) (2001) 329–342.
[154] R.A. Ibrahim, S. Madhavan, S.L. Qiao, W.K. Chang, Experimental investigation of friction-induced noise in disc brake systems, International Journal of Vehicle Design 23 (3–4) (2000) 218–240.
[155] S.L. Qiao, R.A. Ibrahim, Stochastic dynamics of systems with friction-induced vibration, Journal of Sound and Vibration 223 (1) (1999) 115–140.
[156] G.D. Liles, Analysis of disc brake squeal using finite element methods, Technical Report 891150, SAE, Warrendale, PA, SAE, 1989.
[157] W. Liu, J. Pfeiffer, Reducing high frequency disc brake squeal by pad shape optimization, Technical Report 2000-01-0447, SAE, Warrendale, PA, 2000.
[158] L. Reynolds, Got squealing brakes?, http://www.porschepark.org/garage/ta00004.html, April 2000.
[159] R. Singh, A.A. Sheikh, M.J. Mitchell, Viscoelastic damping to control disc brake squeal, Sound and Vibration 32 (10) (1998) 18–22.
[160] J. Storer, R. Maddox, J.H. Haynes, Honda Accord Automotive Repair Manual, Haynes, North America, Newbury Park, CA, 1999.
[161] T.A. Nosseir, M.A.R. Said, N.S. El Nahas, G. Abu El Fetouh, Significance of squeal in disk brake design, International Journal of Vehicle Design 19 (1) (1998) 124–133.
[162] P.C. Brooks, D.A. Crolla, A.M. Lang, D.R. Schafer, Eigenvalue sensitivity analysis applied to disc brake squeal, in: Braking of Road Vehicles, Institution of Mechanical Engineers, Mechanical Engineering Publications Limited, Bury St. Edmunds, England, 1993, pp. 135–143.
[163] P. DiLisio, R. Parisi, J. Rieker, W. Stringham, Brake noise resolution on the 1998 Mercedes-Benz M-Class, Technical Report 982245, SAE, Warrendale, PA, 1998.
[164] Y. Kumemura, Y. Gamo, K. Kono, S. Suga, Analysis for reducing low frequency squeal of a disc brake, Technical Report 2001-01-3137, SAE, Warrendale, PA, 2001.
[165] K.A. Cunefare, A.J. Graf, Disc brake rotor squeal suppression using dither control, Technical Report 2001-01-1605, SAE, Warrendale, PA, 2001.

[166] K.A. Cunefare, A.J. Graf, Experimental active control of automotive disc brake rotor squeal using dither, Journal of Sound and Vibration 250 (4) (2002) 579–590.
[167] D. Sinclair, Frictional vibrations, Transactions of the American Society of Mechanical Engineers Journal of Applied Mechanics 77 (1955) 207–213.
[168] F.P. Bowden, L. Leben, The nature of sliding and the analysis of friction, Royal Society of London Proceedings Series A Mathematics Physics and Engineering Science 169 (1939) 371–391.
[169] N.L. Kajdanowsky, S.E. Chajkin, Mechanical relaxation oscillations, Zhurnal Teknischeskoi Fiziki 3 (1) (1933) 93–109 (in Russian).
[170] F. Morgan, M. Musjat, D.W. Reed, Friction phenomena and the stick-slip process, Journal of Applied Physics 12 (10) (1941) 743–752.
[171] H. Blok, Fundamental mechanical aspects of boundary lubrication, SAE Journal 46 (2) (1940) 54–68.
[172] P. Painlevé, Sur le lois du frottement de glissemment, Comptes Rendus Academie Science Paris 121 (1895) 112–115. Also Vol. 140, pp. 702–707 and Vol. 141, pp. 401–405 & pp. 546–552 (1905) (in French).
[173] D.E. Stewart, Rigid-body dynamics with friction and impact, SIAM Review 42 (1) (2000) 3–39.
[174] J.D.C. Crisp, R.A.C. Fosberry, Z. Holubecki, V.E. Gough, F.D. Hales, G.H. Lanchester, L. Martland, R.T. Spurr, A.J. Wilson, Communications on Jarvis and Mills' paper, Proceedings of the Institution of Mechanical Engineers 178 (32) (1963) 857–864.
[175] J.O. Hultén, Drum Brake Squeal, Ph.D. Thesis, Chalmers University of Technology, Göteborg, Sweden, 1998.
[176] J.O. Hultén, J. Flint, T. Nellemose, Mode shape of a squealing drum brake, Technical Report 972028, SAE, Warrendale, PA, 1997.
[177] Y. Denou, M. Nishiwaki, First order analysis of low frequency disk brake squeal, Technical Report 2001-01-3136, SAE, Warrendale, PA, 2001.
[178] S.N. Chan, J.E. Mottershead, M.P. Cartmell, Parametric resonances at subcritical speeds in discs with rotating frictional loads, Proceedings of the Institution of Mechanical Engineers Part C 208 (C6) (1994) 417–425.
[179] H. Abendroth, A new approach to brake testing, Technical Report 850080, SAE, Warrendale, PA, 1985.
[180] A. Schallamach, How does rubber slide?, Wear 17 (4) (1971) 301–312.
[181] M. Rudolph, K. Popp, Brake squeal, in: K. Popp (Ed.), Detection, Utilization and Avoidance of Nonlinear Dynamical Effects in Engineering Applications: Final Report of a Joint Research Project Sponsored by the German Federal Ministry of Education and Research, Shaker, Aachen, 2001, pp. 197–225.
[182] M. Rudolph, K. Popp, Friction induced brake vibrations, in: CD-ROM Proceedings of DETC'01, DETC2001/VIB-21509, ASME, Pittsburgh, PA, 2001, pp. 1–10.
[183] R.P. Jarvis, B. Mills, Authors' reply, Proceedings of the Institution of Mechanical Engineers 178 (32) (1963) 864–866.
[184] S.W.E. Earles, P.W. Chambers, Predicting some effects of damping on the occurrence of disc-brake squeal noise, in: ASME Dynamic Systems and Control Division, Vol. 1, ASME, New York, 1985, pp. 317–323.
[185] A.M. El-Butch, I.M. Ibrahim, Modeling and analysis of geometrically induced vibration in disc brakes considering contact parameters, Technical Report 1999-01-0143, SAE, Warrendale, PA, 1999.
[186] P.C. Brooks, R.S. Sharp, A computational procedure based on eigenvalue sensitivity theory applicable to linear system design, Journal of Sound and Vibration 114 (1) (1987) 13–18.
[187] H.A. Sherif, On the design of anti-squeal friction pads for disc brakes, Technical Report 910575, SAE, Warrendale, PA, 1991.
[188] J.-S. Chen, D.B. Bogy, Effects of load parameters on the natural frequencies and stability of a flexible spinning disk with a stationary load system, Transactions of the American Society of Mechanical Engineers Journal of Applied Mechanics 59 (2) (1992) S230–S235.
[189] J.-S. Chen, D.B. Bogy, Mathematical structure of modal interactions in a spinning disk-stationary load system, Transactions of the American Society of Mechanical Engineers Journal of Applied Mechanics 59 (2) (1992) 390–397.
[190] S.N. Chan, J.E. Mottershead, M.P. Cartmell, Instabilities at subcritical speeds in discs with rotating frictional follower loads, Transactions of the American Society of Mechanical Engineers Journal of Vibration and Acoustics 117 (2) (1995) 240–242.

[191] H. Ouyang, J.E. Mottershead, Unstable travelling waves in the friction-induced vibration of discs, Journal of Sound and Vibration 248 (4) (2001) 768–779.
[192] H. Ouyang, J.E. Mottershead, M.P. Cartmell, M.I. Friswell, Friction-induced parametric resonances in discs: effect of a negative friction-velocity relationship, Journal of Sound Vibration 209 (2) (1998) 251–264.
[193] J.E. Mottershead, H. Ouyang, M.P. Cartmell, M.I. Friswell, Parametric resonances in an annular disc, with a rotating system of distributed mass and elasticity; and the effects of friction and damping, Royal Society of London Proceedings Series A Mathematics Physics and Engineering Science 453 (1956) (1997) 1–19.
[194] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, Wiley, New York, 1979.
[195] W.V. Nack, Brake squeal analysis by finite elements, Technical Report 1999-01-1736, SAE, Warrendale, PA, 1999.
[196] W.V. Nack, Brake squeal analysis by finite elements, International Journal of Vehicle Design 23 (3–4) (2000) 263–275.
[197] P. Blaschke, M. Tan, A. Wang, On the analysis of brake squeal propensity using finite elements, Technical Report 2000-01-2765, SAE, Warrendale, PA, 2000.
[198] F. Moirot, A. Nehme, Q.S. Ngyuen, Numerical simulation to detect low-frequency squeal propensity, Technical Report 2000-01-2767, SAE, Warrendale, PA, 2000.
[199] M. Chargin, L. Dunne, D. Herting, Nonlinear dynamics of brake squeal, Finite Elements in Analysis and Design 28 (1) (1997) 69–82.
[200] H. Ouyang, J.E. Mottershead, W,Li, Vibration and dynamic instability of moving load systems, in: P. Sar, D. Moens (Eds.), Proceedings of ISMA25: 2000 International Conference on Noise and Vibration Engineering, Katholleke Universiteit Leuven, Belgium, 2000, pp. 1355–1360.
[201] Y.-K. Hu, S. Mahajan, K. Zhang, Brake squeal DOE using nonlinear transient analysis, Technical Report 1999-01-1737, SAE, Warrendale, PA, 1999.
[202] Y.-K. Hu, L.I. Nagy, Brake squeal analysis using nonlinear transient finite element method, Technical Report 971510, SAE, Warrendale, PA, 1997.
[203] L.I. Nagy, J. Cheng, Y.-K. Hu, A new method development to predict brake squeal occurrence, Technical Report 942258, SAE, Warrendale, PA, 1994.
[204] O.N. Hamzeh, W.W. Tworzydlo, H.J. Chang, S.T. Fryska, Analysis of friction-induced instabilities in a simplified aircraft brake, Technical Report 1999-01-3404, SAE, Warrendale, PA, 1999.