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Detecting cracks in a longitudinally vibrating beam with dissipative boundary conditions

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Abstract

This paper focuses on detecting a small open crack in an axially vibrating beam with viscous boundary conditions by using non-destructive dynamical measurements. The damage is simulated by an equivalent linear elastic spring. It is shown that the measurement of the changes in a suitable pair of eigenvalues leads to the solution of the diagnostic problem, namely identification of crack location and severity. Results apply to uniform beams under various sets of boundary conditions.

1. Introduction

Dampers are frequently used in designing structures under dynamic excitation. These devices are employed not only in the field of vibration isolation, but also in the one of passive and active control of the response of a vibrating system. Their aim is to dissipate a part of the vibrating energy, and by way of this they lead to a reduction in response amplitude and in force transmissibility, allowing for simpler and lighter structural designs.

Within the class of viscously damped structures, the system consisting of a bar with viscous end dampers plays an important role, since it represents the critical component of many mechanical systems, such as machine tools, car wheel suspensions, crank shafts of internal combustion engines, etc. [1,2]. The integrity of this kind of mechanical component is often crucial to guarantee the good performance of the whole system. As a consequence, it becomes important to develop non-destructive techniques for such a class of structures, so that possible damages—like small cracks—may be identified as soon as they arise.

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Despite the very extensive literature on damage identification based on dynamic measurements (see [3,4] for recent and complete state of the art), most of earlier works dealing with cracked beams consider undamped systems only. Moreover, even if studies on the dynamic behaviour of (undamaged) beams with dissipative boundary conditions of a viscous type were produced long ago [5–10], research works on the effect of cracks on the dynamic behaviour of such dissipative systems are very rare and, to the best of the authors' knowledge, have not been published yet.

This paper focuses on detecting a small open crack in an axially vibrating beam with viscous boundary conditions from the knowledge of damage-induced shifts in a pair of eigenvalues. As in Freund and Herrmann [11] and in Cabib et al. [12], the crack is simulated by an equivalent massless linear elastic spring, of constant stiffness K, connecting the two segments of the beam adjacent to the damaged cross-section. Assuming that the undamaged system is completely defined, only two parameters need to be determined, namely the stiffness K of the spring and abscissa s of the cracked cross-section. Therefore, it is reasonable to investigate the extent to which the measurement of the crack-induced changes in a pair of eigenvalues can be useful to identify damage. It is found that for uniform free-damped beams under axial vibration, roughly speaking, knowledge of the ratio between the imaginary part of the variations of the $2m$th and mth eigenvalues uniquely determines the position variable $S = \cos 2m\pi s/L$, where L is the length of the rod. Furthermore, the variation of the same pair of eigenvalues enables one to estimate the stiffness K of the damage-simulating elastic spring. In both cases, simple closed-form expressions are deduced for S and for K. The results above also apply to uniform axially vibrating rods with equal viscous dampers at both ends. As for uniform fixed-damped rods, by simultaneously using axial frequencies related to different boundary conditions, it is still possible to determine uniquely the damage parameters S and K.

The present study is along the line of the research developed in Refs. [13,14] to identify cracks in undamped beams from frequency measurements. In particular, the explicit expression for damage sensitivity of eigenvalues, which was obtained following the perturbation method presented in Ref. [15], plays a crucial role in this analysis. As a consequence, the procedure applied here does not call for an explicit solution of the eigenvalue problem for the damaged system, as it uses only knowledge of the eigensolutions corresponding to the undamaged configuration.

Dynamic tests performed on simulated cracked beams supported the proposed method for the solution of the diagnostic problem in practical situations. Numerical results show that if the eigenvalues used as data in identification are affected by errors that are relatively small with respect to damage-induced changes in the eigenvalues, damage identification leads to satisfactory results.

2. Eigenvalue sensitivity to damage

Consider a thin straight elastic rod of length L with constant cross-section of area A, Young's modulus E and uniform linear mass density ρ. Assume, for definiteness, that the beam's left end is free and its right end is constrained by a lumped linear damper of viscous damping coefficient $c > 0$.

The infinitesimal, free longitudinal vibrations of the rod are governed by the following equation:

$$EA\frac{\partial^2 w}{\partial z^2} - \rho\frac{\partial^2 w}{\partial t^2} = 0, \quad z \in (0, L), \quad t > 0, \tag{1}$$

with boundary conditions
Free (F):

$$\frac{\partial w}{\partial z}(0, t) = 0, \quad t > 0, \tag{2}$$

Damped (D):

$$EA\frac{\partial w}{\partial z}(L, t) + c\frac{\partial w}{\partial t}(L, t) = 0, \quad t > 0, \tag{3}$$

where $w = w(z,t)$ is the longitudinal displacement of the bar at the cross-section of abscissa z and at a moment of time t. The dynamic problem is completed by assigning some initial conditions, at the moment of time $t = 0$, on w and $\partial w/\partial t$ in the whole interval $[0,L]$.

The dimensionless eigenvalue problem related to problems (1)–(3) is given by

$$u''(x) - \mu^2 u(x) = 0, \quad x \in (0, 1), \tag{4}$$

$$u'(0) = 0, \tag{5}$$

$$u'(1) + \gamma\, \mu u(1) = 0\ , \tag{6}$$

where $u = u(x)$ is the spatial variation of the free longitudinal vibration of the undamaged rod (see [7,8]) and where $u' \equiv du/dx$.

The dimensionless quantities x, μ^2 and γ are defined as follows:

$$x = \frac{z}{L} \in (0, 1), \qquad \mu^2 = \frac{\rho L^2}{EA}\lambda^2, \qquad \gamma = c(EA\rho)^{-1/2}. \tag{7}$$

Under the assumption that $\gamma < 1$, the mth eigenpair $(u_m(x), \mu_m^2)$ of the undamaged rod is given by

$$\mu_m = \frac{1}{2}\ln\frac{(1-\gamma)}{(1+\gamma)} + \mathrm{i}m\pi \equiv \xi + \mathrm{i}\omega_m, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{8}$$

$$u_m(x) = A_m \cosh \mu_m x, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{9}$$

where $\mathrm{i} \equiv \sqrt{-1}$ is the imaginary unit. Hereafter A_m denotes an arbitrary complex number.

For $\gamma > 1$ the mth eigenpair is given by

$$\mu_m = \frac{1}{2}\ln\frac{(\gamma-1)}{(\gamma+1)} + \mathrm{i}\frac{\pi}{2}(1+2m), \quad u_m(x) = A_m \cosh \mu_m x, \quad m = 0, \pm 1, \pm 2, \ldots .$$

Note that if $\gamma = 1$ there is no spectrum at all, see [6, p. 12]; [7, p. 366] for a physical interpretation of this behaviour. To fix ideas, and considering that this is a common situation in practice, the condition $\gamma < 1$ will be assumed hereafter.

Suppose now that a crack appears at the cross-section of abscissa $s \in (0,L)$. Assuming that the crack remains always open during the longitudinal vibration, by modelling it as a massless translational spring, at $z=s$, as suggested in Refs. [11,12], the dimensionless eigenvalue problem for the damaged rod is as follows:

$$u_d''(x) - \mu_d^2 u_d(x) = 0, \quad x \in (0, \sigma) \cup (\sigma, 1), \tag{10}$$

$$u_d'(0) = 0\ , \tag{11}$$

$$[u_d(\sigma)] = \varepsilon u_d'(\sigma), \tag{12}$$

$$[u_d'(\sigma)] = 0, \tag{13}$$

$$u_d'(1) + \gamma\ \mu_d\ u_d(1) = 0\ , \tag{14}$$

where $u_d = u_d(x)$ is the spatial variation of the longitudinal displacement in the damaged beam and $\sigma = s/L$ is the normalized location of the crack. In Eqs. (12) and (13), $[\phi(\sigma)] \equiv (\phi(\sigma^+) - \phi(\sigma^-))$ denotes the jump of the function $\phi = \phi(x)$ at $x = \sigma$. The expression ε is given by

$$\varepsilon = \frac{(EA/L)}{K}, \tag{15}$$

where the spring stiffness K may be related to crack geometry as suggested, for example, in Refs. [11] or [16]. The undamaged system corresponds to $K \to \infty$ or $\varepsilon \to 0$. It can be shown that also for the eigenvalue problem (10)–(14) there is an infinite sequence of simple eigenvalues corresponding to the roots of the characteristic equation

$$0 = (\sinh \mu + \gamma \cosh \mu) + \varepsilon\mu \sinh \mu\sigma(\sinh \mu(1-\sigma) + \gamma \cosh \mu(1-\sigma)). \tag{16}$$

If the crack is small, namely ε is small enough, then the first variation of the eigenvalue μ_m^2 with respect to ε may be found as shown in Ref. [15]. Assuming

$$\mu_{dm}^2 = \mu_m^2 + \varepsilon(\Delta\mu_m^2), \tag{17}$$

the first order variation of the mth eigenvalue with respect to ε is given by

$$\delta(\mu_m^2) = \varepsilon(\Delta\mu_m^2) = \varepsilon(u_m'(\sigma))^2, \tag{18}$$

$m = 0, \pm 1, \pm 2, \ldots$, where the normalizing condition

$$\int_0^1 u_m^2(x)\,\mathrm{d}x + \frac{\gamma}{2}\frac{(u_m(1))^2}{\mu_m} = 1, \tag{19}$$

is taken into account. Similar to the ideal undamped case, the first order change in an eigenvalue produced by a single small crack may be expressed as the product of two terms, the first of which is proportional to the severity of damage and the second only depends on the location of damage. This second term is the square of the first derivative of the corresponding mode shape of the undamaged rod evaluated at the cracked cross-section. However, concerning the problem of damage identification, there is a substantial difference between the undamped case and the dissipative case considered here. In fact, the mode shape of the undamaged rod is now a complex value function and this fact has a strong influence on the damage detection procedure, as it will be shown in the next section.

3. Crack detection results

The problem of identifying a single small crack from the knowledge of damage-induced changes in the eigenvalues of an initially uniform vibrating rod with dissipative boundary conditions will be considered here. In particular, under the assumption that the undamaged system is completely defined and with damage simulated as in Section 2, Eqs. (12) and (13), only the stiffness K and abscissa s of the cracked cross-section need to be determined. Therefore, changes in a (suitable) pair of eigenvalues will be considered in identifying damage. The key point of the identification technique is the explicit expression (18) for damage sensitivity of eigenvalues derived in the previous section. Eq. (18) has in fact an important consequence: the ratios of the change in two different eigenvalues depend only on damage location, not on its severity.

The case of a free–damped (F–D) rod is considered first. With reference to the dimensionless eigenvalue problems (4)–(6) and (10)–(14) corresponding to the undamaged and damaged rod, respectively, the quantity C_m^{F-D} is defined as follows:

$$C_m^{F-D} = \frac{\delta(\mu_m^{F-D})^2}{(\mu_m^{F-D})^2}, \tag{20}$$

$m = 0, \pm 1, \pm 2, \ldots$, where $\delta(\mu_m^{F-D})^2$ has expression (18). Putting expression (8), (9) of the mth eigenpair of the (F–D) rod into Eq. (20) gives

$$C_m^{F-D} = \varepsilon(\sinh^2 \xi\sigma \cos^2 m\pi\sigma - \cosh^2 \xi\sigma \sin^2 m\pi\sigma + \mathrm{i} \sinh \xi\sigma \cosh \xi\sigma \sin 2m\pi\sigma), \tag{21}$$

where $\sigma = s/L$ and $\varepsilon = (EA/L)/K$ are the dimensionless location and severity of damage, and $\xi \equiv \mathrm{Re}(\mu_m^{F-D})$ is defined as in Eq. (8).

In order to identify damage, it is convenient to deal with the imaginary part of C_m^{F-D}'s, $m \neq 0$. Let it be assumed that $\mathrm{Im}(C_m^{F-D}) \neq 0$ for a certain $m \neq 0$. Then, by using standard trigonometric identities, from expression (21) it follows that

$$S \equiv \cos 2m\pi\sigma = \frac{1}{2}\frac{\mathrm{Im}(C_{2m}^{F-D})}{\mathrm{Im}(C_m^{F-D})}, \tag{22}$$

namely, if $\mathrm{Im}(C_m^{F-D}) \neq 0$, the measurement of the pair $\{\mathrm{Im}(C_m^{F-D}), \mathrm{Im}(C_{2m}^{F-D})\}$, $m \neq 0$, uniquely determines the variable $S \equiv \cos 2m\pi\sigma$ of the normalized damage location σ. Note that the sign of m does not affect the localization result. Moreover, for those values of σ such that $|S| = 1$, it turns out that $\mathrm{Im}(C_m^{F-D}) = 0$, and then $S \in (-1,1)$. Therefore there are exactly $2|m|$ possible crack locations σ_k, $k = 1, \ldots, 2|m|$, corresponding to the same ratio $\mathrm{Im}(C_{2m}^{F-D})/\mathrm{Im}(C_m^{F-D})$. These damage locations are symmetrically placed with respect to the mid-point of the rod, namely to every generic crack location σ_k corresponds its symmetric $\sigma_k^{SYM} = \sigma_{2|m|-k+1}$, $k = 1, \ldots, 2|m|$. As for the "ideal" undamped case discussed in Ref. [13], the number of possible crack locations for a given measurement pair increases as the order of the modes assessed increases. This fact accounts for the recourse to "low" frequencies as optimal setting for the problem of damage localization. In fact, for $|m| = 1$, crack location can be uniquely determined from Eq. (22), except for a symmetrical position.

By inserting the expression of a possible damage location, say σ_k, $k=1,\ldots,2|m|$, into the expression of $\mathrm{Im}(C_m^{F-D})$, the corresponding damage severity ε_k can be determined

$$\mathrm{Im}(C_m^{F-D}) = \varepsilon_k \sinh \xi\sigma_k \cosh \xi\sigma_k \sin 2m\pi\sigma_k. \tag{23}$$

Since $\sin(2m\pi\sigma_{2|m|-k+1}) = -\sin(2m\pi\sigma_k)$, $k=1,\ldots,2|m|-1$, and $\xi<0$, Eq. (23) gives $2|m|$ values of ε_k, half of them are negative, the remaining half positive. More precisely, $\varepsilon_k\varepsilon_{k+1}<0$, $k=1,\ldots,2|m|-1$. Then, since the only physically plausible values of ε_k are the positive ones, half of the possible $2|m|$ damage locations identified from Eq. (22) can be discarded. For example, if $\mathrm{Im}(C_m^{F-D}) > 0$ and $\sin(2m\pi\sigma_1) > 0$, the only possible damage locations are $\sigma_2, \sigma_4, \ldots, \sigma_{2|m|}$. This fact has an important consequence in the simplest situation where $|m|=1$, because it implies the unique determination of crack location from the knowledge of ratio $\mathrm{Im}(C_{2m}^{F-D})/\mathrm{Im}(C_m^{F-D})$. It is worth noticing that there is not a direct analogue of this property in the undamped case.

Suppose now that $\mathrm{Im}(C_m^{F-D}) = 0$ for a certain $m\neq 0$. From expression (21) it easily follows that $S=1$, i.e., the possible damage locations are $\sigma_k = k/2|m|$, $k=1,\ldots,2|m|-1$. Damage severity can be identified by substituting $\sigma=\sigma_k$, $k=1,\ldots,2|m|-1$, into the expression of the real part of C_m^{F-D}. A direct calculation shows that

$$\mathrm{Re}(C_m^{F-D}) = \frac{\varepsilon_k}{2}((-1)^k \cosh 2\xi\sigma_k - 1). \tag{24}$$

The expression within brackets is positive for even k and negative for odd k. Then, $\varepsilon_k\varepsilon_{k+1}<0$, $k=1,\ldots,2|m|-2$, and reasoning as before $(|m|-1)$ or $|m|$, depending on the case, of the $(2|m|-1)$ possible damage locations $\sigma_k = k/2|m|$ can be discarded.

The case when both ends of a rod are constrained by a lumped linear damper of equal viscous coefficient c can be included in this discussion. Within the usual notation, the eigenpairs of the axial vibrations of a damped–damped uniform rod (indicated with D–D hereafter) are given by

$$\mu_m^{D-D} = \ln\frac{(1-\gamma)}{(1+\gamma)} + \mathrm{i}m\pi \equiv \xi^{D-D} + \mathrm{i}\omega_m^{D-D}, \tag{25}$$

$$u_m^{D-D}(x) = A_m(\gamma \sinh \mu_m^{D-D}x + \cosh \mu_m^{D-D}x), \quad m = 0, \pm 1, \pm 2, \ldots, \tag{26}$$

where γ is as in Eq. $(7)_3$, $\gamma<1$. By inserting expressions (25), (26) into Eq. (20) and after some easy calculations, the quantity C_m^{D-D} can be written as

$$\begin{aligned} C_m^{D-D} = \varepsilon[&(\gamma \cosh \xi\sigma \cos m\pi\sigma + \sinh \xi\sigma \cos m\pi\sigma)^2 \\ &- (\gamma \sinh \xi\sigma \sin m\pi\sigma + \cosh \xi\sigma \sin m\pi\sigma)^2 \\ &+ \mathrm{i}(\gamma \cosh \xi\sigma + \sinh \xi\sigma)(\gamma \sinh \xi\sigma + \cosh \xi\sigma)\sin 2m\pi\sigma], \end{aligned} \tag{27}$$

$m=0,\pm 1,\pm 2,\ldots$, where it is set $\xi \equiv \xi^{D-D}$ to simplify notation. Taking into consideration the expression of the imaginary part of C_m^{D-D}'s, we formally have the same situation as the corresponding F–D case, see Eq. (22), so that all the remarks made in that case can be extended to this one. For example, if $\mathrm{Im}(C_m^{D-D})\neq 0$, then the measurement of the pair $\{\mathrm{Im}(C_m^{D-D}), \mathrm{Im}(C_{2m}^{D-D})\}$, $m\neq 0$, uniquely determines the variable $S = \cos 2m\pi\sigma$ of damage location:

$$S = \frac{1}{2}\frac{\mathrm{Im}(C_{2m}^{D-D})}{\mathrm{Im}(C_m^{D-D})}. \tag{28}$$

The estimate of damage severity follows as before from the imaginary part of C_m^{D-D}. Since the system is symmetric in the undamaged state, a crack located at any one of a set of symmetrically placed points of a D–D rod will produce identical changes in eigenvalues. Nevertheless, a careful study of the $\mathrm{Im}(C_m^{D-D})$ expression appearing in Eq. (27), the details of which are omitted for simplicity, enables to show that the factor term $(\gamma \cosh \xi\sigma + \sinh \xi\sigma)(\gamma \sinh \xi\sigma + \cosh \xi\sigma)$ is an odd function of σ with respect to $\sigma = \frac{1}{2}$. Therefore, half of the possible damage locations, for $|m| \geqslant 2$, can be discarded in this case too, because they correspond to negative damage severity.

The analysis has hitherto been related to axially vibrating rods with F–D and D–D ends, and it demonstrated, among other things, that the first two eigenvalues allow the crack to be uniquely identified. Such a result does not prove true for clamped–damped boundary conditions (C–D), a quite common situation in applications. In this case, however, by simultaneously employing eigenvalues related to different boundary conditions it is still possible to identify damage.

The eigenpairs of a C–D uniform rod are given by

$$\mu_m^{C-D} = \frac{1}{2} \ln \frac{(1-\gamma)}{(1+\gamma)} + \mathrm{i}\frac{2m+1}{2}\pi \equiv \xi^{C-D} + \mathrm{i}\omega_m^{C-D}, \tag{29}$$

$$u_m^{C-D}(x) = A_m \sinh \mu_m^{C-D} x, \quad m = 0, \pm 1, \pm 2, \ldots . \tag{30}$$

The quantity $C_m^{C-D} = -\delta(\mu_m^{C-D})^2/(\mu_m^{C-D})^2$ is equal to

$$C_m^{C-D} = \varepsilon \left[\frac{1}{2}(1 + \cosh 2\xi\sigma \cos(2m+1)\pi\sigma) + \mathrm{i} \sinh \xi\sigma \cosh \xi\sigma \sin(2m+1)\pi\sigma \right], \tag{31}$$

where $\xi \equiv \xi^{C-D}$. If, for example, $\mathrm{Im}(C_m^{C-D}) \neq 0$, from (21) and (31) it follows that

$$S' \equiv \cos(2m+1)\pi\sigma = \frac{1}{2} \frac{\mathrm{Im}(C_{2m+1}^{F-D})}{\mathrm{Im}(C_m^{C-D})}, \tag{32}$$

$m = 0, \pm 1, \pm 2, \ldots$. Thus, from the knowledge of mth eigenvalue in a cracked rod under boundary conditions C–D and $(2m+1)$th eigenvalue under boundary conditions F–D, it is possible to uniquely determine the position variable $S' \equiv \cos(2m+1)\pi\sigma$, $m = 0, \pm 1, \pm 2, \ldots$. In particular, Eq. (32) implies that damage location is uniquely determined by the measurement of the pair $\{\mathrm{Im}(C_0^{C-D}), \mathrm{Im}(C_1^{F-D})\}$ corresponding to the fundamental eigenvalues for C–D and F–D cases. Note that C_0^{C-D} is always different from zero. Damage severity can be determined as in the F–D case, see Eq. (23) and following remarks. Also in the present case, by leaving out all the locations σ_k which correspond to negative values of damage severity ε_k, half of the possible $2|m|$ damage locations identified from Eq. (32) can be discarded.

If now $\mathrm{Im}(C_m^{C-D}) = 0$ for a certain $|m| \geqslant 1$, then there are $2|m|$ possible damage locations, i.e., $\sigma_k = k/(1+2|m|)$, $k = 1, \ldots, 2|m|$. As in the analogous situation for the case F–D, the real part of C_m^{C-D} gives damage severity ε_k corresponding to crack position σ_k

$$\mathrm{Re}(C_m^{C-D}) = \frac{\varepsilon_k}{2}[(-1)^k \cosh 2\xi\sigma_k + 1]. \tag{33}$$

As before, considering the sign of ε_k's, half of the possible damage locations can be omitted.

So far, an analysis of the diagnostic problem has been developed under the assumption that the damping coefficient γ is strictly less than one. It can be shown that similar results hold true for the case $\gamma > 1$. In brief, it turns out that the treatment of the C–D case with $\gamma > 1$ formally coincides with that of the F–D case with $\gamma < 1$, and vice versa the treatment of the F–D case with $\gamma > 1$ can be formally extended to the C–D case with $\gamma < 1$. The discussion of the D–D case for $\gamma > 1$ follows exactly the same lines as the D–D case for $\gamma < 1$.

Finally, it should be observed that the assumption of small damages restricts the range of application of the proposed method to cracked configurations that are a perturbation of the undamaged one. However, this is not a severe limitation, because in most practical situations it is crucial to be able to identify damage as soon as it arises.

4. Applications

In the previous section, it was shown how the measurement of a pair of eigenvalues of a dissipative vibrating rod with a single crack could be used to assess the location as well as the magnitude of damage.

As it is well known, eigenvalue estimates in experimental applications are based on the knowledge of the *transfer function* expression of a system (which will be indicated by TF hereafter). Then, the determination of the TF is a crucial point of the analysis. Regarding the undamaged F–D vibrating rod considered in Section 2, for example, it can be shown that the TF-receptance between the excitation point located at the right end, $z_2 = L$, and the axial response at the cross-section of abscissa z_1 is given by the expression

$$H(s, z_1, z_2 = L) = \frac{\rho \cosh s z_1}{s} \frac{2e^s}{(1-\gamma)} \left[\frac{1}{2(s-\xi)} - \frac{1}{2} + (s-\xi) \sum_{m=1}^{+\infty} \frac{1}{(s-\xi)^2 + m^2\pi^2} \right], \tag{34}$$

where s is the complex variable. To prevent the reader's attention from being drawn away from the main line of the analysis, some details on the derivation of Eq. (34) are presented in the appendix. With the exception of $s = 0$, the poles of $H(s, z_1, z_2 = L)$ are the eigenvalues μ_m given in Eq. (8). In practical applications, modal analysis techniques might be applied to extract the eigenvalues from TF measurements in a chosen frequency range, see [17]. It is worth noticing that if the determination of the TF is a standard issue for self-adjoint vibrating systems, the TF evaluation presents additional difficulties for non-self-adjoint systems such as the ones investigated in this paper. This fact may explain a recent interest of the specialized literature in finding closed-form expressions for the TF of dissipative vibrating systems, see, for example, [8,9].

Some numerical applications of the proposed diagnostic technique will be presented now. The inverse problem of damage detection is solved for different cases, using pseudo-experimental data, i.e., the eigenvalues are obtained from the direct problem in undamaged conditions and in some damaged conditions. To give an idea of the results obtained, some applications to rods under F–D boundary conditions will be presented hereafter. Identification results are shown in detail for a damage location corresponding to $\sigma = 0.400$. They are illustrative of the main features of the inverse problem and of the identification technique. Three different levels of damage are considered. The first case, $\varepsilon = 0.02$, is characterized by "small" damage, i.e., the value of stiffness K is such that the variations of the real and imaginary part of the first few eigenvalues are about 10% and 1.5% of the initial values, respectively. The second case ("moderate" damage) and the third case ("severe" damage) correspond to variations of the same quantities of about 15%, 7% and 15%, 16%, respectively.

In order to take into account the effect of the damping coefficient on identification results, three levels of damping, $\gamma = 0.05$, 0.35, 0.50, are selected among several cases studied and considered in detail.

The eigenvalues for the undamaged beam and their values associated with the cases of damage are shown in Table 1. The latter were obtained as complex roots of the characteristic Eq. (16). Concerning the effect of a crack on the eigenvalues of the rod, there is a substantial difference between the ideal undamped case and the case with dissipative boundary conditions considered here. As it is well known, the variational formulation for the undamped case shows that eigenvalues of the system are decreasing functions of ε, i.e., damage decreases the (real) natural frequencies. This monotonicity property in general does not apply to a vibrating system with dissipation. In fact, within the class of small cracks, the explicit expression (18) for damage sensitivity derived in Section 2 shows that no general monotonicity property of the first order variation of eigenvalues is expected to hold.

In Fig. 1 the effect of a crack of "moderate" severity ($\varepsilon = 0.08$) on the real and imaginary parts of the first four eigenvalues of the F–D rod is presented. Plots show that a single crack can reduce or increase, depending on the crack position, the real and the imaginary part of an eigenvalue. An

Table 1
Dimensionless eigenvalues $\mu_{d,m}$ of a uniform rod under F–D boundary conditions for different damage severity ε and damping coefficient γ (eigenvalue problem (10)–(14))

Mode	Undamaged (1)	Damage D1 (2)	Damage D2 (3)	Damage D3 (4)
(a)				
1	−0.050 + i3.142	−0.048 + i3.086	−0.044 + i2.928	−0.038 + i2.658
2	−0.050 + i6.283	−0.052 + i6.241	−0.057 + i6.124	−0.065 + i5.939
3	−0.050 + i9.425	−0.046 + i9.359	−0.035 + i9.163	−0.019 + i8.830
4	−0.050 + i12.566	−0.052 + i12.347	−0.059 + i11.815	−0.069 + i11.236
5	−0.050 + i15.708	−0.050 + i15.708	−0.050 + i15.708	−0.050 + i15.709
6	−0.050 + i18.850	−0.045 + i18.512	−0.037 + i17.671	−0.033 + i16.820
(b)				
1	−0.365 + i3.142	−0.354 + i3.085	−0.322 + i2.926	−0.275 + i2.652
2	−0.365 + i6.283	−0.380 + i6.240	−0.419 + i6.123	−0.474 + i5.937
3	−0.365 + i9.425	−0.336 + i9.361	−0.252 + i9.169	−0.135 + i8.837
4	−0.365 + i12.566	−0.381 + i12.343	−0.431 + i11.803	−0.509 + i11.223
5	−0.365 + i15.708	−0.366 + i15.715	−0.367 + i15.735	−0.375 + i15.779
6	−0.365 + i18.850	−0.330 + i18.505	−0.268 + i17.644	−0.234 + i16.749
(c)				
1	−0.549 + i3.142	−0.531 + i3.085	−0.483 + i2.923	−0.410 + i2.645
2	−0.549 + i6.283	−0.572 + i6.240	−0.632 + i6.121	−0.715 + i5.933
3	−0.549 + i9.425	−0.505 + i9.364	−0.376 + i9.178	−0.197 + i8.845
4	−0.549 + i12.566	−0.573 + i12.337	−0.650 + i11.787	−0.768 + i11.208
5	−0.549 + i15.708	−0.550 + i15.723	−0.555 + i15.771	−0.591 + i15.877
6	−0.549 + i18.850	−0.495 + i18.497	−0.399 + i17.608	−0.323 + i16.650

Abscissa of the cracked cross-section: $\sigma = 0.400$. Configurations: (1) Undamaged, $\varepsilon = 0$; (2) Damage D1, $\varepsilon = 0.02$; (3) Damage D2, $\varepsilon = 0.08$; (4) Damage D3, $\varepsilon = 0.20$. Levels of damping: (a) $\gamma = 0.05$; (b) $\gamma = 0.35$; (c) $\gamma = 0.50$.

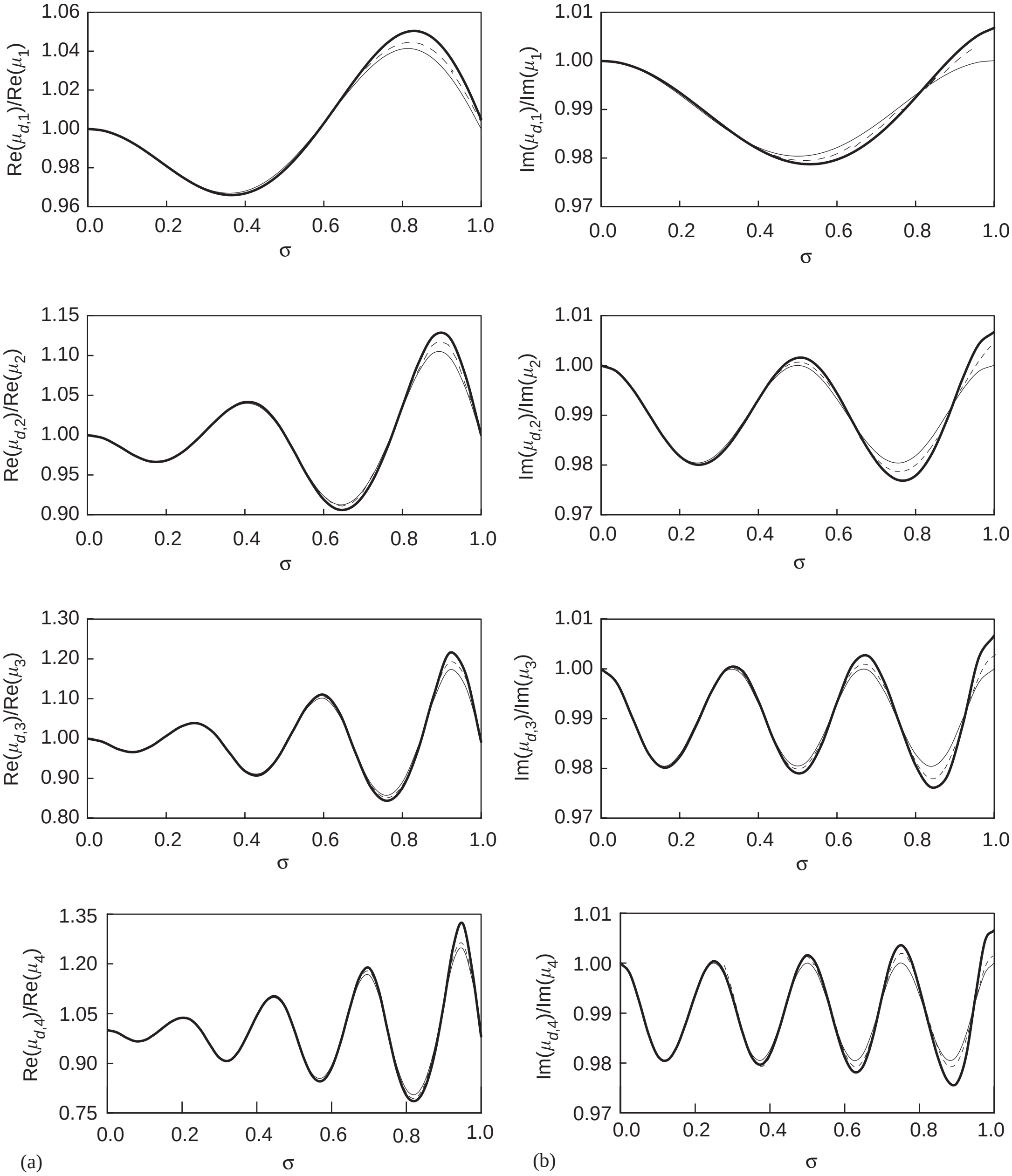

Fig. 1. Variation of the real and imaginary part of the first four eigenvalues versus crack position for ‘‘moderate’’ damage ($\varepsilon = 0.08$) and for different levels of damping. (a) $\mathrm{Re}(\mu_{d,m})/\mathrm{Re}(\mu_m)$, $m = 1$–4; (b) $\mathrm{Im}(\mu_{d,m})/\mathrm{Im}(\mu_m)$, $m = 1$–4. Thin solid line: $\gamma = 0.05$; thin dashed line: $\gamma = 0.35$; thick solid line: $\gamma = 0.50$.

increase of the imaginary part seems to be produced by cracks located in a small neighbourhood of every zero-sensitivity point for the ideal undamped system and, in particular, it is more pronounced for large values of damping coefficient γ.

Table 2
Example of damage identification results (cases free of errors) for different damage severity ε and damping coefficient γ

Pair $m-2m$	Damage D1		Damage D2		Damage D3	
	σ_k	ε_k	σ_k	ε_k	σ_k	ε_k
(a)						
1–2	0.406	0.020	0.430	0.090	(*)	(*)
	0.594	−0.014	0.570	−0.068	(*)	(*)
2–4	0.099	−0.075	0.098	−0.250	0.098	−0.439
	0.401	0.019	0.402	0.061	0.402	0.367
	0.599	−0.012	0.598	−0.041	0.598	−0.247
	0.901	0.008	0.902	0.027	0.902	0.048
3–6	0.069	0.124	0.074	0.566	0.078	1.903
	0.264	−0.032	0.259	−0.162	0.255	−0.583
	0.402	0.021	0.408	0.103	0.412	0.362
	0.598	−0.014	0.592	−0.071	0.588	−0.253
	0.736	0.012	0.741	0.057	0.745	0.200
	0.931	−0.009	0.926	−0.045	0.922	−0.161
(b)						
1–2	0.389	0.020	0.407	0.082	0.479	0.578
	0.611	−0.013	0.593	−0.056	0.521	−0.531
2–4	0.100	−0.078	0.098	−0.257	0.098	−0.450
	0.400	0.019	0.402	0.062	0.402	0.388
	0.600	−0.012	0.598	−0.041	0.598	−0.260
	0.900	0.008	0.902	0.026	0.902	0.044
3–6	0.069	0.127	0.074	0.585	0.087	1.781
	0.264	−0.033	0.259	−0.167	0.246	−0.631
	0.402	0.022	0.408	0.106	0.421	0.369
	0.598	−0.014	0.592	−0.072	0.579	−0.267
	0.736	0.011	0.741	0.057	0.754	0.205
	0.931	−0.009	0.926	−0.045	0.913	−0.169
(c)						
1–2	0.370	0.022	0.384	0.084	0.424	0.216
	0.630	−0.012	0.616	−0.051	0.576	−0.156
2–4	0.100	−0.081	0.099	−0.267	0.099	−0.465
	0.400	0.020	0.401	0.063	0.401	0.416
	0.600	−0.013	0.599	−0.040	0.599	−0.277
	0.900	0.008	0.901	0.024	0.901	0.039
3–6	0.069	0.131	0.074	0.611	0.077	2.144
	0.264	−0.034	0.259	−0.174	0.256	−0.644
	0.403	0.022	0.408	0.110	0.410	0.401
	0.597	−0.014	0.592	−0.074	0.590	−0.278
	0.736	0.011	0.741	0.058	0.744	0.219
	0.931	−0.008	0.926	−0.045	0.923	−0.175
Exact values	0.400	0.020	0.400	0.080	0.400	0.200

Determination of crack location σ_k and damage severity ε_k by using the pair $\{C_m^{F\text{–}D}, C_{2m}^{F\text{–}D}\}$, $m=1, 2, 3$, as data in formulas (22) and (23). (a) $\gamma=0.05$; (b) $\gamma=0.35$; (c) $\gamma=0.50$. The symbol (*) means imaginary solution.

The results of identification based on the mth, $2m$th frequencies, $m=1,2,3$, are summed up in Table 2. With reference to the localization of the cracked cross-section, the method proves satisfactory. In fact, in the absence of errors, the set of solutions predicted by the theory for the mathematical inverse problem contains (a satisfactory estimate of) the actual solution of the damage location problem. Moreover, deviations from the exact damage location are quite stable with respect to large variations of the damage severity and of the damping coefficient. Discrepancies generally are smaller for less severe damages, and this behaviour is expected because the inverse problem is formulated on the assumption that the damaged system is a "small" perturbation of the virgin system. It is worth noticing that, according to the theory developed in Section 3, half of the possible damage locations can be discarded because the corresponding estimate of damage severity ε takes negative values.

The estimate for damage severity ε is less accurate and generally accuracy deteriorates when more severe damages are considered. This trend was already observed in the papers [13,14] for the undamped ideal case.

The analysis was developed in the absence of errors so far, but, as it is well known, the results of most diagnostic techniques strictly depend on possible measurement and modelling errors. To take into account the effect of errors in the experimental data, a comprehensive series of cases in which eigenvalues were corrupted by some random noise was considered. To give an example of the results obtained in this situation, Table 3 refers to the case $\sigma=0.400$ in the presence of random errors having peak values ranging linearly from 0.5% in the ideal value for the first eigenvalue to 3% in the sixth eigenvalue, both on the real and imaginary part. As a general remark, if eigenvalues used as data in identification are affected by errors being relatively small with respect to the variations of the eigenvalues induced by damage, then damage identification leads to satisfactory results. However, the analysis shows that the noise in the data is usually amplified strongly and the estimates of damage parameters seem to be rather sensitive to input errors.

5. Conclusions

This paper focused on detecting a single crack when damage-induced shifts in a pair of eigenvalues of a longitudinally vibrating beam with damped boundary conditions are known. The analysis is based on an explicit expression of the eigenvalue sensitivity to damage and the damaged system is considered as a perturbation of the virgin system. For different sets of boundary conditions it was shown how the knowledge of a suitable pair of eigenvalues might be used to estimate the location of damage. The theoretical results are confirmed by a comparison with numerical tests performed on cracked beams where eigenvalues were corrupted by some random noise. This suggests that it may be possible to use the method in practical situations including, in perspective, more complex damped vibrating systems, such as beams in bending vibration, beams under less restrictive classes of damage and with damper devices with more refined rheological behaviour. For these more complicated situations, testing the proposed diagnostic technique on the basis of experimental data should be an important question to be investigated.

Table 3
Example of damage identification results in the presence of random errors on data

Pair m–$2m$	Damage D1		Damage D2		Damage D3	
	σ_k	ε_k	σ_k	ε_k	σ_k	ε_k
(a)						
1–2	0.363	0.019	0.410	0.077	*	*
	0.637	−0.011	0.590	−0.054	*	*
2–4	0.090	−0.077	0.094	−0.256	0.095	−0.450
	0.410	0.017	0.406	0.059	0.405	0.361
	0.590	−0.012	0.594	−0.041	0.595	−0.246
	0.910	0.008	0.906	0.027	0.905	0.047
3–6	0.058	0.121	0.072	0.537	0.077	1.828
	0.276	−0.025	0.261	−0.149	0.256	−0.554
	0.391	0.018	0.406	0.096	0.411	0.345
	0.609	−0.011	0.594	−0.065	0.589	−0.240
	0.724	0.010	0.739	0.053	0.744	0.190
	0.942	−0.007	0.928	−0.042	0.923	−0.154
(b)						
1–2	0.352	0.027	0.385	0.082	0.437	0.229
	0.648	−0.014	0.615	−0.051	0.563	−0.177
2–4	0.106	−0.083	0.101	−0.256	0.100	−0.443
	0.394	0.022	0.399	0.063	0.400	0.386
	0.606	−0.014	0.601	−0.041	0.600	−0.257
	0.894	0.009	0.899	0.026	0.900	0.044
3–6	0.063	0.120	0.073	0.561	0.087	1.712
	0.270	−0.028	0.260	−0.157	0.246	−0.606
	0.396	0.019	0.406	0.100	0.421	0.354
	0.604	−0.012	0.594	−0.068	0.579	−0.257
	0.730	0.010	0.740	0.054	0.754	0.197
	0.937	−0.008	0.927	−0.043	0.913	−0.162
(c)						
1–2	0.338	0.017	0.374	0.078	0.415	0.196
	0.662	−0.008	0.626	−0.045	0.585	−0.137
2–4	0.064	−0.109	0.092	−0.268	0.096	−0.466
	0.436	0.015	0.408	0.058	0.404	0.402
	0.564	−0.012	0.592	−0.038	0.596	−0.271
	0.936	0.006	0.908	0.022	0.904	0.038
3–6	0.078	0.143	0.077	0.635	0.078	2.211
	0.256	−0.043	0.257	−0.189	0.255	−0.677
	0.411	0.026	0.410	0.117	0.412	0.418
	0.589	−0.018	0.590	−0.080	0.588	−0.291
	0.744	0.014	0.743	0.062	0.745	0.229
	0.922	−0.010	0.923	−0.049	0.922	−0.184
Exact values	0.400	0.020	0.400	0.080	0.400	0.200

Determination of crack location σ_k and damage severity ε_k by using the pair $\{C_m^{F\text{–}D}, C_{2m}^{F\text{–}D}\}$, $m=1, 2, 3$, as data in formulas (22) and (23). (a) $\gamma=0.05$; (b) $\gamma=0.35$; (c) $\gamma=0.50$. The symbol (*) means imaginary solution.

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Appendix A

The aim of this appendix is to provide some details on the derivation of the TF expression (34) for an F–D vibrating rod. For vibrating systems described by self-adjoint boundary value problems, the determination of TF is a standard issue. Due to energy dissipation at the boundary point $z = L$, the system under investigation is non-self-adjoint. So, conventional modal analysis techniques based on eigenfunctions expansion are not directly applicable to obtain a TF expression. Here, the study of the dynamic problem (1)–(3) is formulated within the semigroup theory. Notations are the same as in Section 2. Following Veselic [6], set

$$\mathbf{U} \equiv \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w \\ \dfrac{\partial w}{\partial t} \end{pmatrix}. \tag{A.1}$$

Then Eqs. (1)–(3) can be reformulated as the homogeneous evolution problem

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \mathbf{A}\mathbf{U} = 0, \quad t > 0, \tag{A.2}$$

with boundary conditions

$$u'(0) = 0, \tag{A.3}$$

$$EAu'(L) + cv(L) = 0, \tag{A.4}$$

where operator $\mathbf{A}$ is formally given by the differential expression

$$\mathbf{A} = \begin{vmatrix} 0 & -1 \\ -\dfrac{EA}{\rho}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} & 0 \end{vmatrix}. \tag{A.5}$$

Let $H = H^1(0, L) \times L^2(0, L)$ be the Hilbert space equipped with the usual scalar product given by $\langle \mathbf{U}_1, \mathbf{U}_2 \rangle = \int_0^L u_1' \bar{u}_2' + \int_0^L u_1 \bar{u}_2 + \int_0^L v_1 \bar{v}_2$ for every $\mathbf{U}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $\mathbf{U}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ belonging to H. Here, $H^k(0, L)$, k a non-negative integer, is the Sobolev space of functions defined on $(0,L)$ with square summable derivatives up to the kth order.

Let $D(\mathbf{A}) = \{(f, g) \in H | f \in H^2(0, L), g \in H^1(0, L), EAf'(L) + cg(L) = 0, f'(0) = 0\}$. It is easily seen that $\mathbf{A} : D(\mathbf{A}) \subset H \to H$ is a monotonic dissipative operator, so that problem (A.2)–(A.4) generates a contractive semigroup, whose generator $\mathbf{A}$ has the resolvent $(\lambda\mathbf{1} + \mathbf{A})^{-1}$ given by

$$(\lambda\mathbf{1} + \mathbf{A})^{-1} = \begin{vmatrix} \lambda\Gamma_\lambda + \Phi_\lambda & \Gamma_\lambda \\ \lambda^2\Gamma_\lambda + \lambda\Phi_\lambda - 1 & \lambda\Gamma_\lambda \end{vmatrix}, \tag{A.6}$$

where λ is a complex number and

$$(\Gamma_\lambda[f])(z) = \int_0^L G_\lambda(z,y) f(y)\,\mathrm{d}y, \tag{A.7}$$

$$(\Phi_\lambda[f])(z) = cf(L)\frac{\cosh\sqrt{\rho/EA}\lambda z}{\lambda N(\lambda L)}, \tag{A.8}$$

$$G_\lambda(z,y) = \frac{\sqrt{\rho/EA}}{\lambda N(\lambda L)}\begin{cases} M(\lambda(L-z))\cosh\sqrt{\rho/EA}\lambda y, & 0\leqslant y\leqslant z, \\ M(\lambda(L-y))\cosh\sqrt{\rho/EA}\lambda z, & z\leqslant y\leqslant L, \end{cases} \tag{A.9}$$

$$M(\theta) = \sqrt{\rho EA}\cosh\sqrt{\rho/EA}\theta + c\sinh\sqrt{\rho/EA}\theta, \tag{A.10}$$

$$N(\theta) = \sqrt{\rho EA}\sinh\sqrt{\rho/EA}\theta + c\cosh\sqrt{\rho/EA}\theta. \tag{A.11}$$

Therefore, by applying standard results in the semigroup theory (e.g., the Hille–Yosida Theorem, Ref. [18]), it can be proved that for every $\mathbf{U}_0 \in D(\mathbf{A})$ there exists a unique solution of (A.2)–(A.4) satisfying the initial condition

$$\mathbf{U}(0) = \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \equiv \mathbf{U}_0.$$

Moreover, for every sufficiently regular $\mathbf{F}(t)$, there exists a unique solution for the non-homogeneous problem

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \mathbf{AU} = \mathbf{F}(t), & t>0, \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{A.12}$$

see Ref. [18].

To determine, for example, the frequency response function (FRF) receptance term $H(\mathrm{i}\omega, z_1, z_2 = L)$ an harmonic load $f(t) = f_0 e^{\mathrm{i}\omega t}$ can be applied at the right end of the beam, $z_2 = L$, and the axial response displacement can be measured after a sufficiently large moment of time at another (possibly coincident) point of abscissa z_1. The corresponding dynamic problem is described by Eq. $(A.12)_1$ with initial conditions $(A.12)_2$ and with

$$\mathbf{F}(t) = \mathbf{F}_0 e^{\mathrm{i}\omega t}, \quad \mathbf{F}_0 = \begin{pmatrix} 0 \\ f_0\delta_y(L) \end{pmatrix},$$

where $\delta_y(L)$ is the Dirac delta at the right end $y = L$. Since the system is governed by a dissipative operator, it can be shown that (for $t\to\infty$) there exists a steady state motion

$$\tilde{\mathbf{U}}(t) = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad \tilde{v} \equiv \frac{\partial \tilde{u}}{\partial t},$$

given by

$$\tilde{\mathbf{U}}(t) = (\mathrm{i}\omega\mathbf{1} + \mathbf{A})^{-1}\mathbf{F}_0 e^{\mathrm{i}\omega t}. \tag{A.13}$$

By recalling expression (A.6) of the resolvent $(\lambda\mathbf{1}+\mathbf{A})^{-1}$, $\lambda = \mathrm{i}\omega$, it turns out that

$$\tilde{u}(t) = \Gamma_{\mathrm{i}\omega}(\delta_y(L))f_0\mathrm{e}^{\mathrm{i}\omega t}, \tag{A.14}$$

that is, the FRF between the excitation point $z_2 = L$ and any point of abscissa z_1 is given by the expression

$$H(\mathrm{i}\omega, z_1, z_2 = L) = \frac{\rho\cosh\left(\sqrt{(\rho/EA)}\mathrm{i}\omega z_1\right)}{\mathrm{i}\omega}\frac{1}{\sqrt{\rho EA}\sinh(\sqrt{(\rho/EA)}\mathrm{i}\omega L) + c\cosh(\sqrt{(\rho/EA)}\mathrm{i}\omega L)}. \tag{A.15}$$

The corresponding TF of the rod formally coincides with the previous expression (A.15), where the variable $\mathrm{i}\omega$ is replaced by the complex variable s. To obtain the analogue of the "classical" TF for self-adjoint systems expressed by an infinite series, it is convenient to rewrite the right factor on the right half-side of (A.15) in the following form:

$$\frac{1}{\sinh s + \gamma\cosh s} = \frac{2\mathrm{e}^s}{(1-\gamma)(\mathrm{e}^{2(s-\xi)}-1)}. \tag{A.16}$$

where s is a complex number. Then, application of the identity

$$\frac{1}{(\mathrm{e}^\eta - 1)} = \frac{1}{\eta} - \frac{1}{2} + 2\eta\sum_{n=1}^{\infty}\frac{1}{(\eta^2 + 4n^2\pi^2)}, \tag{A.17}$$

with $\eta = 2(s-\xi)$ leads to the desired expression (34), see Ref. [19, p. 113]. Expression (34) represents a closed-form expression of the TF of the rod, see also Ref. [20] for closed-form evaluation of the TF of (even more general) distributed parameter systems via Laplace transform of the Green system function.

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