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Letter to the Editor

Exact modelling of kinetic energy in truss-type members

K.M. Ahmida, J.R.F. Arruda*

Departamento de Mecânica Computacional, FEM, UNICAMP, Caixa Postal 6122, 13083-970 Campinas, Sao Paulo, Brazil

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1. Introduction

The main advantage of the spectral formulation given here as the spectral element method (SEM) is that the element dynamic stiffness is computed from the exact analytical solution in the frequency domain. A spectral frame element consists of a combination of a bar element, which is used for axial vibrations, a shaft element, which is used for torsional vibrations, and a beam element, which is used for flexural vibrations. Thus, using the SEM formulations helps to calculate the kinetic energy in frame elements in an exact form. Energy methods are gaining widespread applications related to dynamic analyses of structures, especially at higher frequencies. The statistical energy analysis (SEA) is one of the methods gaining increasing attention for dynamic modelling of structural–acoustic systems in medium and high frequencies. Thus, energy calculations could be used, for example, for the estimation of coupling loss factors (CLF) used in SEA, which are considered the critical factors in SEA modelling [1]. Another application could be the use of these energy formulations in the *hybrid* element method, lately presented by Langley and Bremner [2]. The other significant utilization of the presented formulations lies in its use for the estimation of energy using measured displacements. Normally, energy is calculated using translation-only displacements, due to the well-known difficulties encountered in measuring rotational degrees of freedom (d.o.f.). Through mathematical manipulations of the spectral equation describing wave propagation, rotational d.o.f.s can be found and used in total energy calculation in the waveguide. Using the SEM together with the rotational d.o.f.s, Structural Intensity (SI) estimations can also be conducted exactly. The SI estimations found in the literature generally consider the farfield hypothesis but, with the SEM and the rotational d.o.f.s, the near field can also be taken into account.

*Corresponding author. Tel.: +55-19-3788-3386; fax: +55-19-3289-3722.
E-mail address: arruda@fem.unicamp.br (J.R.F. Arruda).

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2. Truss-type structures

Formulations for the calculation of the kinetic energy in rod and beam elements are given below. The simple Bernoulli–Euler theory and the higher order Timoshenko theory are used. The authors have already used these formulations for the estimation of CLFs for SEA [4], where the energies are calculated as time averaged.

2.1. Kinetic energy in rods

The time-averaged kinetic energy due to the propagation of longitudinal waves in rod elements is calculated as

$$\langle E_R^K \rangle_t = \frac{1}{2}\int_0^L \rho A \langle \dot{u}^2 \rangle_t \mathrm{d}x, \tag{1}$$

where $\langle E_R^K \rangle_t$ is the time-averaged kinetic energy, ρ is the mass density, A is the cross-section area, L is the element length and $\dot{u}$ is the velocity along the rod. It is straightforward to show that the time-averaged velocity can be written as

$$\langle \dot{u}^2 \rangle_t = \tfrac{1}{2}\Re\{\hat{\dot{u}}\hat{\dot{u}}^*\} = \tfrac{1}{2}\Re\{(\mathrm{i}\omega\hat{u})(\mathrm{i}\omega\hat{u})^*\} = \tfrac{1}{2}\omega^2\Re\{\hat{u}(x)\hat{u}^*(x)\}, \tag{2}$$

where the * represents the complex conjugate, $\mathrm{i} = \sqrt{-1}$, ω is the frequency component and $\hat{u}(x)$ is the complex amplitude of the displacement along the rod. The velocity at any arbitrary point x along the rod can be calculated through the shape functions as $\hat{u}(x) = \hat{g}_1(x)\hat{u}_1 + \hat{g}_2(x)\hat{u}_2$, and thus,

$$\langle E_R^K \rangle_t = \tfrac{1}{4}\omega^2\rho A\Re\{(E_R^{11}|\hat{u}_1|^2 + E_R^{22}|\hat{u}_2|^2 + E_R^{12}\hat{u}_1\hat{u}_2^* + E_R^{21}\hat{u}_1^*\hat{u}_2)\}, \tag{3}$$

where $\hat{u}_1$ and $\hat{u}_2$ are the nodal displacements of the rod spectral element, and the terms used are defined as

$$E_R^{11} = \int_0^L \hat{g}_1\hat{g}_1^*\,\mathrm{d}x, \quad E_R^{22} = \int_0^L \hat{g}_2\hat{g}_2^*\,\mathrm{d}x, \quad E_R^{12} = \int_0^L \hat{g}_1\hat{g}_2^*\,\mathrm{d}x, \quad E_R^{21} = \int_0^L \hat{g}_2\hat{g}_1^*\,\mathrm{d}x. \tag{4}$$

Note that $E_R^{12} = E_R^{21}$. For shaft elements, the same formulations can be used by exchanging ρA with ρJ, with J defined as the polar moment of inertia and $\hat{u}(x)$ as the angular displacement along the shaft. The shape functions for the rod spectral element are given by [3]

$$\hat{g}_1(x) = \left(\mathrm{e}^{-\mathrm{i}kx} - \mathrm{e}^{-\mathrm{i}k(2L-x)}\right)/\Delta_1, \quad \hat{g}_2(x) = -\left(\mathrm{e}^{-\mathrm{i}k(L+x)} + \mathrm{e}^{-\mathrm{i}k(L-x)}\right)/\Delta_1, \tag{5}$$

where $\Delta_1 = 1 - \mathrm{e}^{-\mathrm{i}2kL}$.

Solving the above integrals for the case of no hysteretic damping results in the following expressions:

$$E_R^{11} = \frac{(-2kLe^{(i2kL)} - \frac{1}{2}ie^{(i4kL)} + i\frac{1}{2})}{\Delta_2},$$
$$E_R^{12} = E_R^{21} = \frac{(ie^{(i3kL)} - ie^{(ikL)} + kLe^{(i3kL)} + kLe^{(ikL)})}{\Delta_2}, \tag{6}$$
$$E_R^{22} = \frac{(-2kLe^{(i2kL)} - \frac{1}{2}ie^{(i4kL)} + \frac{1}{2}i)}{\Delta_2},$$

where $\Delta_2 = (-2ke^{i2kL} + ke^{i4kL} + k)$.

In the case of hysteretic damping Young's modulus is complex $E \to E(1 + i\eta)$ with η defined as the loss factor and, thus, the wave number k is a complex quantity and this should be taken into account when solving the integral equations. Defining $k = \alpha + i\beta$, E_R^{11} is given below. The other expressions are not demonstrated but they can be calculated in the same manner:

$$E_R^{11} = -\frac{1}{2}\frac{i\left(-i\alpha e^{(i2\alpha L)} - \beta e^{(2L(i2\alpha+\beta))} + e^{(2\beta L)}\beta + i\alpha e^{(2L(2\beta+i\alpha))}\right)}{(\alpha\beta e^{(i2\alpha L)} - \alpha\beta e^{(2L(i2\alpha+\beta))} - \alpha\beta e^{(2\beta L)} + \alpha\beta e^{(2L(2\beta+i\alpha))})}. \tag{7}$$

2.2. *Kinetic energy in throw-off rod elements*

For rod throw-off element (single-noded element), the solution is a single-coefficient as waves propagate in only one direction, that is,

$$\hat{u}(x) = Ae^{-ikx}. \tag{8}$$

Using this solution, the shape function of this single-noded element is given by $\hat{g}_1(x) = e^{-ikx}$. Following the same methodology for the 2-noded rod element, the time-averaged kinetic energy for the throw-off element is then given by

$$\langle E^K_{R_{1\text{-}noded}} \rangle_t = \frac{1}{2}\int_0^\infty \rho A \langle \dot{u}^2 \rangle_t \, dx = \frac{1}{4}\omega^2 \rho A \Re\{(E^{11}_{R_{1\text{-}noded}} |\hat{u}_1|^2\} \tag{9}$$

with the term $E^{11}_{R_{1\text{-}noded}}$ defined as

$$E^{11}_{R_{1\text{-}noded}} = \int_0^\infty \hat{g}_1 \hat{g}_1^* \, dx. \tag{10}$$

Observe that the mathematical evaluation of this expression is possible, nevertheless, it depends on whether the dynamic system is hysteretically damped or not. In a damped system the wave number becomes complex. Defining $k = \alpha + i\beta$, for undamped system $\beta = 0$ and thus $E^{11}_{R_{1\text{-}noded}}$ tends to infinity.

For hysteretically damped systems $\beta \neq 0$ and thus the term $E^{11}_{R_{1\text{-}noded}}$ is calculated as

$$E^{11}_{R_{1\text{-}noded}} = \int_0^\infty e^{2\beta x} \, dx = \frac{-1}{2\beta} + \frac{1}{2}\lim_{x\to\infty}\frac{e^{2\beta x}}{\beta}. \tag{11}$$

Observe that the convergence of this integral depends on the values given to β, nevertheless, β always has values less than α, as the hysteretic damping η is implicit in Young's modulus, i.e.,

$E \rightarrow E(1 + i\eta)$. A material with large damping factor η, say Lexan, has values as large as 0.01. The important characteristic that plays a role in solving this integral is that β has always negative values and thus the integral quantity is finally given by

$$E^{11}_{R_{1\text{-}noded}} = \frac{-1}{2\beta}, \tag{12}$$

which is a positive quantity as β is always a negative quantity, and thus the time-averaged kinetic energy in a throw-off rod element can be calculated as

$$\langle E^K_{R_{1\text{-}noded}} \rangle_t = \tfrac{1}{4}\omega^2 \rho A \Re\left\{\frac{-1}{2\beta}|\hat{u}_1|^2\right\}. \tag{13}$$

2.3. Kinetic energy in Bernoulli–Euler beams

The time-averaged kinetic energy in Bernoulli–Euler beams is only due to the flexural displacements and can be calculated as (having the velocity along the beam $\dot{v}$)

$$\langle E^K_B \rangle_t = \tfrac{1}{2}\rho A \int_0^L \langle \dot{v}^2 \rangle_t \, \mathrm{d}x. \tag{14}$$

The velocity along the beam can be calculated by using shape functions of the form

$$\begin{Bmatrix} \hat{g}_1(x) \\ \hat{g}_2(x) \\ \hat{g}_3(x) \\ \hat{g}_4(x) \end{Bmatrix} = \frac{1}{\Delta_3}\begin{bmatrix} r_1 & r_2 & 0 & 0 \\ 0 & 0 & r_1 & r_2 \\ r_2 & r_1 & 0 & 0 \\ 0 & 0 & -r_2 & -r_1 \end{bmatrix}\begin{Bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \\ h_4(x) \end{Bmatrix} = \frac{1}{\Delta_3}[r]\{h(x)\}, \tag{15}$$

where $\Delta_3 = -r_1^2 + r_2^2$ and the terms of the shape functions are not shown here but can be found in Refs. [3–5]. The time-averaged kinetic energy is then given by

$$\begin{aligned}\langle E^K_B \rangle_t = \tfrac{1}{4}\omega^2\rho A \Re\Big\{ & E^{11}_B|\hat{v}_1|^2 + E^{12}_B\hat{v}_1\hat{\phi}_1^* + E^{13}_B\hat{v}_1\hat{v}_2^* + E^{14}_B\hat{v}_1\hat{\phi}_2^* \\ & + E^{21}_B\hat{\phi}_1\hat{v}_1 + E^{22}_B|\hat{\phi}_1|^2 + E^{23}_B\hat{\phi}_1\hat{v}_2 + E^{24}_B\hat{\phi}_1\hat{\phi}_2 \\ & + E^{31}_B\hat{v}_2\hat{v}_1 + E^{32}_B\hat{v}_2\hat{\phi}_1 + E^{33}_B|\hat{v}_2|^2 + E^{34}_B\hat{v}_2\hat{\phi}_2 \\ & + E^{41}_B\hat{\phi}_2\hat{v}_1 + E^{42}_B\hat{\phi}_2\hat{\phi}_1 + E^{43}_B\hat{\phi}_2\hat{v}_2 + E^{44}_B|\hat{\phi}_2|^2\Big\},\end{aligned} \tag{16}$$

with the terms E^{ij}_B defined by

$$E^{ij}_B = \int_0^L \hat{g}_i(x)\hat{g}_j^*(x)\,\mathrm{d}x = \int_0^L \left(\frac{1}{\Delta_3} r_{ij}h_j\right)\left(\frac{1}{\Delta_3^*} r_{ij}^* h_j^*\right)\mathrm{d}x, \tag{17}$$

or in the compact form

$$\begin{bmatrix} E^{11}_B & E^{12}_B & E^{13}_B & E^{14}_B \\ E^{21}_B & E^{22}_B & E^{23}_B & E^{24}_B \\ E^{31}_B & E^{32}_B & E^{33}_B & E^{34}_B \\ E^{41}_B & E^{42}_B & E^{43}_B & E^{44}_B \end{bmatrix} = \frac{1}{|\Delta_3|^2}[r]\left[\int_0^L \{h\}\{h\}^{\mathrm{H}}\,\mathrm{d}x\right][r]^{\mathrm{H}} = \frac{1}{|\Delta_3|^2}[r][I_1][r]^{\mathrm{H}}, \tag{18}$$

The terms of matrix $[I_1]$ are given in Appendix A, noting that the terms E_B^{ij} obey the following properties: $E_B^{ij} = E_B^{ji}$, $\forall i \neq j$; $E_B^{23} = -E_B^{14}$; $E_B^{33} = E_B^{11}$; $E_B^{34} = -E_B^{12}$; $E_B^{44} = E_B^{22}$.

2.4. Kinetic energy in Timoshenko beams

The time-averaged kinetic energy in Timoshenko beams is due to the translational displacements and rotational inertia, and it is given by

$$\langle E_B^K \rangle_t = \tfrac{1}{2}\rho A \int_0^L \langle \dot{v}^2 \rangle_t \, \mathrm{d}x + \tfrac{1}{2}\rho I \int_0^L \langle \dot{\phi}^2 \rangle_t \, \mathrm{d}x.$$

Different shape functions are used for the different d.o.f.s and can be found in Refs. [3–5]. The time-averaged kinetic energy is given by

$$\begin{aligned}\langle E_B^K \rangle_t = \tfrac{1}{4}\omega^2 \rho A \Re \Big\{ & E_B^{11v}|\hat{v}_1|^2 + E_B^{12v}\hat{v}_1\hat{\phi}_1^* + E_B^{13v}\hat{v}_1\hat{v}_2^* + E_B^{14v}\hat{v}_1\hat{\phi}_2^* \\ & + E_B^{21v}\hat{\phi}_1\hat{v}_1 + E_B^{22v}|\hat{\phi}_1|^2 + E_B^{23v}\hat{\phi}_1\hat{v}_2 + E_B^{24v}\hat{\phi}_1\hat{\phi}_2 \\ & + E_B^{31v}\hat{v}_2\hat{v}_1 + E_B^{32v}\hat{v}_2\hat{\phi}_1 + E_B^{33v}|\hat{v}_2|^2 + E_B^{34v}\hat{v}_2\hat{\phi}_2 \\ & + E_B^{41v}\hat{\phi}_2\hat{v}_1 + E_B^{42v}\hat{\phi}_2\hat{\phi}_1 + E_B^{43v}\hat{\phi}_2\hat{v}_2 + E_B^{44v}|\hat{\phi}_2|^2 \Big\} \\ & + \tfrac{1}{4}\omega^2 \rho I \Re \Big\{ E_B^{11\phi}|\hat{v}_1|^2 + E_B^{12\phi}\hat{v}_1\hat{\phi}_1^* + E_B^{13\phi}\hat{v}_1\hat{v}_2^* + E_B^{14\phi}\hat{v}_1\hat{\phi}_2^* \\ & + E_B^{21\phi}\hat{\phi}_1\hat{v}_1 + E_B^{22\phi}|\hat{\phi}_1|^2 + E_B^{23\phi}\hat{\phi}_1\hat{v}_2 + E_B^{24\phi}\hat{\phi}_1\hat{\phi}_2 \\ & + E_B^{31\phi}\hat{v}_2\hat{v}_1 + E_B^{32\phi}\hat{v}_2\hat{\phi}_1 + E_B^{33\phi}|\hat{v}_2|^2 + E_B^{34\phi}\hat{v}_2\hat{\phi}_2 \\ & + E_B^{41\phi}\hat{\phi}_2\hat{v}_1 + E_B^{42\phi}\hat{\phi}_2\hat{\phi}_1 + E_B^{43\phi}\hat{\phi}_2\hat{v}_2 + E_B^{44\phi}|\hat{\phi}_2|^2 \Big\}, \end{aligned} \tag{19}$$

with the terms E_B^{ij} defined by

$$\begin{bmatrix} E_B^{11v} & E_B^{12v} & E_B^{13v} & E_B^{14v} \\ E_B^{21v} & E_B^{22v} & E_B^{23v} & E_B^{24v} \\ E_B^{31v} & E_B^{32v} & E_B^{33v} & E_B^{34v} \\ E_B^{41v} & E_B^{42v} & E_B^{43v} & E_B^{44v} \end{bmatrix} = [P]^{\mathrm{T}} \left[\int_0^L \{N_v(x)\}\{N_v(x)\}^{\mathrm{H}} \, \mathrm{d}x \right] [P^*] = [P]^{\mathrm{T}}[I_2][P^*], \tag{20}$$

$$\begin{bmatrix} E_B^{11\phi} & E_B^{12\phi} & E_B^{13\phi} & E_B^{14\phi} \\ E_B^{21\phi} & E_B^{22\phi} & E_B^{23\phi} & E_B^{24\phi} \\ E_B^{31\phi} & E_B^{32\phi} & E_B^{33\phi} & E_B^{34\phi} \\ E_B^{41\phi} & E_B^{42\phi} & E_B^{43\phi} & E_B^{44\phi} \end{bmatrix} = [P]^{\mathrm{T}} \left[\int_0^L \{N(x)\}\{N(x)\}^{\mathrm{H}} \, \mathrm{d}x \right] [P^*] = [P]^{\mathrm{T}}[I_3][P^*]. \tag{21}$$

The different terms of the matrices $[N_v(x)]$ and $[N(x)]$ can be found in Refs. [3–5], but the elements of $[I_3]$ and $[I_4]$ are given in Appendix B, noting the following properties: $E_B^{ijv} = E_B^{jiv}$, $\forall i \neq j$; $E_B^{ij\phi} = E_B^{ji\phi}$, $\forall i \neq j$; and $E_B^{23v} = -E_B^{14v}$, $E_B^{33v} = E_B^{11v}$, $E_B^{34v} = -E_B^{12v}$, $E_B^{44v} = E_B^{22v}$, $E_B^{23\phi} = -E_B^{14\phi}$, $E_B^{33\phi} = E_B^{11\phi}$, $E_B^{34\phi} = -E_B^{12\phi}$, $E_B^{44\phi} = E_B^{22\phi}$. All the terms E_B^{ij} are complex. When using a hysteretic damping, the shear and Young's modulus become complex quantities and should be taken into account when solving for the analytical expressions of these functions. The use of mathematical manipulation software as Maple® or Mathematica® is advised to obtain the analytical expressions. Observe that the same formulations given above can be used with FEA shape functions, except that these shape functions are real.

3. Determining rotational d.o.f.s

Dynamists, usually, have no difficulties in measuring flexural d.o.f.s on truss-type members. Longitudinal d.o.f.s are somehow harder to measure but still different techniques could be used, as, for example, taking two measurements with a Laser Doppler Vibrometer (LDV) with two different angles and calculating the longitudinal component using ordinary geometry laws, or by using accelerometers glued on their side on the surface of the waveguide, or simply using tri-axial accelerometers. Nevertheless, on the other hand, measuring rotational d.o.f.s is not an easy task. There exist special rotational accelerometers to accomplish the job but they might have cost restrictions, not forgetting the fact that non-contact measurements (LDV techniques) are normally preferred, especially on lightweight structures, where accelerometer weight and cabling might affect the structure dynamics.

When energy is calculated using flexural-only measured data, energy representation is not exact, as rotational d.o.f.s are not included. These rotational d.o.f.s ($\hat{\phi}(x)$) can still be obtained in an exact form using flexural data, and then used to estimate the total energy. The equations given as follows describe the methodology. Observe that these formulations depend upon the beam waveguide theory adopted, whether it is Bernoulli–Euler where $\hat{\phi}(x) = \mathrm{d}\hat{v}/\mathrm{d}x$, or Timoshenko, where $\hat{v}(x)$ and $\hat{\phi}(x)$ are not related by one equation but instead described by two different equations. For a beam waveguide of length L, and using Bernoulli–Euler theory, the displacement can be described by the following equation, which is an exact solution to the Bernoulli–Euler equation of motion of the waveguide:

$$\hat{v}(x) = \mathbf{A}\mathrm{e}^{-\mathrm{i}kx} + \mathbf{B}\mathrm{e}^{-kx} + \mathbf{C}\mathrm{e}^{\mathrm{i}kx} + \mathbf{D}\mathrm{e}^{kx}. \tag{22}$$

Consider a straight beam with length L with four measurements of flexural displacements taken at four points along the beam $[x_1, x_2, x_3, x_4]$. The frequency-dependent constants **A**, **B**, **C** e **D** are then calculated for each frequency component using the equation

$$\begin{bmatrix} \mathrm{e}^{-\mathrm{i}kx_1} & \mathrm{e}^{-kx_1} & \mathrm{e}^{\mathrm{i}kx_1} & \mathrm{e}^{kx_1} \\ \mathrm{e}^{-\mathrm{i}kx_2} & \mathrm{e}^{-kx_2} & \mathrm{e}^{\mathrm{i}kx_2} & \mathrm{e}^{kx_2} \\ \mathrm{e}^{-\mathrm{i}kx_3} & \mathrm{e}^{-kx_3} & \mathrm{e}^{\mathrm{i}kx_3} & \mathrm{e}^{kx_3} \\ \mathrm{e}^{-\mathrm{i}kx_4} & \mathrm{e}^{-kx_4} & \mathrm{e}^{\mathrm{i}kx_4} & \mathrm{e}^{kx_4} \end{bmatrix} \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{Bmatrix} = \begin{Bmatrix} \hat{v}_1(x) \\ \hat{v}_2(x) \\ \hat{v}_3(x) \\ \hat{v}_4(x) \end{Bmatrix}. \tag{23}$$

These constants are then used for the calculation of the rotational d.o.f.s $\hat{\phi}(x)$, according to the beam theory adopted (in this case, $\hat{\phi}(x) = \mathrm{d}\hat{v}/\mathrm{d}x$). It should be noted that a minimum of four measurements is needed for every span of straight beam with different geometrical or material properties. Normally, more points are measured, which implies that the super-determined system of equations is to be solved by a least-squares technique. It may be important to mention that

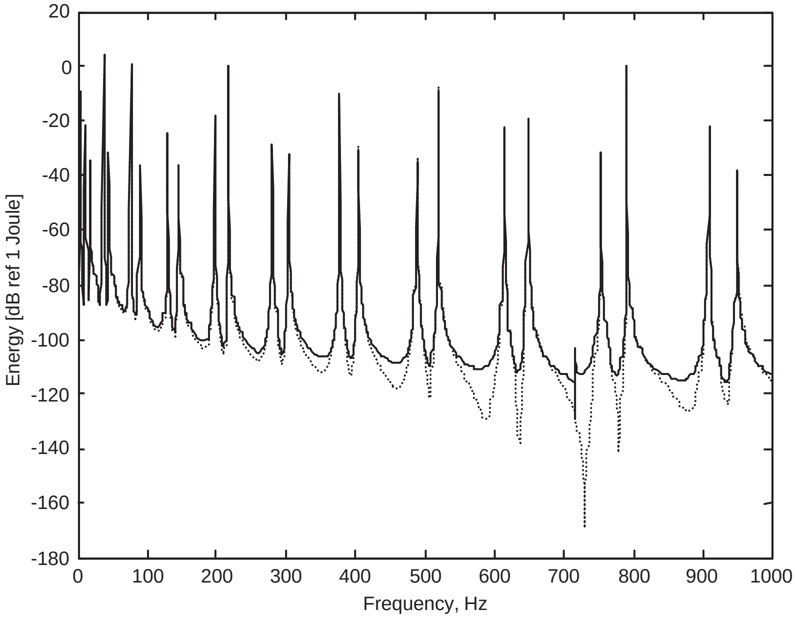

Fig. 1. Total kinetic energy in L-shaped beam: - - - - -, without rotational d.o.f.; ——, with rotational d.o.f.

using more points implies higher numerical cost, as a matrix inversion would be calculated for every frequency component. If the Timoshenko theory is adopted, the rotational d.o.f.s are then calculated using equations different from those given above. To show the importance of rotational d.o.f.s in dynamic analysis, an L-shaped beam having the following characteristics is used: $L_1 = L_2 = 1\,\mathrm{m}$, $E = 2.1 \times 10^{11}\,\mathrm{N/m^2}$, $\rho = 7800\,\mathrm{kg/m^3}$, $A = 5.836 \times 10^{-5}\,\mathrm{m^2}$. The structure is excited transversally at beam 1 and the total kinetic energy is calculated at 21 points along the L-shaped beam for two cases, that is, with and without the $\hat{\phi}(x)$ d.o.f.s. These calculations were conducted using the aforementioned spectral energy formulations. The results are shown in Fig. 1.

4. Conclusions

Formulations have been presented for exactly calculating kinetic energy in truss-type structures. Rod, Bernoulli–Euler beam and Timoshenko beam are covered. A methodology is also presented for obtaining the rotational degrees of freedom using measured transverse degrees of freedom.

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Appendix A. Bernoulli–Euler beam

$$r_1 = \mathrm{i}(k_1 - k_2)(1 - \mathrm{e}^{-\mathrm{i}k_1 L})\mathrm{e}^{-\mathrm{i}k_2 L}, \quad r_2 = \mathrm{i}(k_1 + k_2)\mathrm{e}^{-\mathrm{i}k_1 L} - \mathrm{e}^{-\mathrm{i}k_2 L},$$

$$r = \begin{bmatrix} r_1 & r_2 & 0 & 0 \\ 0 & 0 & r_1 & r_2 \\ r_2 & r_1 & 0 & 0 \\ 0 & 0 & -r_2 & -r_1 \end{bmatrix},$$

$$\begin{aligned} I_1(1,1) = \tfrac{3}{2}(&\alpha \mathrm{e}^{-2(\alpha+\mathrm{i}\beta)L} + \mathrm{i}\beta \mathrm{e}^{-2(\alpha+\mathrm{i}\beta)L} - \alpha \mathrm{e}^{2(-\beta+\mathrm{i}\alpha)L} - \mathrm{i}\beta \mathrm{e}^{-2(-\beta+\mathrm{i}\alpha)L} \\ &+ \mathrm{i}\alpha \mathrm{e}^{-2(-\beta+\mathrm{i}\alpha)L} - \beta \mathrm{e}^{2(-\beta+\mathrm{i}\alpha)L} - i\alpha \mathrm{e}^{-2(\alpha+\mathrm{i}\beta)L} + \beta \mathrm{e}^{-2(\alpha+\mathrm{i}\beta)L} \\ &- \alpha - \mathrm{i}\beta + \alpha \mathrm{e}^{-2(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)L} + \mathrm{i}\alpha \mathrm{e}^{-2(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)L} - \mathrm{i}\alpha \\ &+ \beta + \mathrm{i}\alpha \mathrm{e}^{-2(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)L} - \beta \mathrm{e}^{-2(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)L}), \end{aligned}$$

$$\begin{aligned} I_1(1,2) = {}& L\beta^2 \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} - L\alpha^2 \mathrm{e}^{-(\alpha+\mathrm{i}\beta)L} - L\alpha^2 \mathrm{e}^{-(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)L} \\ &- 2\mathrm{i}L\beta\alpha \mathrm{e}^{-(\alpha+\mathrm{i}\beta)L} + 2\mathrm{i}L\beta\alpha \mathrm{e}^{-(-\beta+\mathrm{i}\alpha)L} - 2\mathrm{i}L\beta\alpha \mathrm{e}^{-(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)L} \\ &+ 2\mathrm{i}L\beta\alpha \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} - L\beta^2 \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} + L\beta^2 \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} \\ &+ L\alpha^2 \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} + L\alpha^2 \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + L\beta^2 \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} - 3\beta \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} \\ &+ 3\alpha \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + 3\beta \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} - 3\alpha \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} + 3\mathrm{i}\alpha \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} \\ &- 3\mathrm{i}\beta \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} + 3\mathrm{i}\beta \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} - 3\mathrm{i}\alpha \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)}, \end{aligned}$$

$$I_1(1,3) = 4\mathrm{e}^{-L(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)} - \mathrm{e}^{-2L(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)} - \mathrm{e}^{-2L(-\beta+\mathrm{i}\alpha)} - \mathrm{e}^{-2L(\alpha+\mathrm{i}\beta)} - 1,$$

$$\begin{aligned} I_1(1,4) = {}& \mathrm{i}L\alpha \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} + \mathrm{i}L\alpha \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} - L\beta \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} \\ &+ \mathrm{i}L\beta \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + L\alpha \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} - \mathrm{i}L\beta \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} \\ &- L\alpha \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} + L\beta \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} \\ &+ \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} + \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} - \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} \\ &- \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\beta+\mathrm{i}\alpha)} - \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} - \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} \\ &+ \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)}, \end{aligned}$$

$$\begin{aligned} &I_1(2,1) = I_1(1,2), \quad I_1(2,2) = I_1(1,1), \quad I_1(2,3) = I_1(1,4), \\ &I_1(2,4) = I_1(1,3), \quad I_1(3,1) = I_1(1,3), \quad I_1(3,2) = I_1(2,3), \end{aligned}$$

$$\begin{aligned} I_1(3,3) = \frac{1}{\alpha+\mathrm{i}\beta}\big(&4L\alpha \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)} + 4\mathrm{i}L\beta \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)} \\ &+ \tfrac{1}{2}\mathrm{e}^{-2L(\alpha+\mathrm{i}\beta)} - \tfrac{1}{2}\mathrm{e}^{-2L(-\beta+\mathrm{i}\alpha)} - \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-2L(-\beta+\mathrm{i}\alpha)} + \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-2L(\alpha+\mathrm{i}\beta)} \\ &- \tfrac{1}{2} + \tfrac{1}{2}\mathrm{e}^{-2L(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)} + \tfrac{1}{2}\mathrm{i} - \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-2L(\mathrm{i}\beta+\alpha+\mathrm{i}\alpha-\beta)}\big), \end{aligned}$$

$$
\begin{aligned}
I_1(3,4) = \frac{1}{\alpha + \mathrm{i}\beta}(& L\alpha \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + \mathrm{i}L\beta \mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + L\alpha \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} \\
& + \mathrm{i}L\beta \mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} + L\alpha \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} + \mathrm{i}L\beta \mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} \\
& + L\alpha \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} + \mathrm{i}L\beta \mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} \\
& - \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} + \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} + \tfrac{1}{2}\mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)} \\
& - \tfrac{1}{2}\mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\alpha+\mathrm{i}\beta)} - \tfrac{1}{2}\mathrm{i}\mathrm{e}^{-L(\mathrm{i}\beta+\alpha+2\mathrm{i}\alpha-2\beta)} \\
& - \tfrac{1}{2}\mathrm{e}^{-L(-\beta+\mathrm{i}\alpha)} + \tfrac{1}{2}\mathrm{e}^{-L(2\mathrm{i}\beta+2\alpha+\mathrm{i}\alpha-\beta)}),
\end{aligned}
$$

$$
\begin{aligned}
& I_1(4,1) = I_1(1,4), \quad I_1(4,2) = I_1(2,4), \\
& I_1(4,3) = I_1(3,4), \quad I_1(4,4) = I_1(3,3).
\end{aligned}
$$

Appendix B. Timoshenko beam

$$
\begin{aligned}
& \xi_i = k_i L,\ i = 1,2; \quad P_i = \frac{\mathrm{i}L}{GA\kappa\xi_i}\left(\frac{EI\xi_i^2}{L^2} + GA\kappa - \rho I\omega^2\right), \\
& i = 1,2;\ e_i = \mathrm{e}^{-\mathrm{i}k_\mathrm{i}L},\ i = 1,2;
\end{aligned}
$$

$$
\begin{aligned}
\Delta_4 = & -(P_1^2 - 2P_2P_1 + 8P_2e_2P_1e_1 - 2P_2e_2^2P_1 - e_2^2P_1^2 \\
& + P_2^2 - P_2^2e_2^2 - 2P_1e_1^2P_2 - P_1^2e_1^2 + P_1^2e_1^2e_2^2 \\
& - 2P_1e_1^2P_2e_2^2 - e_1^2P_2^2 + P_2^2e_2^2e_1^2),
\end{aligned}
$$

$$
\begin{aligned}
& P(1,1) = (-P_1 + P_2 - 2P_2e_2e_1 + P_2e_2^2 + e_2^2P_1)/\Delta_4, \\
& P(1,2) = (P_1 - P_2 + P_1e_1^2 - 2P_1e_1e_2 + e_1^2P_2)/\Delta_4, \\
& P(1,3) = (2P_2e_2 - P_1e_1 + P_1e_1e_2^2 - e_1P_2 - P_2e_2^2e_1)/\Delta_4, \\
& P(1,4) = (-P_2e_2 - e_2P_1 + 2P_1e_1 - P_1e_1^2e_2 + P_2e_2e_1^2)/\Delta_4,
\end{aligned}
$$

$$
\begin{aligned}
& P(2,1) = (-P_2(-P_1 + P_2 + 2P_1e_1e_2 - P_2e_2^2 - e_2^2P_1))/\Delta_4, \\
& P(2,2) = (P_1(-P_1 + P_2 + P_1e_1^2 - 2P_2e_2e_1 + e_1^2P_2))/\Delta_4, \\
& P(2,3) = (P_2(e_1P_2 - 2e_2P_1 + P_1e_1 + P_1e_1e_2^2 - P_2e_2^2e_1))/\Delta_4, \\
& P(2,4) = (-P_1(-P_2e_2 - e_2P_1 + 2e_1P_2 + P_1e_1^2e_2 - P_2e_2e_1^2))/\Delta_4,
\end{aligned}
$$

$$
\begin{aligned}
& P(3,1) = (-2P_2e_2 + P_1e_1 - P_1e_1e_2^2 + e_1P_2 + P_2e_2^2e_1)/\Delta_4, \\
& P(3,2) = (P_2e_2 + e_2P_1 - 2P_1e_1 + P_1e_1^2e_2 - P_2e_2e_1^2)/\Delta_4, \\
& P(3,3) = (P_1 - P_2 + 2P_2e_2e_1 - P_2e_2^2 - e_2^2P_1)/\Delta_4, \\
& P(3,4) = (-P_1 + P_2 - P_1e_1^2 + 2P_1e_1e_2 - e_1^2P_2)/\Delta_4,
\end{aligned}
$$

$$P(4,1) = (P_2(e_1P_2 - 2e_2P_1 + P_1e_1 + P_1e_1e_2^2 - P_2e_2^2e_1))/\Delta_4,$$
$$P(4,2) = (-P_1(-P_2e_2 - e_2P_1 + 2e_1P_2 + P_1e_1^2e_2 - P_2e_2e_1^2))/\Delta_4,$$
$$P(4,3) = (-P_2(-P_1 + P_2 + 2P_1e_1e_2 - P_2e_2^2 - e_2^2P_1))/\Delta_4,$$
$$P(4,4) = (P_1(-P_1 + P_2 + P_1e_2^2 - 2P_2e_2e_1 + e_1^2P_2))/\Delta_4;$$

$$\chi_i = \Re(P_i), \quad \gamma_i = \Im(P_i), \quad i = 1,2;$$

$$I_3(1,1) = \frac{1}{2}\frac{(\mathrm{i}\chi_1^2 - 2\chi_1\gamma_1 - \mathrm{i}\gamma_1^2)}{(\alpha_1 + \mathrm{i}\beta_1)}(1 + \mathrm{e}^{2L(\mathrm{i}\alpha-\beta)}),$$
$$I_3(1,2) = \frac{(\chi_1\chi_2 + \mathrm{i}\chi_1\gamma_2 + \mathrm{i}\gamma_1\chi_2 - \gamma_1\gamma_2)}{(-\mathrm{i}\alpha_1 + \beta - \mathrm{i}\alpha_2 + \beta_2)}(1 + \mathrm{e}^{L(-\mathrm{i}\alpha_1+\beta_1-\mathrm{i}\alpha_2+\beta_2)}),$$
$$I_3(1,3) = -L(\chi_1^2 + 2\mathrm{i}\chi_1\gamma_1 - \gamma_1^2)\mathrm{e}^{-L(\mathrm{i}\alpha_1-\beta_1)},$$
$$I_3(1,4) = \frac{(-\chi_1\chi_2 - \mathrm{i}\gamma_1\chi_1 - \mathrm{i}\gamma_1\chi_1 + \gamma_1\gamma_2)}{(-\mathrm{i}\alpha_2 + \beta_2 + \mathrm{i}\alpha_1 - \beta_1)}(\mathrm{e}^{-L(\mathrm{i}\alpha_2-\beta_2)} + \mathrm{e}^{-L(\mathrm{i}\alpha_1-\beta_1)}),$$

$$I_3(2,1) = I_3(1,2),$$
$$I_3(2,2) = \frac{1}{2}\frac{(\mathrm{i}\chi_2^2 - 2\chi_2\gamma_2 - \mathrm{i}\gamma_2^2)}{(\alpha_2 + \mathrm{i}\beta_2)}(\mathrm{e}^{2L(-\mathrm{i}\alpha_2+\beta_2)} - 1),$$
$$I_3(2,3) = \frac{(-\chi_1\chi_2 - \mathrm{i}\gamma_1\chi_2 - \mathrm{i}\chi_1\gamma_2 + \gamma_1\gamma_2)}{(-\mathrm{i}\alpha_2 + \beta_2 + \mathrm{i}\alpha_1 - \beta_1)}(\mathrm{e}^{L(-\mathrm{i}\alpha_2+\beta_2)} + \mathrm{e}^{-L(\mathrm{i}\alpha_1-\beta_1)}),$$
$$I_3(2,4) = -L(\chi_2^2 + 2\mathrm{i}\chi_2\gamma_2 - \gamma_2^2)\,\mathrm{e}^{L(-\mathrm{i}\alpha_2+\beta_2)},$$

$$I_3(3,1) = I_3(1,3),$$
$$I_3(3,2) = I_3(2,3),$$
$$I_3(3,3) = \frac{1}{2}\frac{(\mathrm{i}\chi_1^2 - 2\chi_1\gamma_1 - \mathrm{i}\gamma_1^2)}{(\alpha_1 + \mathrm{i}\beta_1)}(\mathrm{e}^{-2L(\mathrm{i}\alpha_1-\beta_1)} - 1),$$
$$I_3(3,4) = \frac{(\chi_1\chi_2 + \mathrm{i}\chi_1\gamma_2 - \mathrm{i}\gamma_1\chi_2)}{(-\mathrm{i}\alpha_1 + \beta_1 - \mathrm{i}\alpha_2 + \beta_2)}(\mathrm{e}^{L(-\mathrm{i}\alpha_1+\beta_1+\mathrm{i}\alpha_2+\beta_2)} - 1),$$

$$I_3(4,1) = I_3(1,4),$$
$$I_3(4,2) = I_3(2,4),$$
$$I_3(4,3) = I_3(3,4),$$
$$I_3(4,4) = \frac{1}{2}\frac{(\mathrm{i}\chi_2^2 - 2\chi_2\gamma_2 - \mathrm{i}\gamma_2^2)}{(\alpha_2 + \mathrm{i}\beta_2)}(\mathrm{e}^{2L(-\mathrm{i}\alpha_2+\beta_2)} - 1),$$

$$I_4(1,1) = \frac{1}{2}\frac{\mathrm{i}}{(\alpha_1 + \mathrm{i}\beta_1)}(\mathrm{e}^{-2L(\mathrm{i}\alpha_1-\beta_1)} + 1),$$
$$I_4(1,2) = \frac{1}{(-\mathrm{i}\alpha_1 + \beta_1 - \mathrm{i}\alpha_2 + \beta_2)}(\mathrm{e}^{L(-\mathrm{i}\alpha_1+\beta_1-\mathrm{i}\alpha_2+\beta_2)} + 1),$$
$$I_4(1,3) = L\mathrm{e}^{-L(\mathrm{i}\alpha_1-\beta_1)},$$
$$I_4(1,4) = \frac{1}{(-\mathrm{i}\alpha_2 + \beta_2 + \mathrm{i}\alpha_1 - \beta_1)}(\mathrm{e}^{L(-\mathrm{i}\alpha_2+\beta_2)} - \mathrm{e}^{-L(\mathrm{i}\alpha_1-\beta_1)}),$$

$$I_4(2,1) = I_4(1,2),$$
$$I_4(2,2) = \frac{1}{2}\frac{\mathrm{i}}{(\alpha_2 + \mathrm{i}\beta_2)}(\mathrm{e}^{2L(-\mathrm{i}\alpha_2+\beta_2)} - 1),$$
$$I_4(2,3) = \frac{1}{(-\mathrm{i}\alpha_2 + \beta_2 + \mathrm{i}\alpha_1 - \beta_1)}(\mathrm{e}^{L(-\mathrm{i}\alpha_2+\beta_2)} - \mathrm{e}^{-L(\mathrm{i}\alpha_1-\beta_1)}),$$
$$I_4(2,4) = L\mathrm{e}^{L(-\mathrm{i}\alpha_2+\beta_2)},$$

$$I_4(3,1) = I_4(1,3),$$
$$I_4(3,2) = I_4(2,3),$$
$$I_4(3,3) = \frac{1}{2}\frac{\mathrm{i}}{(\alpha_1 + \mathrm{i}\beta_1)}(\mathrm{e}^{-2L(\mathrm{i}\alpha_1-\beta_1)} - 1),$$
$$I_4(3,4) = \frac{1}{(-\mathrm{i}\alpha_1 + \beta_1 - \mathrm{i}\alpha_2 + \beta_2)}(\mathrm{e}^{L(-\mathrm{i}\alpha_1+\beta_1-\mathrm{i}\alpha_2+\beta_2)} - 1),$$

$$I_4(4,1) = I_4(1,4),$$
$$I_4(4,2) = I_4(2,4),$$
$$I_4(4,3) = I_4(3,4),$$
$$I_4(4,4) = \frac{1}{2}\frac{\mathrm{i}}{(\alpha_2 + \mathrm{i}\beta_2)}(\mathrm{e}^{2L(-\mathrm{i}\alpha_2+\beta_2)} - 1),$$

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