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Compensation for discarded singular values in vibro-acoustic inverse methods

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Abstract

The paper addresses the inverse problem where source strengths are back-calculated from a sound pressure field sampled at several points. Regularization techniques, such as singular value discarding or Tikhonov regularization, are commonly used to improve estimates of source strength in such situations. However, over-regularization can result in even worse errors. A simple procedure is proposed here to compensate for errors of over-regularization. The basis is to constrain the solution such that the spatial mean of the measured and reconstructed sound pressure are equal. In other words, to set the overall sound power of the equivalent (calculated) sources equal to that of the real source. It is argued that the overall sound power is the most stable and reliable quantity on which to base source strength estimates. Examples of both singular value discarding and Tikhonov regularization are given.

1. Introduction

There has recently been considerable interest in acoustic inverse problems [1–3] in which unknown acoustic source strengths are calculated from an acoustic field sampled at several points. The analogous problem also occurs in inverse vibration problems where unknown forces are inferred from a measured velocity field [4–6].

Such inverse solutions usually require some form of regularization to prevent amplification of measurement errors due to ill conditioning. Two common methods are singular value discarding and Tikhonov regularization. In the former, 'unreliable' singular values are discarded, and in the latter singular values are weighted according to their reliability using a regularization parameter β. In both cases, regularization can improve the accuracy of the calculated source strengths, but too

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much regularization can result in even worse errors [1,3].Therefore, the choice of regularization parameter is critical.

This paper is based on the observation that with both forms of regularization the spatial average of the reconstructed and measured sound pressure are never equal. (For convenience, the spatial average sound pressure will henceforth be referred to simply as the 'mean' pressure.) It is argued that the mean sound pressure is the most stable measured quantity relating to a sound source. For example, depending on the positioning of the measurement points, the sound power of a source is proportional to the mean sound pressure, and nearly all methods of determining sound power rely on this fact. It would therefore seem sensible to constrain the regularized solution such that the reconstructed and measured mean pressure are identical.

In this paper, a simple procedure is described to compensate for the 'missing' or 'de-emphasized' singular values. This yields a robust solution to the source strengths, even when there is over-regularization that would normally result in significant errors.

2. Singular value discarding

An example of the inverse problem to be solved is shown in Fig. 1: the sound field, sampled at NR positions around the source, is assumed to be the result of NS monopoles. The reconstructed sound pressure $\hat{\mathbf{p}}$ is obtained from

$$\hat{\mathbf{p}} = \mathbf{Hq}, \tag{1}$$

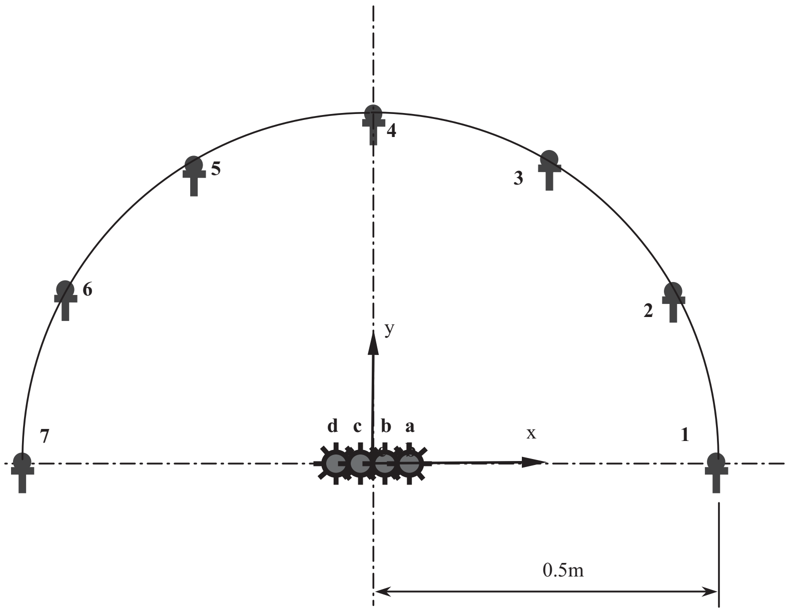

Fig. 1. Motor showing positions of monopoles forming equivalent source, and microphone positions for free field measurements.

where $\mathbf{q}$ is the as yet unknown vector of source strengths, and $\mathbf{H}$ is the $(N_R \times N_S)$ matrix of transfer functions. The vector $\mathbf{q}$ effectively defines an equivalent source system, in that $\mathbf{q}$ is solved such that the sound field it generates is as close as possible to that of the real source.

The usual method of solution involves minimizing the squared error between the reconstructed pressure vector $\hat{\mathbf{p}}$ and the measured pressure vector $\mathbf{p}$. Eq. (1) is written in terms of the singular value decomposition of the transfer function matrix $\mathbf{H}$, i.e.,

$$\hat{\mathbf{p}} = \mathbf{Hq} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathrm{V}^{\mathrm{H}}\mathbf{q}, \tag{2}$$

in which the columns of $\mathbf{U}$ and $\mathbf{V}$ are the output and input singular vectors, respectively, and the superscript H indicates Hermitian transpose (conjugate transpose). The matrix $\mathbf{\Sigma}$ is diagonal, containing the singular values of $\mathbf{H}$. Pre-multiplying by $\mathbf{U}^{\mathrm{H}}$ and using the unitary property ($\mathbf{U}^{\mathrm{H}}\mathbf{U} = \mathbf{I}$) the equation is recast into a system of uncoupled input–output equations:

$$\hat{\mathbf{r}} = \mathbf{\Sigma s}, \tag{3}$$

where $\hat{\mathbf{r}} = \mathbf{U}^{\mathrm{H}}\hat{\mathbf{p}}$ is a $(N_R \times 1)$ column vector quantifying the participation of each of the output singular modes in the overall reconstructed sound field. $\mathbf{s} = \mathbf{V}^{\mathrm{H}}\mathbf{q}$ is a $(N_S \times 1)$ vector, as yet unkonwn, which quantifies the participation of the input modes to the excitation. The transformed source strength vector $\mathbf{s}$ is obtained by substituting in the measured pressure vector and inverting Eq. (3):

$$\mathbf{s} = \mathbf{\Sigma}^{+}\mathbf{r}, \tag{4}$$

where $\mathbf{r} = \mathbf{U}^{\mathrm{H}}\mathbf{p}$ is now obtained from the measured pressure, and $\mathbf{\Sigma}^{+}$ is the pseudo-inverse of $\mathbf{\Sigma}$. Eq. (4) can also be expressed term by term as

$$\mathbf{s}^{\mathrm{T}} = \left[\frac{1}{\sigma_1} r_1 \quad \frac{1}{\sigma_2} r_2 \quad \cdots \quad \frac{1}{\sigma_{N_S}} r_{N_S} \right], \tag{5}$$

in which r_i is the ith element of $\mathbf{r}$, and σ_i the corresponding singular value. Large errors may occur in the solution when some of the σ_i are small because the terms $1/\sigma_i$ become large and may amplify errors in the corresponding r_i.

In singular value discarding such errors are avoided by discarding unreliable terms (those with smallest singular values) to give a truncated vector $\mathbf{s_n}$ given by

$$\mathbf{s_n^T} = \left[\frac{1}{\sigma_1} r_1 \quad \frac{1}{\sigma_2} r_2 \quad \cdots \quad \frac{1}{\sigma_{n_S}} r_{n_S} \right] n_S < N_S. \tag{6}$$

The reconstructed sound pressure at the measurement points $\hat{\mathbf{p}}$ is obtained by substituting a solution $\mathbf{s}$ or $\mathbf{s_n}$ into Eq. (2):

$$\hat{\mathbf{p}} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{s}, \tag{7a}$$

$$\hat{\mathbf{p}}_\mathbf{n} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{s_n}, \tag{7b}$$

where $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}_{\mathbf{n}}$ are the reconstructed sound fields obtained from the unregularized and regularized solution, respectively.

At this point consider the mean reconstructed pressure which, for the unregularized solution, is obtained by pre-multiplying Eq. 7(a) by its own conjugate transpose:

$$\sum_{i=1}^{N_R} |\hat{p}_i|^2 = \hat{\mathbf{p}}^{\mathrm{H}} \cdot \hat{\mathbf{p}} = \sum_{j=1}^{N_S} \sigma_j^2 |s_j|^2 = \sum_{j=1}^{N_S} \sigma_j^2 \left(\frac{1}{\sigma_j^2}\right) |r_j|^2 = \sum_{j=1}^{N_S} |r_j|^2, \tag{8}$$

where r_j are the elements of $\mathbf{r}$ from Eq. (6), and s_j the corresponding elements of $\mathbf{s}$. The term $\sum_{j=1}^{N_S} |r_j|^2$ is the contribution of the first N_S output modes to the pressure. For a determined system ($N_R = N_S$), $\sum_{j=1}^{N_S} |r_j|^2 = \sum_{k=1}^{N_R} |p_k|^2$, i.e., the mean measured and reconstructed pressures are equal. For an overdetermined system($N_R > N_S$), $\sum_{j=1}^{N_S} |r_j|^2 \leqslant \sum_{k=1}^{N_R} |p_k|^2$, i.e., the mean reconstructed pressure is less than or equal to the mean measured pressure, although in practice the difference is usually small. (Under-determined systems ($N_R < N_S$) are not commonly used and will not be considered.)

In a similar way, the mean reconstructed pressure from the regularized solution at the same points is given by

$$\sum_{i=1}^{N_R} |\hat{p}_i|^2 = \hat{\mathbf{p}}^{\mathrm{H}} \cdot \hat{\mathbf{p}} = \sum_{j=1}^{n_S} |r_j|^2, \quad n_S < N_S \leq N_R. \tag{9}$$

Comparing Eqs. (8) and (9) it is seen that when singular values are discarded as part of the regularization, the reconstructed mean pressure is expressed as a truncated series. Since all terms in the series are positive, the regularized solution always underestimates the measured mean pressure, and increasingly so the more terms are discarded.

At this point it should be appreciated that in most laboratory measurements it is not background noise that causes error in the measured pressure spectrum. Rather, it is random errors, particularly phase error due to imprecise positioning of microphones and their inherent phase mismatches. Such errors do not introduce additional energy into the system, so the measured mean pressure is energetically correct. However, the random errors manifest as an incorrect distribution of this energy amongst the output vectors, in particular the contributions of higher order output modes tend to be exaggerated. Thus, the measurements can be said to contain errors in 'shape' rather than in 'level'.

This being the case, the underestimate in the mean sound pressure of the regularized solution is equivalent to saying that the equivalent source always has a lower sound power than the real source when singular value discarding is employed. This is not ideal because the sound power is the most stable and reliable quantity relating to the source. We would like to ensure that whatever the differences between the real and equivalent sources, at least their sound power is equal.

This can be achieved by defining a correction factor c to make the right side of Eq. (9) equal to the measured mean pressure. Thus,

$$c^2 \sum_{j=1}^{n_s} |r_j|^2 = \sum_{i=1}^{N_R} |p_i|^2 \tag{10}$$

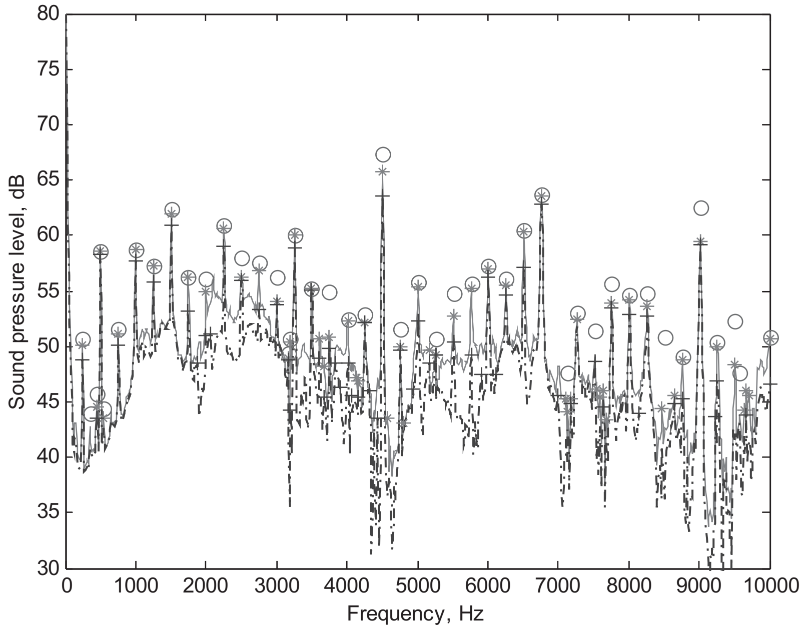

Fig. 2. Mean reconstructed sound pressure: ..... unregularized; —— one; –·–·– two singular values discarded.

or

$$c = \left[\sum_{i=1}^{N_R} |p_i|^2 \Big/ \sum_{j=1}^{n_s} |r_j|^2\right]^{1/2}. \tag{11}$$

Thus, by multiplying the remaining singular values by the factor c prior to evaluating Eq. (6), the sound power of the equivalent and real sources will always be equal for any degree of truncation. This compensation procedure is the main novel feature of this paper.

An example of this procedure now follows, using the set-up shown in Fig. 1. The equivalent source consists of four monopoles whose complex volume velocity is to be determined from seven microphone readings. The measurements are conducted in free field so **H** is calculated from the freefield Green function. Fig. 2 shows the mean measured pressure and that reconstructed with 1 and 2 singular values discarded. This illustrates Eq. (9) in that the more singular values are discarded, the more the mean pressure is underestimated: the discarding of certain modes has resulted in a loss of energy to the system. The compensated solution (not shown), however, coincides exactly with the measured mean.

3. Application to Tikhonov regularization

A similar analysis to the above can be carried out for Tikhonov regularization. Here rather than discarding singular values they are de-emphasized by introducing the parameter β. The solution corresponding to Eq. (6) is then given by

$$\mathbf{s}_t^{\mathrm{T}} = \left[\left(\frac{\sigma_1}{\sigma_1^2+\beta}\right) r_1 \quad \left(\frac{\sigma_2}{\sigma_2^2+\beta}\right) r_2 \quad \cdots \quad \left(\frac{\sigma_{N_S}}{\sigma_{N_S}^2+\beta}\right) r_{N_S} \right], \tag{12}$$

where the subscript t indicates that the solution is Tikhonov regularized. Following the same steps that led to Eq. (8), the mean reconstructed pressure is given by

$$\sum_{i=1}^{N_R} |\hat{p}_i|^2 = \hat{\mathbf{p}}^{\mathrm{H}} \cdot \hat{\mathbf{p}} = \sum_{j=1}^{N_S} \sigma_j^2 |s_j|^2 = \sum_{j=1}^{N_S} \left(\frac{\sigma_j^2}{\sigma_j^2+\beta}\right)^2 |r_j|^2. \tag{13}$$

Thus, $\sum_{i=1}^{N_R} |\hat{p}_i|^2 \to \sum_{j=1}^{N_S} |r_j|^2$ as $\beta \to 0$ giving the same result as in Eq. (8) for the unregularized case. However, $\sum_{i=1}^{N_R} |\hat{p}_i|^2 \to 0$ as $\beta \to \infty$, indicating that a large underestimate of mean pressure could result from choosing too high a value for β.

A similar compensation procedure can be employed, for which the correction factor corresponding to that in Eq. (11) is

$$c_t = \left[\sum_{i=1}^{N_R} |p_i|^2 \Big/ \sum_{j=1}^{N_S} \left(\frac{\sigma_j^2}{\sigma_j^2+\beta}\right)^2 |r_j|^2 \right]^{1/2}. \tag{14}$$

By way of example, Fig 3a compares the reconstructed sound pressure around the source with the measured pressure for varying values of β/σ^2. The case $\beta = 0$ is the unregularizsed solution and gives the most detailed spatial patterns. The more regularization is employed, the more the solution is smoothed, giving less emphasis to the detail (which is most likely to contain noise). However, at the same time there is also an undesirable decrease in the mean value away from the measured mean. In Fig 3b the loss of energy from the system due to regularization has been compensated so the pressure patterns are smoother (equivalent to having been filtered in k space [1]), but are energetically equal to the measured pattern.

4. Concluding remarks

In both singular value discarding and Tikhonov regularization the question of how many modes to exclude from the solution is critical: too few and there remains too much emphasis on unreliable terms, too many and the sound power of the source is underestimated. The compensation procedure presented ensures that the sound power of the equivalent source is always in agreement with the measured sound power. Thus, the main source of error due to over-regularization is avoided.

The most obvious application is to reduce errors where the use of optimization procedures for choosing the regularization parameter [1,3] is too costly to perform. However, there are also situations where reducing the complexity of a source model is desirable. In such cases, compensation allows regularization to be thought of rather flexibly, as a means of smoothing, such that the reconstructed pressure field is energetically correct, but contains less detail than the measured field.

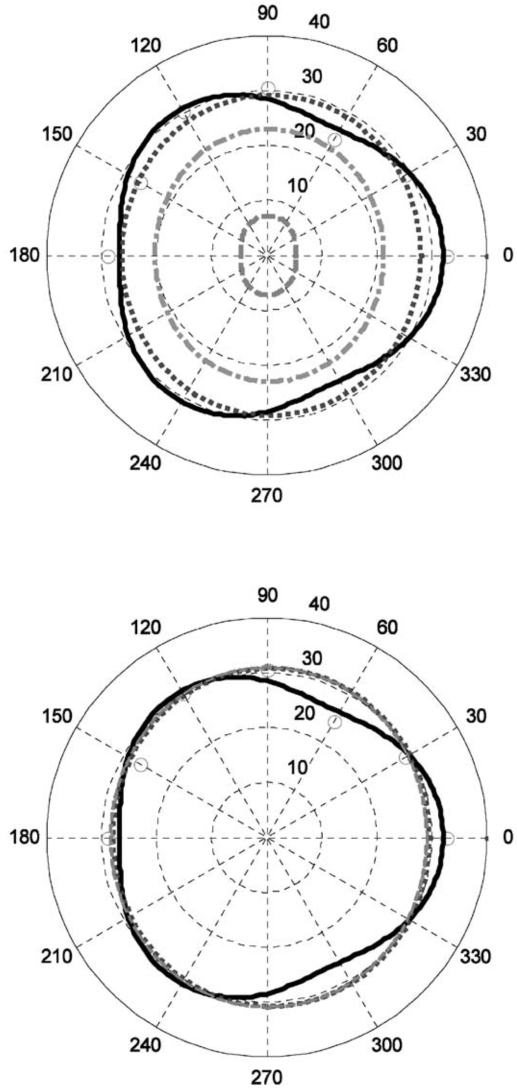

Fig. 3. Directivity patterns at 3 kHz reconstructed from equivalent source with various degrees of Tikhonov regularization ___ $\beta=0$, ... $\beta = 0.1\sigma^2$, - · - $\beta = \sigma^2$, - - - $\beta = 10\sigma^2$. (a) No compensation, and (b) with compensation. Circles are measured results.

The same arguments and the same procedures can be employed in inverse vibration problems where unknown forces are calculated from measured vibration at a number of points. Although the spatial averaged velocity in a system is not necessarily related to the power of a source in the same way as for acoustic systems, it is nevertheless arguably the single most reliable indicator of the overall energy in the system. Therefore, it is argued that a priority should be to predict it correctly.

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