ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Sound and Vibration 267 (2003) 191–207

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Forced vibration of Euler–Bernoulli beams by means of dynamic Green functions

M. Abu-Hilal*

Department of Mechanical and Industrial Engineering, Applied Sciences University, Amman 11931 Jordan

Received 21 June 2002; accepted 10 October 2002

Abstract

A method for determining the dynamic response of prismatic damped Euler–Bernoulli beams subjected to distributed and concentrated loads is presented. The method yields exact solutions in closed form and may be used for single and multi-span beams, single and multi-loaded beams, and statically determinate and indeterminate beams. Also Green functions for various beams with different homogenous and elastic boundary conditions are given. In order to demonstrate the use of the Green functions method, several examples are given. Some of the obtained results are compared with those given in the references.

1. Introduction

The forced transverse vibration of beams due to steady and moving loads is a very important research topic in all branches of engineering. The most used method for determining these vibrations is the expansion of the applied loads and the dynamic responses in terms of the eigenfunctions of the undamped beams [1,2]. This method leads to solutions presented as infinite series, which will be truncated after a number of terms and approximate solutions are then obtained. Frýba [3] used the Fourier sine (finite) integral transformation and the Laplace–Carson integral transformation to determine the dynamic response of beams due to moving loads and obtained this response in the form of series solutions. Leissa [4] presented an exact method for determining the dynamic deflection of single-span Euler–Bernoulli beams subjected to distributed

*Fax: +1-652-32-899.

E-mail address: hilall22@yahoo.com (M. Abu-Hilal).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00178-0

loads of the form $P(x)\sin\Omega t$. He used a solution for the deflection similar to the exciting load function and obtained a fourth order spatially dependent ordinary differential equation. Then he solved the ordinary differential equation and applied the boundary conditions to evaluate the integration constants. Although the dynamic response of the undamped beam was readily obtained, the determination of the response of the damped system is more complicated. Finally, he compared the obtained solution with the series solution obtained through expansion in normal modes. Hamada [5] solved the response problem of a simply supported and damped Euler–Bernoulli uniform beam of finite length traversed by a constant force moving at a uniform speed by applying the double Laplace transformation with respect to both time and the length co-ordinate along the beam. He obtained in closed form an exact solution for the dynamic deflection of the considered beam. Nicholson and Bergman [6,7] used Green functions for analyzing the free vibration of a combined linear undamped dynamical system consisting of beams and discrete spring–mass oscillators. They used the method of separation of variables to separate the governing partial differential equation for the system into a second order time-dependent and a fourth order spatially dependent ordinary differential equations and derived the characteristic equation for the eigenfrequencies of the system. Finally, the characteristic equation was solved and exact natural frequencies and exact normal modes obtained. Also they determined in Ref. [7] the dynamic response of the studied system due to an arbitrary excitation using the method of expansion in eigenforms. Broome [8] used a Green function approach and a particular integral approach to study the economical analysis of combined dynamical systems. Bergman and McFarland [9] used Green functions to study the free vibration of an Euler–Bernoulli beam with homogeneous boundary conditions, supported in its interior by arbitrarily located pin supports and translational and torsional linear springs. Furthermore, they determined the forced response of the beam to an arbitrary excitation by modal analysis method. Kukla and Posiadala [10] utilized the Green function method to study the free transverse vibration of Euler–Bernoulli beams with many elastically mounted masses. They obtained closed form expressions of the equations for the natural frequencies. Also Kukla [11] applied the Green function method to determine the natural frequencies of a Timoshenko beam with attached multi-mass oscillators. Foda and Abduljabbar [12] used a Green function approach to determine the dynamic deflection of an undamped simply supported Euler–Bernoulli beam of finite length subject to a moving mass traversing its span at constant speed.

In this paper a Green functions method for determining the dynamic response of Euler–Bernoulli beams subjected to distributed and concentrated loads is presented. This method may be used for single and multi-span beams, single and multi-loaded beams, and statically determinate and indeterminate beams. Also Green functions for various beams with different homogeneous and elastic boundary conditions are determined. Several examples are given to illustrate the use of the Green functions method.

The method of Green functions is more efficient than the series methods because this method yields exact solutions in closed forms. This is in particular essential for calculating dynamic stresses and determining the dynamic response of beams other than simply supported. Also by the use of the Green functions method, the boundary conditions are embedded in the Green functions of the corresponding beams. Furthermore, by using this method, it is not necessary to solve the free vibration problem in order to obtain the eigenvalues and the corresponding eigenfunctions which are required while using series solutions.

2. Green functions of beams with homogeneous boundary conditions

The transverse vibration of a uniform elastic homogeneous isotropic Euler–Bernoulli beam is described by the partial differential equation

$$EIv'''' + \mu\ddot{v} + r_a\dot{v} + r_i\dot{v}'''' = p(x,t), \tag{1}$$

where EI, μ, r_a, and r_i are the flexural rigidity of the beam, the mass per unit length of the beam, the coefficient of external damping of the beam, and the coefficient of internal damping of the beam, respectively. $v(x,t)$ is the deflection of the beam at point x and time t and $p(x,t)$ denotes the load per unit length of the beam at point x and time t. A prime denotes differentiation with respect to position x and an overdot denotes differentiation with respect to time t. Assuming the load function $p(x,t)$ is given in the form

$$p(x,t) = f(x)\cos\Omega t, \tag{2}$$

where $f(x)$ is an arbitrary but deterministic distributed load, then Eq. (1) becomes

$$EIv'''' + \mu\ddot{v} + r_a\dot{v} + r_i\dot{v}'''' = f(x)\cos\Omega t. \tag{3}$$

By the action of a concentrated harmonic force $F(t)$ at a position ξ, the load $p(x,t)$ is given as

$$p(x,t) = \delta(x-\xi)F_0\cos\Omega t \tag{4}$$

and by the action of a harmonic moment load $M(t)$ at a position ξ, the load $p(x,t)$ is given as

$$p(x,t) = \delta'(x-\xi)M_0\cos\Omega t, \tag{5}$$

where $\delta(.)$ is the Dirac delta function.

Since the beam is damped, it is recommended to write Eq. (3) in the complex form

$$EIw'''' + \mu\ddot{w} + r_a\dot{w} + r_i\dot{w}'''' = f(x)e^{i\Omega t}, \tag{6}$$

where $i = \sqrt{-1}$ is the imaginary unit and

$$v(x,t) = Re\{w(x,t)\}. \tag{7}$$

The solution of Eq. (6) is assumed in the form

$$w(x,t) = X(x)e^{i\Omega t}. \tag{8}$$

Substituting this solution into Eq. (6) and dividing by $e^{i\Omega t}$ yields

$$X'''' - \kappa^4 X = \frac{f(x)}{(EI + ir_i\Omega)}, \tag{9}$$

where

$$\kappa^4 = \frac{(\mu\Omega^2 - ir_a\Omega)}{(EI + ir_i\Omega)}. \tag{10}$$

The solution of Eq. (9) may be given as

$$X(x) = \int_0^L f(\xi)G(x,\xi)\,d\xi, \tag{11}$$

where L is the length of the beam and $G(x,\xi)$ is a Green function which is to be determined. Since a Green function of a beam is its response due to a unit concentrated force acting at an arbitrary position ξ, we write Eq. (9) in the form

$$X'''' - \kappa^4 X = \frac{\delta(x-\xi)}{(EI + \mathrm{i}r_i\Omega)} \tag{12}$$

to get the desired Green function.

Although any method for solving differential equations of the form given in Eq. (12), for example the method of variation of parameters or the method of undetermined coefficients, may be used to obtain the desired Green function, in this work the method of Laplace transform will be used since this method seems to be easier to use. The Laplace transformed solution of Eq. (12) with respect to position variable x is

$$\hat{X}(s) = \frac{1}{(s^4 - \kappa^4)}\left[\frac{\mathrm{e}^{-s\xi}}{(EI + \mathrm{i}r_i\Omega)} + s^3X(0) + s^2X'(0) + sX''(0) + X'''(0)\right], \tag{13}$$

where s is a suitable transform parameter which is in general a complex variable and $X(0)$, $X'(0)$, $X''(0)$, and $X'''(0)$ are the values of the function X and their derivatives at $x = 0$. In general, only two of these conditions are known for beam problems with homogeneous boundary conditions. Therefore, the two unknown conditions will be left as parameters which can be then evaluated by applying two boundary conditions at $x = L$ to the obtained inverted solution. The inverse transform of Eq. (13) is found to be

$$\begin{aligned} X(x,\xi) = {} & \frac{\phi_4(x-\xi)u(x-\xi)}{\kappa^3(EI + \mathrm{i}r_i\Omega)} + X(0)\phi_1(x) + \frac{X'(0)}{\kappa}\phi_2(x) \\ & + \frac{X''(0)}{\kappa^2}\phi_3(x) + \frac{X'''(0)}{\kappa^3}\phi_4(x), \end{aligned} \tag{14}$$

where $u(x)$ is the unit step function and

$$\begin{aligned} \phi_1(x) &= \tfrac{1}{2}(\cosh\kappa x + \cos\kappa x), \quad \phi_2(x) = \tfrac{1}{2}(\sinh\kappa x + \sin\kappa x), \\ \phi_3(x) &= \tfrac{1}{2}(\cosh\kappa x - \cos\kappa x), \quad \phi_4(x) = \tfrac{1}{2}(\sinh\kappa x - \sin\kappa x). \end{aligned} \tag{15}$$

Eq. (14) represents the sought Green function of Eq. (1), i.e.,

$$G(x,\xi) = X(x,\xi). \tag{16}$$

Knowing that

$$\begin{array}{lll} \phi_1' = \kappa\phi_4 & \phi_1'' = \kappa^2\phi_3 & \phi_1''' = \kappa^3\phi_2 \\ \phi_2' = \kappa\phi_1 & \phi_2'' = \kappa^2\phi_4 & \phi_2''' = \kappa^3\phi_3 \\ \phi_3' = \kappa\phi_2 & \phi_3'' = \kappa^2\phi_1 & \phi_3''' = \kappa^3\phi_4 \\ \phi_4' = \kappa\phi_3 & \phi_4'' = \kappa^2\phi_2 & \phi_4''' = \kappa^3\phi_1 \end{array} \tag{17}$$

then the first, second, and third derivatives of $X(x,\xi)$ with respect to x for $x \geqslant \xi$ are

$$\begin{aligned} X'(x,\xi) = {} & \frac{\phi_3(x-\xi)}{\kappa^2(EI + \mathrm{i}r_i\Omega)} + \kappa X(0)\phi_4(x) + X'(0)\phi_1(x) \\ & + \frac{X''(0)}{\kappa}\phi_2(x) + \frac{X'''(0)}{\kappa^2}\phi_3(x), \end{aligned} \tag{18}$$

$$X''(x,\xi) = \frac{\phi_2(x-\xi)}{\kappa(EI + \mathrm{i}r_i\Omega)} + \kappa^2 X(0)\phi_3(x) + \kappa X'(0)\phi_4(x) + X''(0)\phi_1(x) + \frac{X'''(0)}{\kappa}\phi_2(x), \tag{19}$$

$$X'''(x,\xi) = \frac{\phi_1(x-\xi)}{(EI + \mathrm{i}r_i\Omega)} + \kappa^3 X(0)\phi_2(x) + \kappa^2 X'(0)\phi_3(x) + \kappa X''(0)\phi_4(x) + X'''(0)\phi_1(x). \tag{20}$$

The relationship between the boundary vectors of the left-hand end ($x = 0$) and the right-hand end ($x = L$) of the beam are obtained using Eqs. (14), (18)–(20):

$$\begin{bmatrix} \phi_1(L) & \dfrac{\phi_2(L)}{\kappa} & \dfrac{\phi_3(L)}{\kappa^2} & \dfrac{\phi_4(L)}{\kappa^3} \\ \kappa\phi_4(L) & \phi_1(L) & \dfrac{\phi_2(L)}{\kappa} & \dfrac{\phi_3(L)}{\kappa^2} \\ \kappa^2\phi_3(L) & \kappa\phi_4(L) & \phi_1(L) & \dfrac{\phi_2(L)}{\kappa} \\ \kappa^3\phi_2(L) & \kappa^2\phi_3(L) & \kappa\phi_4(L) & \phi_1(L) \end{bmatrix} \begin{bmatrix} X(0) \\ X'(0) \\ X''(0) \\ X'''(0) \end{bmatrix} = \begin{bmatrix} X(L) - f_1(\xi) \\ X'(L) - f_2(\xi) \\ X''(L) - f_3(\xi) \\ X'''(L) - f_4(\xi) \end{bmatrix} \tag{21}$$

or

$$\mathbf{TX_0} = \mathbf{X_L} - \mathbf{f}, \tag{22}$$

where

$$f_1(\xi) = \frac{\phi_4(L-\xi)}{\kappa^3(EI + \mathrm{i}r_i\Omega)}, \quad f_2(\xi) = \frac{\phi_3(L-\xi)}{\kappa^2(EI + \mathrm{i}r_i\Omega)},$$
$$f_3(\xi) = \frac{\phi_2(L-\xi)}{\kappa(EI + \mathrm{i}r_i\Omega)}, \quad f_4(\xi) = \frac{\phi_1(L-\xi)}{EI + \mathrm{i}r_i\Omega}. \tag{23}$$

The Green function for a certain beam may be now determined using Eqs. (14) and (21) and appropriate boundary conditions. As an example, a simply supported beam is considered. Because the deflection and the internal bending moment of this beam must vanish at $x = 0$, i.e., $X(0) = EIX''(0) = 0$, then the first and third columns of the matrix $\mathbf{T}$ may be omitted. On the other hand, since the slope (X') and shear force (EIX''') are unknown at $x = L$, the second and fourth rows of the matrix $\mathbf{T}$ may be ignored. Knowing that $X(L) = X''(L) = 0$, Eq. (21) becomes

$$\begin{bmatrix} \dfrac{\phi_2(L)}{\kappa} & \dfrac{\phi_4(L)}{\kappa^3} \\ \kappa\phi_4(L) & \dfrac{\phi_2(L)}{\kappa} \end{bmatrix} \begin{bmatrix} X'(0) \\ X'''(0) \end{bmatrix} = \begin{bmatrix} -\dfrac{\phi_4(L-\xi)}{\kappa^3(EI + \mathrm{i}r_i\Omega)} \\ -\dfrac{\phi_2(L-\xi)}{\kappa(EI + \mathrm{i}r_i\Omega)} \end{bmatrix}. \tag{24}$$

Solving this equation yields for the unknown conditions

$$X'(0) = \frac{1}{\kappa^2(EI + \mathrm{i}r_i\Omega)} \frac{\phi_4(L)\phi_2(L-\xi) - \phi_2(L)\phi_4(L-\xi)}{\phi_2^2(L) - \phi_4^2(L)}. \tag{25}$$

$$X'''(0) = \frac{1}{EI + \mathrm{i}r_i\Omega} \frac{\phi_4(L)\phi_4(L-\xi) - \phi_2(L)\phi_2(L-\xi)}{\phi_2^2(L) - \phi_4^2(L)}. \tag{26}$$

With $X'(0)$ and $X'''(0)$ defined in Eqs. (25) and (26), the Green function for a simply supported beam is obtained from Eq. (14) as

$$G(x,\xi) = \frac{\phi_4(x-\xi)u(x-\xi)}{\kappa^3(EI + \mathrm{i}r_i\Omega)} + \frac{X'(0)}{\kappa}\phi_2(x) + \frac{X'''(0)}{\kappa^3}\phi_4(x). \tag{27}$$

Using Eqs. (15), (25) and (26) yields

$$G(x,\xi) = \frac{1}{2\kappa^3(EI + \mathrm{i}r_i\Omega)}[\sin h\kappa(x-\xi) - \sin\kappa(x-\xi)]u(x-\xi),$$

$$+\frac{1}{2\kappa^3(EI + \mathrm{i}r_i\Omega)}\frac{\sin\kappa x \sin h\kappa L \sin\kappa(L-\xi) - \sin h\kappa x \sin\kappa L \sin h\kappa(L-\xi)}{\sin\kappa L \sin h\kappa L}, \tag{28}$$

or in the more familiar form

$$G(x,\xi) = A\begin{cases} \sin\kappa x \sinh\kappa L \sin\kappa(L-\xi) - \sinh\kappa x \sin\kappa L \sinh\kappa(L-\xi) & x \leqslant \xi, \\ \sin\kappa\xi \sinh\kappa L \sin\kappa(L-x) - \sinh\kappa\xi \sin\kappa L \sinh\kappa(L-x) & \xi \leqslant x, \end{cases} \tag{29}$$

where

$$A = \frac{1}{2\kappa^3(EI + \mathrm{i}r_i\Omega)\sin\kappa L \sinh\kappa L}. \tag{30}$$

Table 1 includes Green functions for beams with different boundary conditions derived by using Eqs. (14) and (21) and appropriate boundary conditions.

3. Green functions of beams with elastic boundary conditions

Consider a beam supported on torsional and translational springs at both ends. The torsional and translational spring constants at the left-hand (right-hand) end of the beam are denoted by k_{Lt} (k_{Rt}) and $k_L(k_R)$, respectively. The boundary conditions of this beam at the left-hand side

Table 1
Green functions for different Euler–Bernoulli beams

Beam type	$G(x,\xi)$
Pinned–pinned	$C\phi_4(x-\xi)u(x-\xi) + C[g_1(\xi)\phi_4(x) + g_2(\xi)\phi_2(x)]$
Fixed–fixed	$C\phi_4(x-\xi)u(x-\xi) + C[g_3(\xi)\phi_4(x) + g_4(\xi)\phi_3(x)]$
Fixed–pinned	$C\phi_4(x-\xi)u(x-\xi) + C[g_5(\xi)\phi_4(x) + g_6(\xi)\phi_3(x)]$
Pinned–fixed	$C\phi_4(x-\xi)u(x-\xi) + C[g_7(\xi)\phi_4(x) + g_8(\xi)\phi_2(x)]$
Fixed–free	$C\phi_4(x-\xi)u(x-\xi) + C[g_9(\xi)\phi_4(x) + g_{10}(\xi)\phi_3(x)]$
Free–fixed	$C\phi_4(x-\xi)u(x-\xi) + C[g_{11}(\xi)\phi_2(x) + g_{12}(\xi)\phi_1(x)]$
Pinned–sliding	$C\phi_4(x-\xi)u(x-\xi) + C[g_{13}(\xi)\phi_4(x) + g_{14}(\xi)\phi_2(x)]$
Fixed–sliding	$C\phi_4(x-\xi)u(x-\xi) + C[g_{15}(\xi)\phi_4(x) + g_{16}(\xi)\phi_3(x)]$
Sliding–free	$C\phi_4(x-\xi)u(x-\xi) + C[g_{17}(\xi)\phi_3(x) + g_{18}(\xi)\phi_1(x)]$
Pinned–free	$C\phi_4(x-\xi)u(x-\xi) + C[g_{19}(\xi)\phi_4(x) + g_{20}(\xi)\phi_2(x)]$
Sliding–sliding	$C\phi_4(x-\xi)u(x-\xi) + C[g_{21}(\xi)\phi_3(x) + g_{22}(\xi)\phi_1(x)]$
Free–free	$C\phi_4(x-\xi)u(x-\xi) + C[g_{23}(\xi)\phi_2(x) + g_{24}(\xi)\phi_1(x)]$

$(x = 0)$ are given as

$$V = -k_L v, \quad M = k_{Lt} v' \tag{31}$$

and at the right-hand side $(x = L)$ as

$$V = k_R v, \quad M = -k_{Rt} v', \tag{32}$$

where $V = EIv'''$ is the shear force and $M = EIv''$ is the bending moment of the beam. Using the product solution $v(x,t) = X(x)T(t)$ and their derivatives, One obtains for the boundary conditions at $x = 0$

$$\begin{aligned} X(0) &= \frac{-EI}{k_L} X'''(0), \\ X'(0) &= \frac{EI}{k_{Lt}} X''(0) \end{aligned} \tag{33}$$

and at $x = L$

$$\begin{aligned} X(L) &= \frac{EI}{k_R} X'''(L), \\ X'(L) &= \frac{-EI}{k_{Rt}} X''(L). \end{aligned} \tag{34}$$

Using Eqs. (14), (18)–(20), (33) and (34) yields the unknown boundary conditions at $x = 0$

$$X(0) = \frac{f_1(\xi)}{C_1}, \tag{35}$$

$$X'(0) = \frac{f_2(\xi)}{\kappa C_1}, \tag{36}$$

$$X''(0) = \frac{k_{Lt} f_2(\xi)}{EI\kappa C_1}, \tag{37}$$

$$X'''(0) = \frac{-k_L f_1(\xi)}{EIC_1}, \tag{38}$$

where

$$\begin{aligned} C_1 = &-\phi_1^2(L)(EI)^2\kappa^4(k_L+k_R)(k_{Rt}+K_{Lt}) \\ &+EI\kappa^3[\phi_1(L)\phi_2(L)-\phi_3(L)\phi_4(L)][(EI)^2\kappa^4(k_{Lt}+k_{Rt})-k_{Lt}k_{Rt}(k_L+k_R)] \\ &-EI\kappa[\phi_1(L)\phi_4(L)-\phi_2(L)\phi_3(L)][(EI)^2\kappa^4(k_L+k_R)-k_Lk_R(k_{Lt}+k_{Rt})] \\ &+(EI\kappa)^2[\phi_2^2(L)-\phi_4^2(L)][\kappa^4k_{Lt}k_{Rt}-k_Lk_R] \\ &-\phi_3^2(L)[(EI)^4\kappa^8-(EI)^2\kappa^4(k_{Lt}k_R+k_Lk_{Rt})+k_Lk_Rk_{Lt}k_{Rt}] \\ &+\phi_2(L)\phi_4(L)[(EI)^4\kappa^8+(EI)^2\kappa^4(k_Lk_{Lt}+k_Rk_{Rt})+k_Lk_Rk_{Lt}k_{Rt}], \end{aligned} \tag{39}$$

$$\begin{aligned} f_1(\xi) = &[k_R\phi_4(L-\xi)-EI\kappa^3\phi_1(L-\xi)][EI\kappa(k_{Lt}+k_{Rt})\phi_1(L)+k_{Lt}k_{Rt}\phi_2(L) \\ &+(EI\kappa)^2\phi_4(L)]-[EI\kappa\phi_2(L-\xi)+k_{Rt}\phi_3(L-\xi)][EI\kappa k_R\phi_2(L) \\ &-((EI)^2\kappa^4-k_{Lt}k_R)\phi_3(L)-EI\kappa^3k_{Lt}\phi_4(L)], \end{aligned} \tag{40}$$

$$\begin{aligned} f_2(\xi) = &[k_R\phi_4(L-\xi)-EI\kappa^3\phi_1(L-\xi)][EI\kappa k_L\phi_2(L)-((EI)^2\kappa^4-k_Lk_{Rt})\phi_3(L) \\ &-EI\kappa^3k_{Rt}\phi_4(L)]+[EI\kappa\phi_2(L-\xi)+k_{Rt}\phi_3(L-\xi)][EI\kappa^3(k_L+k_R)\phi_1(L) \\ &-(EI)^2\kappa^6\phi_2(L)-k_Lk_R\phi_4(L)]. \end{aligned} \tag{41}$$

Substituting Eqs. (35) to (38) with $C_1, f_1(\xi)$, and $f_2(\xi)$ defined in Eq. (39) to (40) into Eq. (14) yields the Green function of a beam with elastic boundary conditions:

$$\begin{aligned} X(x,\xi) = &\frac{\phi_4(x-\xi)u(x-\xi)}{\kappa^3(EI+\mathrm{i}r_i\Omega)} \\ &+\frac{1}{C_1}\left[f_1(\xi)\phi_1(\xi)+\frac{f_2(\xi)}{\kappa^2}\phi_2(x)+\frac{k_{Lt}f_2(\xi)}{EI\kappa^3}\phi_3(x)-\frac{k_Lf_1(\xi)}{EI\kappa^3}\phi_4(x)\right]. \end{aligned} \tag{42}$$

In order to obtain the Green function for a certain beam, we let the stiffness constants at the right-hand and/or the left-hand ends of the beam go to zero or infinity. Consider, for example, a beam fixed supported at the left-hand end and elastically supported in translational direction at the right-hand end. To obtain the Green function of this beam from the equations above, let k_L and k_{Lt} go to infinity and k_{Rt} go to zero. Doing so, one obtains for the unknown boundary conditions at $x=0$:

$$X(0)=X'(0)=0,$$

$$X''(0)=\frac{[k_R\phi_4(L)-EI\kappa^3\phi_1(L)]\phi_2(L-\xi)+\phi_2(L)[EI\kappa^3\phi_1(L-\xi)-k_R\phi_4(L-\xi)]}{EI\kappa[EI\kappa^3\phi_1^2(L)-k_R\phi_1(L)\phi_4(L)+\phi_2(L)(k_R\phi_3(L)-EI\kappa^3\phi_4(L))]}, \tag{43}$$

$$X'''(0)=\frac{[EI\kappa^3\phi_4(L)-k_R\phi_3(L)]\phi_2(L-\xi)-\phi_1(L)[EI\kappa^3\phi_1(L-\xi)-k_R\phi_4(L-\xi)]}{EI[EI\kappa^3\phi_1^2(L)-k_R\phi_1(L)\phi_4(L)+\phi_2(L)(k_R\phi_3(L)-EI\kappa^3\phi_4(L))]}. \tag{44}$$

Substituting these values into Eq. (14) yields the desired Green function.

4. Numerical examples and discussion

The following examples demonstrate the use of Green functions for determining the dynamic response of different beams

4.1. Statically indeterminate cantilevered beam

In this example a cantilevered beam with intermediate simple support as shown in Fig. 1 is considered. The dynamic response of this beam was studied by Gürgöze and Erol [13]. They used for the solution the receptance matrix method, which uses an approximate series solution and therefore yields an approximate solution. In order to prove the validity of this method, they used a solution through boundary value problem formulation. In doing so, they divided the beam into three portions and used a product solution for each portion. This leads to a solution with twelve unknown coefficients, which may be determined from the boundary and continuity conditions. Although this method gives an exact solution, it is circumstantial and time-consuming. The use of the Green function method as we see in the following, gives an exact solution and leads to the same results but in a simpler and faster way. By applying this method the support reaction at B was considered as an external force, which can be determined from the condition $v(a,t) = 0$ after the dynamic response of the beam has been found. Since the exciting load $((F\cos\Omega t))$ is harmonic and the studied beam behaves linearly, the reaction at B must be harmonic. Thus, the load may be given as

$$p(x,t) = [F\delta(x-b) - B\delta(x-a)]\cos\Omega t = f(x)\cos\Omega t. \tag{45}$$

The Green function of a cantilevered beam is obtained from Table 1 as

$$G(x,\xi) = \frac{1}{\kappa^3 EI}[\phi_4(x-\xi)u(x-\xi) + g_9(\xi)\phi_4(x) + g_{10}(\xi)\phi_3(x)] \tag{46}$$

with $g_9(\xi)$, $g_{10}(\xi)$ defined in the appendix and $\phi_3(x)$, $\phi_4(x)$ defined in Eq. (15). Substituting $f(x)$ as given in Eq. (45) and Eq. (46) into Eq. (11) yields

$$\begin{aligned} X(x) = {} & \frac{1}{\kappa_3 EI}\int_0^L \phi_4(x-\xi)[F\delta(\xi-b) - B\delta(\xi-a)]u(x-\xi)\,\mathrm{d}\xi \\ & + \frac{1}{\kappa_3 EI}\int_0^L [g_9(\xi)\phi_4(x) + g_{10}(\xi)\phi_3(x)][F\delta(\xi-b) - B\delta(\xi-a)]\,\mathrm{d}\xi \end{aligned} \tag{47}$$

The evaluation of the first integral in this equation gives three solutions depending upon the product of the step and the delta-functions. This product gives for all ξ over the interval $0 \leqslant \xi \leqslant L$

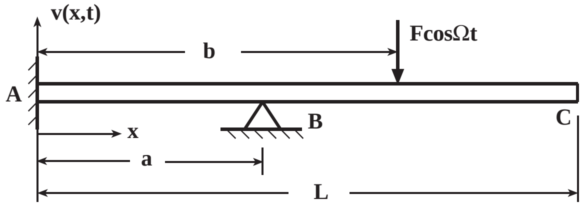

Fig. 1. Statically indeterminate cantilevered beam.

the value of 0 for $0 \leqslant x \leqslant a$, $-B\delta(\xi - a)$ for $a \leqslant x \leqslant b$, and $F\delta(\xi - b) - B\delta(\xi - a)$ for $b \leqslant x \leqslant L$. Thus, using the relation

$$\int_{\alpha}^{\beta} f(x)\delta(x-\xi)\,\mathrm{d}x = \begin{cases} 0 & \text{for} \quad \xi < \alpha < \beta \\ f(\xi) & \text{for} \quad \alpha \leqslant \xi \leqslant \beta \\ 0 & \text{for} \quad \alpha < \beta < \xi \end{cases} \tag{48}$$

One obtains for the first integral in Eq. (47)

$$I_1(x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant x < a \\ -\dfrac{B}{\kappa^3 EI}\phi_4(x-a) & \text{for} \quad a \leqslant x < b \\ \dfrac{1}{\kappa^3 EI}[F\phi_4(x-b) - B\phi_4(x-a)] & \text{for} \quad b \leqslant x \leqslant L \end{cases} \tag{49}$$

The second integral in Eq. (47) can be readily evaluated by using Eq. (48)

$$I_2(x) = \frac{1}{\kappa^3 EI}\{F[g_9(b)\phi_4(x) + g_{10}(b)\phi_3(x)] - B[g_9(a)\phi_4(x) + g_{10}(a)\phi_3(x)]\}. \tag{50}$$

Combining the obtained results in Eqs. (49) and (50) gives for the dynamic response of the beam

$$v(x,t) = \cos\Omega t \begin{cases} I_2(x) & \text{for} \quad 0 \leqslant x < a \\ I_2(x) - \dfrac{B}{\kappa^3 EI}\phi_4(x-a) & \text{for} \quad a \leqslant x < b \\ I_2(x) + \dfrac{1}{\kappa^3 \mathrm{EI}}[F\phi_4(x-b) - B\phi_4(x-a)] & \text{for} \quad b \leqslant x \leqslant L \end{cases} \tag{51}$$

The unknown support reaction B may be now obtained from the condition $v(a,t) = 0$. This gives for B the value

$$B = \frac{[g_9(b)\phi_4(a) + g_{10}(b)\phi_3(a)]}{[g_9(a)\phi_4(a) + g_{10}(a)\phi_3(a)]}F. \tag{52}$$

In order to compare the results obtained from Eq. (51) with those in Ref. [13], the non-dimensionalized displacement ($\bar{v}(x,t) = v(x,t)/(FL^3/EI)$) of the beam at three positions ($x = 0.5L,\ x = 0.8L,\ x = L$) are plotted in Fig. 2 for values given in [13]; namely: $\Omega = 5\sqrt{EI/\mu L^4}$, $b = L$, and $a = 0.1L$. The amplitudes are found to be $\bar{v}_{max}(x = 0.5L) = 0.1618507$, $\bar{v}_{max}(x = 0.8L) = 0.3790165$, and $\bar{v}_{max}(x = L) = 0.5304795$. Comparison of these values with those given in Ref. [13, Table 1] yields excellent agreement.

4.2. Comparison of Green functions solution with series solution

In order to compare the method of eigenfunction expansion with the method of Green functions, an undamped cantilevered beam acted upon by a force $P_0 \cos\Omega t$ at the right-hand side is considered. For that, the dynamic deflection of the beam at $x = L$ and the bending moment and shear force at $x = 0$, will be calculated using both methods and compared with each others. A cantilevered beam is chosen to make the difference between both methods clearer, because the series solution of beams others than simply supported converges slower [3]. Using the

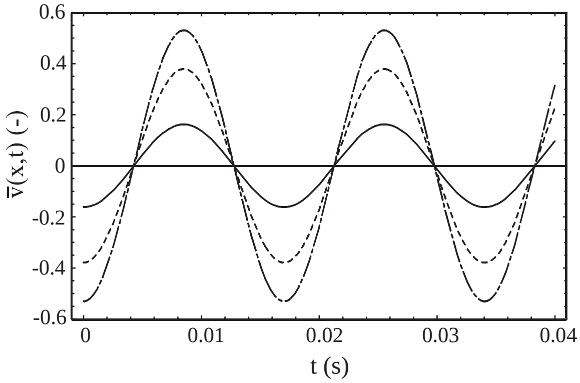

Fig. 2. Dimensionless dynamic deflection versus time of the beam shown in Fig. 1 at three positions: (——) $x = 0.5L$, (- - - -) $x = 0.8L$, (— · —) $x = L$.

eigenfunction expansion, the deflection of the beam is given in series solution as

$$v(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) \cos \Omega t, \tag{53}$$

where $X_n(x)$ is the nth eigenfunction of the cantilevered beam given as

$$X_n(x) = \sin \lambda_n x - \sinh \lambda_n x - \frac{\sin \lambda_n L + \sinh \lambda_n L}{\cos \lambda_n L + \cosh \lambda_n L}(\cos \lambda_n x - \cosh \lambda_n x) \tag{54}$$

and

$$c_n = \frac{P_0 X_n(L)}{m_n(\omega_n^2 - \Omega^2)} \tag{55}$$

is the nth expansion coefficient where

$$m_n = \mu \int_0^L X_n^2(x)\,\mathrm{d}x \tag{56}$$

is the generalized mass and ω_n is the circular eigenfrequency of the beam associated with the nth mode. The used eigenvalues λ_n are correct up to 12 decimal places.

On the other hand, the dynamic deflection of the cantilevered beam by means of Green functions is given as

$$v(x,t) = \cos \Omega t \int_0^L f(\xi) G(x,\xi)\,\mathrm{d}\xi, \tag{57}$$

where $f(\xi) = P_0 \delta(\xi - L)$ and $G(x,\xi)$ is obtained from Table 1 as

$$G(x,\xi) = \frac{\phi_4(x-\xi)}{\kappa^3 EI} u(x-\xi) + \frac{1}{\kappa^3 EI}[g_9(\xi)\phi_4(x) + g_{10}(\xi)\phi_3(x)] \tag{58}$$

with $g_9(\xi), g_{10}(\xi)$ defined in the Appendix and $\phi_3(x)$, $\phi_4(x)$ defined in Eq. (15). Substituting $f(\xi)$ and $G(x,\xi)$ into Eq. (57) and carrying out the integration yields

$$v(x,t) = \frac{P_0}{\kappa^3 EI}[g_9(L)\phi_4(x) + g_{10}(L)\phi_3(x)] \cos \Omega t. \tag{59}$$

Table 2
Non-dimensionalized amplitudes of dynamic deflection $\bar{v}_{max}$ at $x = L$ and non-dimensionalized amplitudes of bending moment $\bar{M}_{max}$ and shear force $\bar{V}_{max}$ at $x = 0$ for a cantilevered beam

	$\bar{v}_{max}$	$\bar{M}_{max}$	$\bar{V}_{max}$
Green functions solution (exact)	6.6885	7.6624	10.1581
Series solution: $n = 1$	6.6586	7.8039	10.7422
$n = 2$	6.6839	7.6184	9.8550
$n = 3$	6.6871	7.6834	10.3653
$n = 4$	6.6879	7.6503	10.0012
$n = 5$	6.6882	7.6703	10.2843
$n = 10$	6.6885	7.6604	10.0946
$n = 20$	6.6885	7.6619	10.1253
$n = 30$	6.6885	7.6622	10.1359
$n = 40$	6.6885	7.6623	10.1412
$n = 50$	6.6885	7.6623	10.1444

The bending moment $M(x,t)$ and shear force $V(x,t)$ are obtained from Eqs. (53) and (59) through differentiation with respect to x as

$$M(x,t) = EIv''(x,t), \tag{60}$$

$$V(x,t) = EIv'''(x,t). \tag{61}$$

To make numerical calculations, the following data are used: $L = 2\,\text{m}$, $\mu = 117.45\,\text{kg/m}$, $EI = 640625\ \text{Nm}^2$, and $\Omega = 60\,\text{rad/s}$.

In Table 2, the non-dimensionalized amplitudes of dynamic deflection ($\bar{v}_{max}$), bending moment ($\bar{M}_{max}$) and shear force ($\bar{V}_{max}$) are given, where

$$\bar{v}_{max} = \frac{v_{max}(L)}{(P_0L^3/3EI)}, \quad \bar{M}_{max} = \frac{M_{max}(0)}{P_0L}, \quad \bar{V}_{max} = \frac{V_{max}(0)}{P_0}. \tag{62}$$

In the first row of Table 2, the exact values are given. These are obtained from the Green functions solution. In the remaining rows partial sums for several terms of the series (53) and their derivatives multiplied by EI are set out. As the table indicates, the convergence of the series solution is comparatively very fast for the dynamic deflection, but much less so for the bending moment, and particularly for the shear force. That is because each differentiation of the dynamic deflection with respect to x impairs the convergence of the series [3].

4.3. Cantilevered beam with elastic support

As a third example, a damped cantilevered beam with elastic support at the right-hand end as shown in Fig. 3 is presented. The Green function of this beam may be given as

$$G(x,t) = C\phi_4(x-\xi)u(x-\xi) + h_1(\xi)\phi_3(x) + h_2(\xi)\phi_4(x), \tag{63}$$

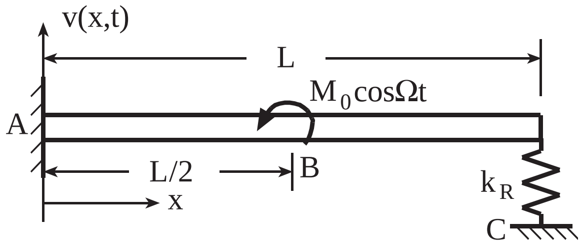

Fig. 3. Fixed elastic-supported beam.

where C is given in the appendix and

$$h_1 = \frac{X''(0)}{\kappa^2}, \quad h_2 = \frac{X'''(0)}{\kappa^3} \tag{64}$$

with $X''(0)$ and $X'''(0)$ defined in Eqs. (43) and (44), respectively. The acting load is given as

$$p(x,t) = \delta'(x - L/2)M_0 \cos \Omega t. \tag{65}$$

According to Eqs. (8) and (11), the dynamic response of the damped beam may be given in the complex form

$$w(x,t) = M_0 \mathrm{e}^{\mathrm{i}\Omega t} \int_0^L \delta'(\xi - L/2)G(x,\xi)\,\mathrm{d}\xi, \tag{66}$$

where the real-valued dynamic deflection corresponding to Eq. (7) is given as $v(x,t) = Re\{w(x,t)\}$. Making use of the relation

$$\int_a^c \delta^{(n)}(\xi - b)f(\xi)\,\mathrm{d}\xi = (-1)^n f^{(n)}(b) \quad a \leqslant b \leqslant c \tag{67}$$

gives for the complex dynamic deflection

$$w(x,t) = \frac{M_0 \mathrm{e}^{\mathrm{i}\Omega t}}{\kappa^2 EI}[\phi_3(x - L/2)u(x - L/2) + r_1(L/2)\phi_3(x) + r_2(L/2)\phi_4(x)], \tag{68}$$

where

$$r_1(\xi) = \frac{[k_R\phi_4(L) - EI\kappa^3\phi_1(L)]\phi_1(L-\xi) + \phi_2(L)[EI\kappa^3\phi_4(L-\xi) - k_R\phi_3(L-\xi)]}{EI\kappa^3\phi_1^2(L) - k_R\phi_1(L)\phi_4(L) + \phi_2(L)[k_R\phi_3(L) - EI\kappa^3\phi_4(L)]}, \tag{69}$$

$$r_2(\xi) = \frac{[EI\kappa^3\phi_4(L) - k_R\phi_3(L)]\phi_1(L-\xi) - \phi_1(L)[EI\kappa^3\phi_4(L-\xi) - k_R\phi_3(L-\xi)]}{EI\kappa^3\phi_1^2(L) - k_R\phi_1(L)\phi_4(L) + \phi_2(L)[k_R\phi_3(L) - EI\kappa^3\phi_4(L)]}. \tag{70}$$

In order to make numerical calculations, the following data for the beam are used: beam length $L = 2.5\,\mathrm{m}$, width of the beam's cross-section $b = 300\,\mathrm{mm}$, height of the cross-section $h = 50\,\mathrm{mm}$, density of the used material $\rho = 7830\,\mathrm{kg/m^3}$, and modulus of elasticity $E = 205\ (10^9)\,\mathrm{N/m^2}$. Using these values yields for the flexural rigidity $EI = 640625\,\mathrm{Nm^2}$ and for the mass per unit length $\mu = 117.45\,\mathrm{kg/m}$. In Table 3, the first five circular natural frequencies for different values of

Table 3
The first five natural frequencies of the beam shown in Fig. 3 in rad/s for different values of dimensionless stiffness $\bar{k} = k_R/(72EI/L^3)$

$\bar{k}$	ω_1	ω_2	ω_3	ω_4	ω_5
0	41.548	260.375	729.056	1428.659	2361.676
0.1	73.925	268.338	731.842	1430.072	2362.529
0.25	100.444	280.988	736.124	1432.214	2363.816
0.5	125.176	302.865	743.528	1435.844	2365.978
1	148.083	344.552	759.266	1443.322	2370.369
2	163.994	409.008	793.626	1459.150	2379.424
5	174.792	500.846	899.350	1512.858	2408.760
∞	182.192	590.419	1231.862	2106.556	3214.503

stiffness k_R are set out. These frequencies are obtained by using the frequency equation

$$\lambda^3 = \frac{k_R L^3}{EI} \frac{(\cos\lambda \sinh\lambda - \sin\lambda \cosh\lambda)}{(1 + \cos\lambda \cosh\lambda)}, \tag{71}$$

where

$$\omega_n = (\lambda_n/L)^2 \sqrt{EI/\mu}. \tag{72}$$

The frequency Eq. (71) is obtained by setting $\kappa = \lambda/L$ in the denominator of Eq. (69) or Eq. (70) and then equating it to zero.

The first and last rows in the table give the natural frequencies of a cantilevered beam ($k_R = 0$) and a fixed-pinned beam ($k_R = \infty$), respectively. The table shows that increasing the stiffness k_R leads to an increase in the natural frequencies. Furthermore, the effect of this stiffness on the lower frequencies is greater than on the higher frequencies.

Fig. 4 shows the dimensionless dynamic deflection $\bar{v} = v/v_{max}$ at $x = x_{max}$ for different values of the dimensionless stiffness $k = k_R/k_0$ and different excitation frequencies Ω, where v_{max}, x_{max}, and k_0 are defined as

$$v_{max} = \frac{M_0 L^2}{72EI}, \quad x_{max} = 2L/3, \quad k_0 = 72EI/L^3. \tag{73}$$

The quantities v_{max} and x_{max} are the maximum static deflection of a fixed-pinned beam ($k_R = \infty$) due to a static moment load acting at $x = L/2$ and the position at which v_{max} occurs, respectively. The figure shows clearly the effect of the stiffness k_R on the dynamic deflection of the beam. Varying the value of this stiffness leads to increasing or decreasing the amplitudes of the dynamic deflection of the beam. This is dependent upon, weather the natural frequencies of the beam thereby approaches the excitation frequency Ω or they will be removed from it.

Fig. 5 shows the dimensionless dynamic response of the beam at $x = x_{\max}$ due to the acting moment for the excitation frequency $\Omega = 160$ rad/s, the stiffness ratio $\bar{k} = 1$, and different values of damping ζ, where only the external damping r_a, is considered ($r_i = 0$). For that, a damping ratio

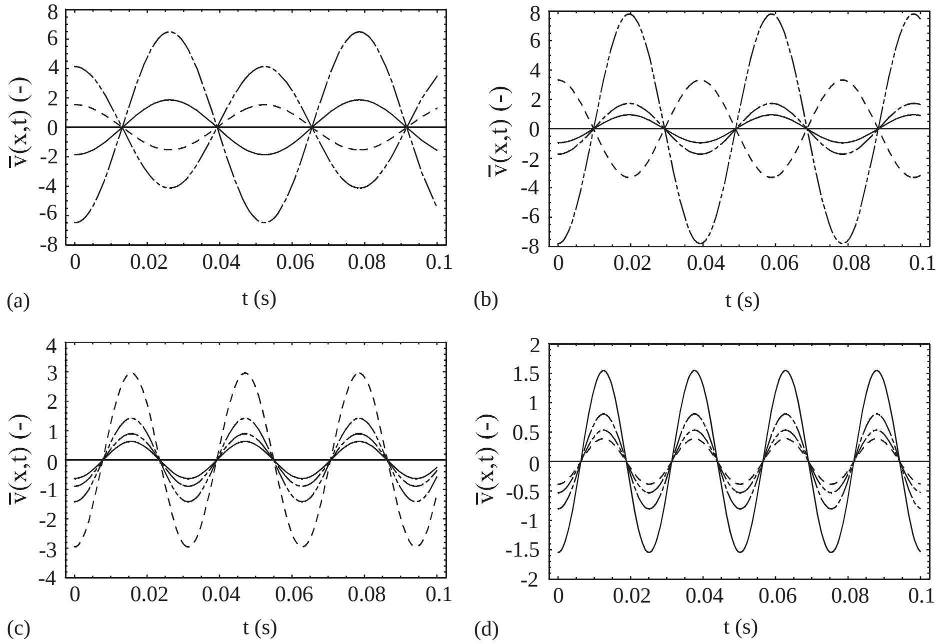

Fig. 4. Dimensionless dynamic deflection versus time of the beam shown in Fig. 3 at $x = 2L/3$ for different frequencies and different dimensionless stiffnesses: (a) $\Omega = 120$ rad/s, (b) $\Omega = 160$ rad/s, (c) $\Omega = 200$ rad/s, (d) $\Omega = 250$ rad/s; (——) $\bar{k} = 0$, (— · —) $\bar{k} = 0.25$, (— · · —) $\bar{k} = 1$, (- - - -) $\bar{k} = \infty$.

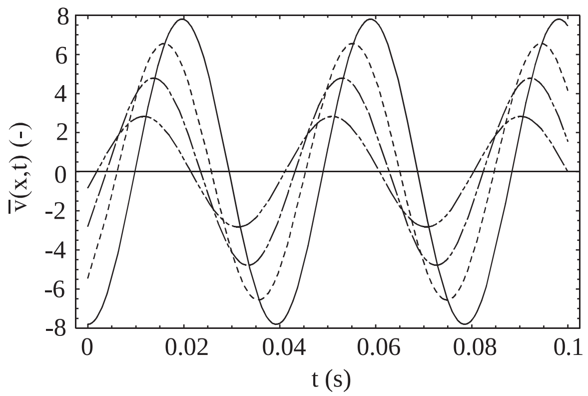

Fig. 5. Dimensionless dynamic deflection versus time of the beam shown in Fig. 3 at $x = 2L/3$ for different values of damping: (——) $\zeta = 0$, (- - - -) $\zeta = 0.05$, (— · —) $\zeta = 0.1$, (— · · —) $\zeta = 0.2$.

is defined as

$$\zeta = \frac{r_a}{2\mu\omega_1}, \tag{74}$$

where ω_1 is the first natural frequency of the considered beam and is equal to 148.083 rad/s in the studied case. The figure shows that the damping ratios $\zeta = 0.05$, 0.1 and 0.2 reduces the amplitudes of the deflection down to 84%, 61% and 37% of the amplitude without damping, respectively.

5. Conclusions

A method for determining the dynamic response of damped Euler–Bernoulli beams is presented. This method is based on the use of Green functions and yields exact solutions. It can be used to study the dynamic behavior of single and multi-span beams, single and multi-loaded beams, and statically determinate and indeterminate beams. Also Green functions for different beams are given. To verify the analysis performed, three numerical examples are presented and discussed.

Appendix A

$$C = \frac{1}{\kappa^3(EI + \mathrm{i}r_i\Omega)},$$

$$g_1(\xi) = \frac{\phi_4(L)\phi_4(L-\xi) - \phi_2(L)\phi_2(L-\xi)}{\phi_2^2(L) - \phi_4^2(L)}, \quad g_2(\xi) = \frac{\phi_4(L)\phi_2(L-\xi) - \phi_2(L)\phi_4(L-\xi)}{\phi_2^2(L) - \phi_4^2(L)},$$

$$g_3(\xi) = \frac{\phi_2(L)\phi_4(L-\xi) - \phi_3(L)\phi_3(L-\xi)}{\phi_3^2(L) - \phi_2(L)\phi_4(L)}, \quad g_4(\xi) = \frac{\phi_4(L)\phi_3(L-\xi) - \phi_3(L)\phi_4(L-\xi)}{\phi_3^2(L) - \phi_2(L)\phi_4(L)},$$

$$g_5(\xi) = \frac{\phi_1(L)\phi_4(L-\xi) - \phi_3(L)\phi_2(L-\xi)}{\phi_2(L)\phi_3(L) - \phi_1(L)\phi_4(L)}, \quad g_6(\xi) = \frac{\phi_4(L)\phi_2(L-\xi) - \phi_2(L)\phi_4(L-\xi)}{\phi_2(L)\phi_3(L) - \phi_1(L)\phi_4(L)},$$

$$g_7(\xi) = \frac{\phi_1(L)\phi_4(L-\xi) - \phi_2(L)\phi_3(L-\xi)}{\phi_2(L)\phi_3(L) - \phi_1(L)\phi_4(L)}, \quad g_8(\xi) = \frac{\phi_4(L)\phi_3(L-\xi) - \phi_3(L)\phi_4(L-\xi)}{\phi_2(L)\phi_3(L) - \phi_1(L)\phi_4(L)},$$

$$g_9(\xi) = \frac{\phi_4(L)\phi_2(L-\xi) - \phi_1(L)\phi_1(L-\xi)}{\phi_1^2(L) - \phi_2(L)\phi_4(L)}, \quad g_{10}(\xi) = \frac{\phi_2(L)\phi_1(L-\xi) - \phi_1(L)\phi_2(L-\xi)}{\phi_1^2(L) - \phi_2(L)\phi_4(L)},$$

$$g_{11}(\xi) = \frac{\phi_4(L)\phi_4(L-\xi) - \phi_1(L)\phi_3(L-\xi)}{\phi_1^2(L) - \phi_2(L)\phi_4(L)}, \quad g_{12}(\xi) = \frac{\phi_2(L)\phi_3(L-\xi) - \phi_1(L)\phi_4(L-\xi)}{\phi_1^2(L) - \phi_2(L)\phi_4(L)},$$

$$g_{13}(\xi) = \frac{\phi_3(L)\phi_3(L-\xi) - \phi_1(L)\phi_1(L-\xi)}{\phi_1^2(L) - \phi_3^2(L)}, \quad g_{14}(\xi) = \frac{\phi_3(L)\phi_1(L-\xi) - \phi_1(L)\phi_3(L-\xi)}{\phi_1^2(L) - \phi_3^2(L)},$$

$$g_{15}(\xi) = \frac{\phi_4(L)\phi_3(L-\xi) - \phi_2(L)\phi_1(L-\xi)}{\phi_1(L)\phi_2(L) - \phi_3(L)\phi_4(L)}, \quad g_{16}(\xi) = \frac{\phi_3(L)\phi_1(L-\xi) - \phi_1(L)\phi_3(L-\xi)}{\phi_1(L)\phi_2(L) - \phi_3(L)\phi_4(L)},$$

$$g_{17}(\xi) = \frac{\phi_3(L)\phi_1(L-\xi) - \phi_2(L)\phi_2(L-\xi)}{\phi_1(L)\phi_2(L) - \phi_3(L)\phi_4(L)}, \quad g_{18}(\xi) = \frac{\phi_4(L)\phi_2(L-\xi) - \phi_1(L)\phi_1(L-\xi)}{\phi_1(L)\phi_2(L) - \phi_3(L)\phi_4(L)},$$

$$g_{19}(\xi) = \frac{\phi_4(L)\phi_1(L-\xi) - \phi_3(L)\phi_2(L-\xi)}{\phi_2(L)\phi_3(L) - \phi_1(L)\phi_4(L)}, \quad g_{20}(\xi) = \frac{\phi_1(L)\phi_2(L-\xi) - \phi_2(L)\phi_1(L-\xi)}{\phi_2(L)\phi_3(L) - \phi_1(L)\phi_4(L)},$$

$$g_{21}(\xi) = \frac{\phi_4(L)\phi_1(L-\xi) - \phi_2(L)\phi_3(L-\xi)}{\phi_2^2(L) - \phi_4^2(L)}, \quad g_{22}(\xi) = \frac{\phi_4(L)\phi_3(L-\xi) - \phi_2(L)\phi_1(L-\xi)}{\phi_2^2(L) - \phi_4^2(L)},$$

$$g_{23}(\xi) = \frac{\phi_2(L)\phi_2(L-\xi) - \phi_3(L)\phi_1(L-\xi)}{\phi_3^2(L) - \phi_2(L)\phi_4(L)}, \quad g_{24}(\xi) = \frac{\phi_4(L)\phi_1(L-\xi) - \phi_3(L)\phi_2(L-\xi)}{\phi_3^2(L) - \phi_2(L)\phi_4(L)}.$$

References

[1] W. Weaver Jr., et al., Vibration Problems in Engineering, 5th Edition, Wiley, New York, 1990.
[2] W.T. Thomson, Theory of Vibration with Applications, 4th Edition, Prentice Hall, Englewood Cliffs, NJ, 1993.
[3] L. Frýba, Vibration of Solids and Structures Under Moving Loads, Noordhoff International, Groningen, 1972.
[4] A.W. Leissa, Closed form exact solutions for the steady state vibrations of continuous systems subjected to distributed exciting forces, Journal of Sound and Vibration 134 (1989) 435–453.
[5] R. Hamada, Dynamic analysis of a beam under a moving force: a double Laplace transform solution, Journal of Sound and Vibration 74 (1981) 221–233.
[6] J.W. Nicholson, L.A. Bergman, Free vibration of combined dynamical systems, American Society of Civil Engineers Journal of Engineering Mechanics 112 (1986) 1–13.
[7] L.A. Bergman, J.W. Nicholson, Forced vibration of a damped combined linear system, American Society of Mechanical Engineers Journal of Vibration, Acoustics, Stress, and Reliability in Design 107 (1985) 275–281.
[8] T.H. Broome Jr., Economical analysis of combined dynamical systems, American Society of Civil Engineers Journal of Engineering Mechanics 115 (1989) 2122–2135.
[9] L.A. Bergman, D.M. McFarland, On the vibration of a point-supported linear distributed system, American Society of Mechanical Engineers Journal of Vibration, Acoustics, Stress, and Reliability in Design 110 (1988) 485–592.
[10] S. Kukla, B. Posiadala, Free vibration of beams with elastically mounted masses, Journal of Sound and Vibration 175 (1994) 557–564.
[11] S. Kukla, Application of Green functions in frequency analysis of Timoshenko beams with oscillators, Journal of Sound and Vibration 205 (1997) 355–363.
[12] M.A. Foda, Z. Abduljabbar, A dynamic Green function formulation for the response of a beam structure to a moving mass, Journal of Sound and Vibration 210 (1998) 295–306.
[13] M. Gürgöze, H. Erol, Determination of the frequency response function of a cantilevered beam simply supported in-span, Journal of Sound and Vibration 247 (2001) 372–378.