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An efficient method for switching branches of period-doubling bifurcations of strongly non-linear autonomous oscillators with many degrees of freedom

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Abstract

Popular algorithms for switching branches at a bifurcation point of strongly non-linear oscillators are generally quite involved as they require the computation of the tangent of a new branch and second derivatives. In this paper, a simple but efficient algorithm is presented by using a perturbation-incremental method for switching branches at a period-doubling bifurcation of strongly non-linear autonomous oscillators with many degrees of freedom. To switch to a new branch at a bifurcation point, a parameter is simply turned on from zero to a small positive value so as to obtain an initial solution on the emanating branch for subsequent continuation. The parametric value at a period-doubling bifurcation can also be determined accurately. Furthermore, limit cycles of period 2^k $(k \geqslant 1)$ can be calculated to any desired degree of accuracy.

1. Introduction

Period-doubling sequences leading to chaos are observed in many science phenomena and engineering problems, and have been the subject of many analytical and numerical investigations, see Refs. [1–8]. In the computation of a new branch from a period-doubling bifurcation, one of the main problems is to find a starting point on the emanating branch which serves for a

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subsequent tracing of the entire branch. The calculation of an emanating solution is called branch switching. All methods for switching branches consist of the following two steps:

(I) An initial guess of a starting point on the emanating branch is to be constructed.
(II) An iteration that should converge to the new branch must be established.

Therefore, a branch switching may be regarded as a problem of establishing suitable predictors and correctors. Popular algorithms for switching branches include the construction of a predictor via the tangent [7,9,10] or the construction of correctors with selective properties [11]. However, such algorithms are generally quite involved since they require the computation of second derivatives [12].

In this paper, we present a simple and efficient method for switching branches of a period-doubling bifurcation of strongly non-linear autonomous oscillators. With this new method, neither the tangent of a new branch nor the second derivatives have to be calculated. A parameter is simply turned on from zero to a small positive value so that a solution on the new branch is obtained. Furthermore, the parametric value at which a period-doubling bifurcation occurs can be determined accurately.

In Refs. [13–15] a perturbation-incremental (PI) method was developed, which works extremely well for single strongly non-linear autonomous oscillators. Later, this method was extended to calculate the limit cycles of quadratic differential systems [16,17] and coupled strongly non-linear oscillators [18]. In this paper, the PI method is extended to the study of period-doubling bifurcations of strongly non-linear autonomous oscillators with many degrees of freedom. The numerical results will be compared with those from the Runge–Kutta method and the bifurcation package AUTO 97 [19,20]. In most cases, up to only the second period-doubling bifurcation can be obtained by using AUTO 97 whereas higher period-doubling bifurcations can be found by using the PI method. The advantage of the PI method lies in its simplicity and ease of application.

2. The perturbation-incremental method

In this section, we outline the PI method for strongly non-linear autonomous oscillators with many degrees of freedom. A detailed description can be found in [18].

Consider the following strongly non-linear autonomous oscillators with many degrees of freedom in which internal resonance may occur

$$\ddot{x}_i + g_i(x_i) = \lambda f_i(x_1, x_2, \ldots, x_N, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N), \quad i = 1, 2, \ldots, N, \tag{2.1}$$

where g_i and f_i are non-linear functions of their arguments, and λ is a parameter of arbitrary magnitude. We introduce a time transformation of the form

$$\frac{d\varphi}{dt} = \Phi(\varphi), \quad \Phi(\varphi + 2\pi) = \Phi(\varphi), \tag{2.2}$$

where φ is the new time. In the φ domain, Eq. (2.1) has the form

$$\Phi \frac{d}{d\varphi}(\Phi x_i') + g_i(x_i) = \lambda f_i(x_1, \ldots, x_N, \Phi x_1', \ldots, \Phi x_N'), \tag{2.3}$$

where prime denotes differentiation with respect to φ. Assume that the system possesses at least one limit cycle solution for $\lambda \simeq 0$, the origin of the x_1–$\dot{x}_1$ phase plane is an interior point of the projected limit cycle and M harmonics provide a sufficiently accurate representation, then the limit cycle may be expressed as

$$x_1 = a \cos \varphi + b, \tag{2.4a}$$

and

$$x_i = \sum_{j=0}^{M} (c_{ij} \cos j\varphi + d_{ij} \sin j\varphi), \quad d_{i0} = 0, \;\; i = 2, \ldots, N, \tag{2.4b}$$

where a is the amplitude, b the bias, c_{ij} and d_{ij} reals.

The procedure of the PI method is divided into two steps. The first step is the perturbation method. For $0 < \lambda \ll 1$, the initial values for Φ, a, b, c_{ij} and d_{ij} defined in Eqs. (2.2) and (2.4) can be obtained by the averaging method using generalized harmonic functions described in Ref. [21].

The second step is the parameter incremental method. Small increments are added to the initial solution $a^*, b^*, \Phi^*, c_{ij}^*$ and d_{ij}^* to obtain a neighbouring solution corresponding to $\lambda = \lambda^* + \Delta\lambda$, $\Phi = \Phi^* + \Delta\Phi$ and $x_i = x_i^* + \Delta x_i$ $(i = 1, \ldots, N)$.

Eq. (2.3) is expanded in Taylor's series about the initial state and linearized incremental equations are derived by ignoring all the non-linear terms of small increments as below

$$\begin{aligned}
&\left(2\Phi x_i'' + \Phi' x_i' - \lambda \frac{\partial f_i}{\partial \Phi}\right)\Delta\Phi + (x_i'\Phi)\Delta\Phi' \\
&+ \sum_{j=1}^{N} \left(-\lambda \frac{\partial f_i}{\partial x_j}\right)\Delta x_j + \sum_{j=1}^{N} \left(-\lambda \frac{\partial f_i}{\partial x_j'}\right)\Delta x_j' \\
&+ \left(\frac{\mathrm{d}g_i}{\mathrm{d}x_i}\right)\Delta x_i + (\Phi\Phi')\Delta x_i' + \left(\Phi^2\right)\Delta x_i'' - f_i\Delta\lambda \\
&= -\Phi^2 x_i'' - \Phi\Phi' x_i' - g_i(x_i) + \lambda f_i, \quad \text{for } i = 1, \ldots, N.
\end{aligned} \tag{2.5}$$

From Eq. (2.4), the terms Δx_i, $\Delta x_i'$, and $\Delta x_i''$ are expressed as, respectively,

$$\Delta x_1 = \Delta a \cos \varphi + \Delta b, \tag{2.6a}$$

$$\Delta x_1' = -\Delta a \sin \varphi, \tag{2.6b}$$

$$\Delta x_1'' = -\Delta a \cos \varphi, \tag{2.6c}$$

$$\Delta x_i = \sum_{j=0}^{M} (\Delta c_{ij} \cos j\varphi + \Delta d_{ij} \sin j\varphi), \quad \Delta d_{i0} = 0, \tag{2.6d}$$

$$\Delta x_i' = \sum_{j=1}^{M} j(\Delta d_{ij} \cos j\varphi - \Delta c_{ij} \sin j\varphi), \tag{2.6e}$$

$$\Delta x_i'' = -\sum_{j=1}^{M} j^2(\Delta c_{ij}\cos j\varphi + \Delta d_{ij}\sin j\varphi), \quad i = 2, 3, \ldots, N. \tag{2.6f}$$

Since Φ is a periodic function in φ with period 2π, we write

$$\Phi = \sum_{j=0}^{M} (p_j \cos j\varphi + q_j \sin j\varphi), \quad q_0 = 0, \tag{2.7a}$$

$$\Delta\Phi = \sum_{j=0}^{M} (\Delta p_j \cos j\varphi + \Delta q_j \sin j\varphi), \quad \Delta q_0 = 0, \tag{2.7b}$$

$$\Delta\Phi' = \sum_{j=1}^{M} j(\Delta q_j \cos j\varphi - \Delta p_j \sin j\varphi). \tag{2.7c}$$

λ is usually taken as a control parameter. By applying the harmonic balance method to the linearized incremental equation (2.5), a system of linear equations is obtained with unknowns $\Delta a, \Delta b$, Δp_j, Δq_j, Δc_{ij} and Δd_{ij} in the form

$$\begin{aligned} &A_n\Delta a + B_n\Delta b + P_{n0}\Delta p_0 + \sum_{j=1}^{M} (P_{nj}\Delta p_j + Q_{nj}\Delta q_j) \\ &+ \sum_{i=2}^{N} \left[C_{ni0}\Delta c_{i0} + \sum_{j=1}^{M} (C_{nij}\Delta c_{ij} + D_{nij}\Delta d_{ij}) \right] = R_n \end{aligned} \tag{2.8}$$

for $n = 1, 2, \ldots, 2MN + N + 2$. Derivation of the coefficients follows the same procedure as shown in Refs. [13–15] and R_n are residue terms.

Eq. (2.8) is to be solved by an equation solver such as the Gaussian elimination procedure. The values $a^*, b^*, p_j^*, q_j^*, c_{ij}^*$ and d_{ij}^* are updated by adding together the original values and the corresponding incremental values. The iteration process continues until $R_n \to 0$ for all n, (in practice, $|R_n|$ is less than a desired degree of accuracy). The incremental process proceeds mainly by adding $\Delta\lambda$ increment to the converged value of λ, using previous solution as initial approximation until a new converged solution is obtained. In case a saddle-node bifurcation occurs in the continuation, the incremental process will proceed by changing the control parameter from λ to a.

To determine the stability of a limit cycle by the Floquet method [12], we rewrite Eq. (2.1) as

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= \lambda f_i(x_1, \ldots, x_N, y_1, \ldots, y_N) - g_i(x_i), \end{aligned} \tag{2.9}$$

where $i = 1, 2, \ldots, N$. Let $\mathbf{A}$ be the Jacobian matrix of Eq. (2.9), i.e.,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \lambda\dfrac{\partial f_1}{\partial x_1} - \dfrac{\mathrm{d}g_1}{\mathrm{d}x_1} & \lambda\dfrac{\partial f_1}{\partial y_1} & \cdots & \lambda\dfrac{\partial f_1}{\partial x_N} & \lambda\dfrac{\partial f_1}{\partial y_N} \\ 0 & 0 & \cdots & 0 & 0 \\ \lambda\dfrac{\partial f_2}{\partial x_1} & \lambda\dfrac{\partial f_2}{\partial y_1} & \cdots & \lambda\dfrac{\partial f_2}{\partial x_N} & \lambda\dfrac{\partial f_2}{\partial y_N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ \lambda\dfrac{\partial f_N}{\partial x_1} & \lambda\dfrac{\partial f_N}{\partial y_1} & \cdots & \lambda\dfrac{\partial f_N}{\partial x_N} - \dfrac{dg_N}{dx_N} & \lambda\dfrac{\partial f_N}{\partial y_N} \end{bmatrix}. \tag{2.10}$$

Let $\underset{\sim}{\zeta} \in R^{2N}$ be a disturbance superimposed on a periodic solution of Eq. (2.9). Then

$$\frac{\mathrm{d}\underset{\sim}{\zeta}}{\mathrm{d}\varphi} = \frac{1}{\Phi}\mathbf{A}(\varphi)\,\underset{\sim}{\zeta}\,. \tag{2.11}$$

Let $\underset{\sim}{\zeta}^{(i)}(0)$ $(i = 1, 2, \ldots, 2N)$ be the $2N \times 1$ column matrix with unity at the ith row and zero elsewhere. By using numerical integration, we obtain the monodromy matrix M

$$M = \left[\underset{\sim}{\zeta}^{(1)}(2\pi), \underset{\sim}{\zeta}^{(2)}(2\pi), \ldots, \underset{\sim}{\zeta}^{(2N)}(2\pi)\right]. \tag{2.12}$$

The eigenvalues of M are used to determine the stability of the limit cycle. One of the eigenvalues or Floquet multipliers of M must be unity which provides a check for the calculation. If all the other eigenvalues are inside the unit circle, then the limit cycle under consideration is stable; otherwise, it is unstable. A period-doubling bifurcation occurs if one of the eigenvalues enters or leaves the unit circle through -1.

3. Period-doubling bifurcation

In the continuation of periodic solutions by the PI method, assume that a period-doubling bifurcation is detected at $\lambda = \lambda^*$ and the solution of the limit cycle at λ^* is given by

$$x_1 = a^* \cos\varphi + b^*, \tag{3.1a}$$

$$x_i = \sum_{j=0}^{M} (c_{ij}^* \cos j\varphi + d_{ij}^* \sin j\varphi), \quad d_{i0}^* = 0, \ \ i = 2, \ldots, N, \tag{3.1b}$$

$$\Phi = \sum_{j=0}^{M} (p_j^* \cos j\varphi + q_j^* \sin j\varphi), \quad q_0^* = 0. \tag{3.1c}$$

To calculate the period-2 limit cycles on the emanating branch, we rescale φ to 2φ and replace the original solution from M harmonics by $2M$ harmonics as

$$x_1 = a_2 \cos 2\varphi + a_1 \cos \varphi + a_0 + b_1 \sin \varphi, \tag{3.2a}$$

$$x_i = \sum_{j=0}^{2M} (c_{ij} \cos j\varphi + d_{ij} \sin j\varphi), \quad d_{i0} = 0, \ \ i = 2, \dots, N, \tag{3.2b}$$

$$\Phi = \sum_{j=0}^{2M} (p_j \cos j\varphi + q_j \sin j\varphi), \quad q_0 = 0. \tag{3.2c}$$

Then, the limit cycle at λ^* is expressed in the form of Eq. (3.2) as

$$\begin{aligned} &a_2 = a^*, \quad a_1 = 0, \quad a_0 = b^* \quad \text{and} \quad b_1 = 0, \\ &c_{i2j} = c_{ij}^*, \quad d_{i2j} = d_{ij}^*, \quad p_{2j} = \frac{p_j^*}{2} \quad \text{and} \quad q_{2j} = \frac{q_j^*}{2} \quad \text{for } j = 1, \dots, M, \\ &\text{and} \quad c_{ij} = d_{ij} = p_j = q_j = 0 \quad \text{for odd} \quad j. \end{aligned} \tag{3.2d}$$

On the emanating branch after a period-doubling bifurcation has occurred, at least one of the parameters a_1 and b_1 will become non-zero. In Eq. (3.2), if $x_i(\varphi) = (a_0, a_1, a_2, b_1;\ c_{ij}, d_{ij};\ p_j, q_j)$ $(i = 1, \dots, N; j = 1, \dots, 2M)$ is a periodic solution, then it can also be expressed as $x^+(\varphi) = x_i(\pi + \varphi) = (a_0, -a_1, a_2, -b_1;\ (-1)^j c_{ij}, (-1)^j d_{ij};\ (-1)^j p_j, (-1)^j q_j)$. In particular, if φ is replaced by $\pi + \varphi$ in Eq. (3.2a), both a_1 and b_1 in x_1 change sign while a_0 and a_2 remain the same. It follows that if $x_i(\varphi)$ is a periodic solution with $b_1 = \varepsilon > 0$, the same solution can also be expressed in the form of Eq. (3.2) with $b_1 = -\varepsilon < 0$ and the signs of $a_1, c_{ij}, d_{ij}, p_j, q_j$ reversed for odd j. With this observation, we propose a simple but efficient method for branch switching of a period-doubling bifurcation as follows. To switch to the emanating branch, Eq. (3.2) is used as an initial solution and b_1 is chosen as the continuation parameter which is simply turned on from zero to a small positive value $\varepsilon > 0$. (If b_1 is turned on from zero to a small negative value $-\varepsilon < 0$, the same solution as $b_1 = \varepsilon$ will be obtained.) The incremental process used for branch switching and subsequent continuation is again the Newton–Raphson method used in Section 2. For an initial solution sufficiently close to a period-doubling bifurcation, the converged solution with $b_1 = \varepsilon > 0$ will be a period-2 limit cycle on the emanating branch.

For the incremental step, the terms $\Delta x_1, \Delta x_1'$ and $\Delta x_1''$ in Eqs. (2.6a–c) are now rewritten as

$$\begin{aligned} \Delta x_1 &= \Delta a_2 \cos 2\varphi + \Delta a_1 \cos \varphi + \Delta a_0 + \Delta b_1 \sin \varphi, \\ \Delta x_1' &= -2\Delta a_2 \sin 2\varphi - \Delta a_1 \sin \varphi + \Delta b_1 \cos \varphi, \\ \Delta x_1'' &= -4\Delta a_2 \cos 2\varphi - \Delta a_1 \cos \varphi - \Delta b_1 \sin \varphi. \end{aligned}$$

The incremental process proceeds mainly by adding Δb_1 increment to the converged value of b_1 using previous solution as initial approximation until a new converged solution is obtained. (By a similar reasoning, a_1 may also be chosen as the continuation parameter in the emanating branch.)

Emanating branch of a higher period-doubling bifurcation can be traced in a similar way. Assume that the kth $(k \geqslant 1)$ period-doubling bifurcation is detected at $\lambda = \lambda^*$ and the solution of the limit cycle at λ^* is given by

$$x_1 = \sum_{j=0}^{2^{k-1}} a_j^* \cos j\varphi + \sum_{j=1}^{2^{k-1}-1} b_j^* \sin j\varphi, \tag{3.3a}$$

$$x_i = \sum_{j=0}^{2^{k-1}M} (c_{ij}^* \cos j\varphi + d_{ij}^* \sin j\varphi), \quad d_{i0}^* = 0, \;\; i = 2, \ldots, N, \tag{3.3b}$$

$$\Phi = \sum_{j=0}^{2^{k-1}M} (p_j^* \cos j\varphi + q_j^* \sin j\varphi), \quad q_0^* = 0. \tag{3.3c}$$

To calculate the limit cycles on the emanating branch after the kth period-doubling bifurcation, we replace the original solution from $2^{k-1}M$ harmonics by $2^k M$ harmonics as

$$\begin{aligned} x_1 &= \sum_{j=0}^{2^k} a_j^* \cos j\varphi + \sum_{j=1}^{2^k-1} b_j^* \sin j\varphi, \\ x_i &= \sum_{j=0}^{2^k M} (c_{ij}^* \cos j\varphi + d_{ij}^* \sin j\varphi), \quad d_{i0}^* = 0, \;\; i = 2, \ldots, N, \\ \Phi &= \sum_{j=0}^{2^k M} (p_j^* \cos j\varphi + q_j^* \sin j\varphi), \quad q_0^* = 0, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} & a_{2j} = a_j^* \quad \text{for } j = 0, 1, \ldots, 2^{k-1}, \\ & b_{2j} = b_j^* \quad \text{for } j = 1, 2, \ldots, 2^{k-1} - 1, \\ & c_{i2j} = c_{ij}^*, \quad d_{i2j} = d_{ij}^*, \quad p_{2j} = \frac{p_j^*}{2} \quad \text{and} \quad q_{2j} = \frac{q_j^*}{2} \text{ for } j = 0, 1, \ldots, 2^{k-1}M, \\ & \quad \text{and} \quad a_j = b_j = c_{ij} = d_{ij} = p_j = q_j = 0 \quad \text{for odd } j. \end{aligned}$$

To switch to the new emanating branch, Eq. (3.4) is used as an initial solution and b_1 is chosen as the continuation parameter which is simply turned on from zero to a small positive value $\varepsilon > 0$. The incremental process proceeds mainly by adding Δb_1 increment to the converged value of b_1, using previous solution as initial approximation until a new converged solution is obtained.

4. Coupled generalized van der Pol oscillators

First we consider the coupled generalized van der Pol oscillators

$$\ddot{x}_1 + x_1^3 = \lambda[(1 - x_1^2)\dot{x}_1 + 2x_2], \tag{4.1a}$$

$$\ddot{x}_2 + 4x_2 - x_2^2 = \lambda[0.5x_1 + (1 - x_2^2)\dot{x}_2]. \tag{4.1b}$$

For $0<\lambda\ll 1$, this system has been investigated in Ref. [21] using a generalized averaging method based on the generalized harmonic functions. The analytical approximation of a limit cycle obtained in that paper is only valid for small λ. The calculation of limit cycles for arbitrary large λ was considered in Ref. [18] by using the PI method described in Section 2. From Ref. [18], an approximate solution of Eq. (4.1) for $\lambda\simeq 0$ in the form of Eq. (2.4) is given as

$$\begin{aligned}
&a = 2.1148, \quad b = 0,\\
&c_{20} = 0.0069, \quad c_{21} = 1.9564, \quad c_{22} = 0.0776,\\
&d_{21} = -0.3982, \quad d_{22} = 0.1740,\\
&c_{2j} = d_{2j} = 0 \quad \text{for } j > 2,\\
&\text{and} \quad \Phi = 1.4954(1+\cos^2\varphi)^{1/2}. \qquad (4.2)
\end{aligned}$$

Eq. (4.2) is an initial solution for the incremental step in which limit cycles of large λ can be obtained. Fig. 1 shows the amplitude a versus the parameter λ of the limit cycles. To investigate the stability of a limit cycle, we reduce the Jacobian matrix of Eq. (2.10) to

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\lambda x_1 y_1 - 3x_1^2 & \lambda(1-x_1^2) & 2\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0.5\lambda & 0 & -2\lambda x_2 y_2 + 2x_2 - 4 & \lambda(1-x_2^2) \end{bmatrix}$$

and calculate the corresponding Floquet multipliers.

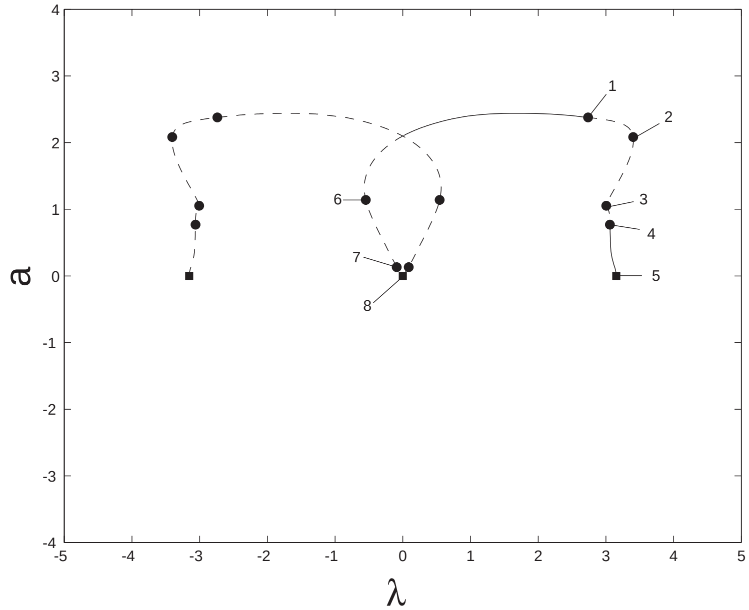


Fig. 1. The amplitude a versus parameter λ of the limit cycles of Eq. (4.1). (——) Stable limit cycle, (- - -) unstable limit cycle, (■) Hopf bifurcation, (•) period-doubling bifurcation.

Table 1
Amplitude and bias of the limits cycle at $\lambda = 2.7353$ (label 1 of Fig. 1) which is near a period-doubling bifurcation, and Fourier coefficients of x_2 and $\Phi(\varphi)$ for Eq. (3.1)

$\lambda = 2.7353,\ a = 2.3769,\ b = 0.0799$				
j	c_{2j}	d_{2j}	p_j	q_j
0	0.39712	0	2.50417	0
1	0.39562	−1.19854	−0.06731	−0.40979
2	0.12317	0.18285	−0.28982	−1.84187
3	0.24654	0.04692	−0.02669	−0.01841
4	−0.09000	0.07335	−0.15181	−0.01974
5	−0.00911	0.06208	0.00666	−0.00014
6	−0.03845	−0.04005	−0.00831	−0.02528
7	−0.01167	−0.00344	−0.00378	0.00403
8	0.01647	−0.01816	0.00061	−0.00454
9	0.00251	0.00056	−0.00255	−0.00228
10	0.00777	0.00638	0.00116	−0.00184

Starting from the approximate solution (4.2) with λ increasing from zero, the periodic solution follows the path containing the labels 1–5 and vanishes at $\lambda \simeq 3.1539$ (label 5) which is a Hopf bifurcation. On the other hand, starting from the approximate solution with λ decreasing from zero, the solution follows the path containing the labels 6–8 and vanishes at $\lambda = 0$ (label 8) which is also a Hopf bifurcation from the trivial solution $x = 0$. From the symmetry property of Eq. (4.1), if $(x_1, y_1, x_2, y_2, \lambda)$ is a solution, then $(-x_1, y_1, x_2, -y_2, -\lambda)$ is also a solution. It follows that if (λ, a) in Fig. 1 is a solution, so is $(-\lambda, a)$. Period-doubling bifurcation occurs at the points labelled 1–4, 6–7 in which a Floquet multiplier either enters or leaves the unit circle at -1.

We next consider the branch switching at label 1 where $\lambda = 2.7353$. From the incremental step, the explicit form of the limit cycle at that point is given in Eq. (3.1) where $a^*, b^*, c^*_{ij}, d^*_{ij}, p^*_j$ and q^*_j are shown in Table 1. To switch to the emanating branch, this limit cycle is rewritten in the form of Eq. (3.2) and is used as an initial solution for the continuation of the new branch. Fig. 2 shows the parameter a_2 versus the parameter λ of the emanating branch. Second period-doubling bifurcation occurs at the points labelled 10–15.

The continuation of periodic solutions obtained by the PI method is compared with that obtained from the bifurcation package AUTO 97. In AUTO, the periodic solutions are found by reformulating the continuation process as boundary value problems and a pseudo-arclength procedure is imposed for the continuation of solutions. Floquet multipliers along branches of periodic solutions are then monitored continuously for the locations of various bifurcation points. Fig. 3 shows the emanating branch switched from label 1 using AUTO 97. If the ordinate of Fig. 2 is changed to $\max[x_1(\varphi)] = a_2 \cos 2\varphi + a_1 \cos \varphi + a_0 + b_1 \sin \varphi$ for all φ, the emanating branch is the same as that of Fig. 3. It follows that the result obtained by the PI method is in good agreement with that obtained by using AUTO 97. Up to second period-doubling bifurcation can be located by using AUTO 97. On the contrary, higher period-doubling bifurcations can be obtained by using the PI method. By using Eq. (3.4) as an initial solution, emanating branch from the second period-doubling bifurcation at $\lambda \simeq 2.7664$ (label 10) of Fig. 2 can be traced, as depicted in Fig. 4. Third period-doubling bifurcation occurs at the points labelled 16–19.

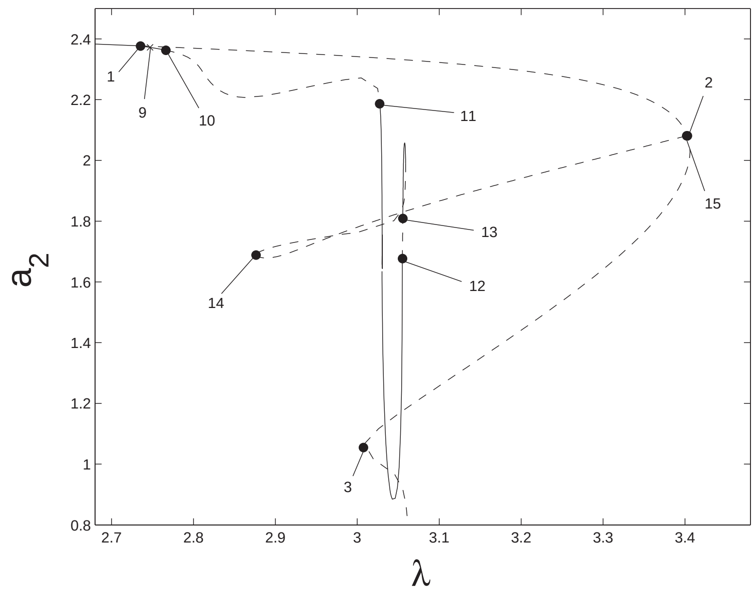


Fig. 2. Emanating branch from first period-doubling bifurcation at $\lambda \simeq 2.7353$ (label 1) of Fig. 1. (—) Stable limit cycle, (- - -) unstable limit cycle, (•) period-doubling bifurcation.

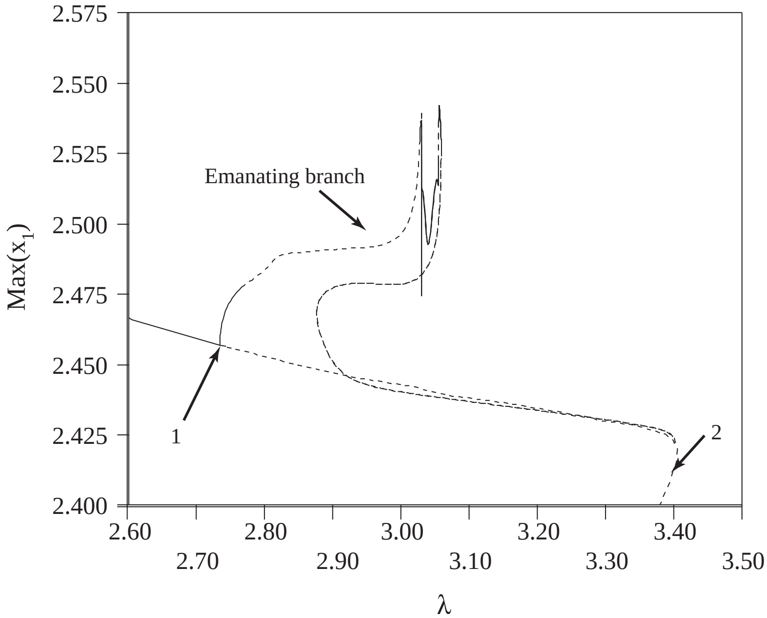


Fig. 3. Emanating branch switched from label 1 of Fig. 1. obtained by using AUTO 97. (—) Stable limit cycle, (- - -) unstable limit cycle.

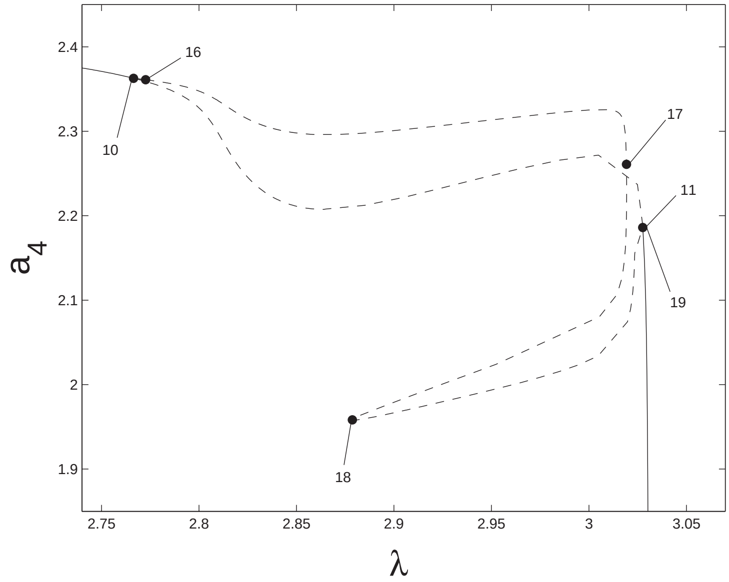


Fig. 4. Emanating branch from second period-doubling bifurcation at $\lambda \simeq 2.7664$ (label 10) of Fig. 2. (—) Stable limit cycle, (- - -) unstable limit cycle, (•) period-doubling bifurcation.

The parametric value at which a period-doubling bifurcation occurs can be determined accurately by means of the PI method. For instance, in Fig. 2, we choose a period-2 limit cycle on the emanating branch with b_1 non-zero. Then, b_1 is decreased gradually to zero by the incremental process. The first period-doubling bifurcation value $\lambda_1 \simeq 2.73441908$ is obtained at $b_1 = 0$. This value is more accurate than that of label 1 which is obtained by the continuation of λ. In a similar way, the second period-doubling bifurcation value in Fig. 4 is found to be $\lambda_2 \simeq 2.76640121$. The third and fourth period-doubling bifurcation values are found to be $\lambda_3 \simeq 2.77248908$ and $\lambda_4 \simeq 2.77381023$, respectively. Feigenbaum [1] showed that a sequence of period-doubling parameters scales according to the law

$$\lim_{i \to \infty} \frac{\lambda_i - \lambda_{i-1}}{\lambda_{i+1} - \lambda_i} = \delta = 4.66292016\ldots \quad . \tag{4.3}$$

The universal constant δ is called the Feigenbaum number. From λ_i ($i = 1, 2, 3, 4$), we determine two values of the sequence equation (4.3),

$$\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2} \simeq 5.25342573, \quad \frac{\lambda_3 - \lambda_2}{\lambda_4 - \lambda_3} \simeq 4.60797767$$

which come close to the limit δ (compared with those values of the Lonenz's fourth order system discussed in Ref. [10, p. 272]).

From the PI method, explicit form of a limit cycle of period 2^n can be obtained for arbitrary value of λ. For instance, when $\lambda = 2.7472$ (label 9 of Fig. 2, a point on the emanating branch switched from label 1), the explicit form of the stable period-2 limit cycle is given in the form of Eqs. (3.2a–c) where $a_j, b_j, c_{ij}, d_{ij}, p_j, q_j$ and M are shown in Table 2. Phase portrait of the limit cycle is shown in Fig. 5 and is compared to the result of the numerical integration obtained by using the

Table 2
Fourier coefficients of x_1, x_2 and $\Phi(\varphi)$ for the period-2 limit cycle at $\lambda = 2.7472$ (label 9 of Fig. 2)

$M = 5,\ \lambda = 2.7472,\ b_1 = 0.0059$
$a_2 = 2.3721,\ a_1 = 0.0192,\ a_0 = 0.0802$

j	c_{2j}	d_{2j}	p_j	q_j
0	0.39371	0	1.24997	0
1	0.11192	−0.10051	−0.00728	−0.05952
2	0.38985	−1.17873	−0.03230	−0.20431
3	−0.00312	−0.09913	0.00193	−0.01689
4	0.12031	−0.17407	−0.14355	−0.92173
5	0.01434	0.00568	0.00682	−0.00238
6	0.23889	−0.04683	−0.01448	−0.00808
7	−0.05165	0.00577	−0.00366	−0.00202
8	−0.08322	0.00697	−0.07585	−0.00936
9	0.00415	−0.00321	0.00149	0.00298
10	−0.00993	0.05927	0.00256	−0.00054

fourth order Runge–Kutta method. It can be seen that they are in good agreement. Parameter–frequency curve of an emanating branch can be obtained from the equation on frequency ω which is given by $\omega = 2\pi / \int_0^{2\pi} \mathrm{d}\varphi/\Phi$.

5. A three-dimensional model of a feedback control system

The algorithm presented in Section 3 for switching branches of a period-doubling bifurcation is not restricted to non-linear autonomous oscillators of the form of Eq. (2.1). It can also be applied to a system mixed with first and second order autonomous oscillators. As a second example, we consider the following three-dimensional model of a feedback control system [22–24]

$$\dot{x} = \mu x - y - xz, \tag{5.1a}$$

$$\dot{y} = \mu y + x, \tag{5.1b}$$

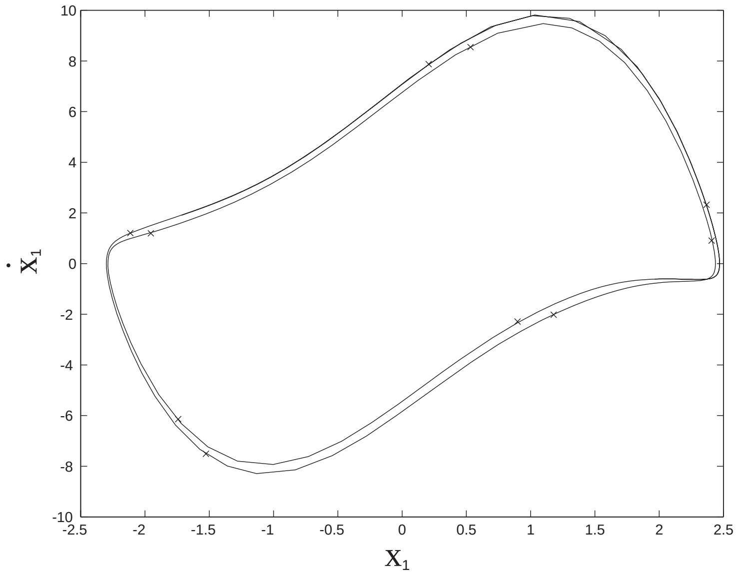
$$\dot{z} = -z + y^2 + x^2 z, \tag{5.1c}$$

where μ is the control parameter. We note that this system is invariant under the transformation $(x, y, z) \Leftrightarrow (-x, -y, z)$. Therefore, if (x, y, z) is a solution of Eq. (5.1), so is $(-x, -y, z)$. Hence, all solutions occur in pairs because of the transformation. A solution of Eq. (5.1) that is invariant under this transformation is called a symmetric solution. If a solution of Eq. (5.1) is not invariant under the transformation, it is called an asymmetric solution.

By letting $(x_1, x_2) = (y, z)$, we rewrite Eq. (5.1) as

$$\ddot{x}_1 + x_1 = \mu \dot{x}_1 + (\dot{x}_1 - \mu x_1)(\mu - x_2), \tag{5.2a}$$

$$\dot{x}_2 + x_2 = x_1^2 + (\dot{x}_1 - \mu x_1)^2 x_2. \tag{5.2b}$$



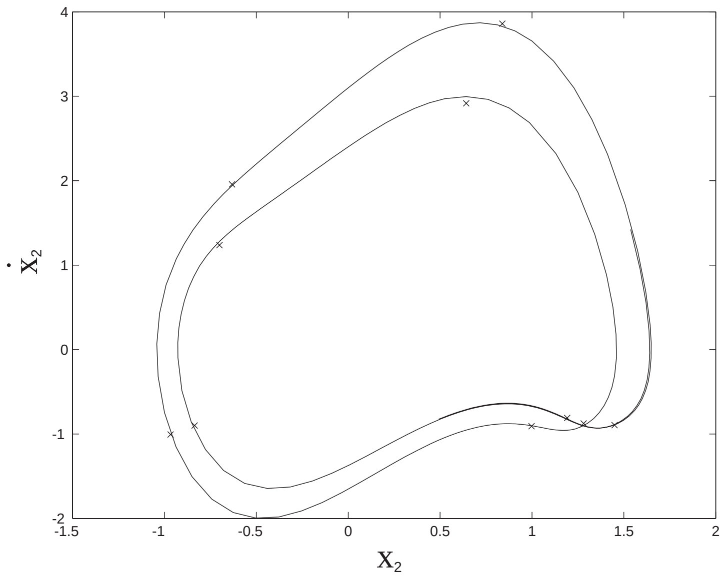


Fig. 5. Limit cycle of period 2 at $\lambda = 2.7472$ of Eq. (4.1). (—) Runge–Kutta method, ($\times$) perturbation-incremental method.

It can be shown easily that a Hopf bifurcation occurs at the origin of Eq. (5.1). To obtain an approximate solution of Eq. (5.2) for small $\mu > 0$, we introduce a new parameter λ to Eq. (5.2a) as

$$\ddot{x}_1 + x_1 = \lambda[\mu\dot{x}_1 + (\dot{x}_1 - \mu x_1)(\mu - x_2)]. \tag{5.3}$$

Eq. (5.3) is reduced to Eq. (5.2) when $\lambda = 1$.

For the first step of the PI method, we use a perturbation method similar to the KBM method [25]. For $\lambda \ll 1$, an approximate solution of Eqs. (5.3) and (5.2b) may be expressed in the form

$$x_1 = a\cos\varphi + \lambda X_1(a) + O(\lambda^2), \tag{5.4a}$$

$$x_2 = \sum_{j=0}^{M} (c_{2j}\cos j\varphi + d_{2j}\sin j\varphi), \tag{5.4b}$$

where X_1 is φ independent. a and φ are assumed to vary with time t in such a way that they satisfy the equations

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \lambda A_1(a) + O(\lambda^2), \tag{5.4c}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 + \lambda \Phi_1(\varphi) + O(\lambda^2), \tag{5.4d}$$

where Φ_1 is a periodic function of φ with period 2π. Substituting Eq. (5.4) into Eq. (5.2b) and using the harmonic balance method, we can expressed the Fourier coefficients c_{2j}, d_{2j} $(j = 0, 1, \dots, M)$ in terms of the amplitude a. For instance, for $M = 2$,

$$\begin{aligned}
c_{20} &= \frac{a^2[40 - 2(5 - 4\mu + 3\mu^2)a^2 + (3 + 4\mu^2 + \mu^4)a^4]}{[2 - (1+\mu^2)a^2][40 - 8(1+\mu^2)a^2 + (1+\mu^2)^2a^4]},\\
c_{22} &= \frac{4a^2[2 - (3 + 4\mu + \mu^2)a^2 + a^4]}{80 - 56(1+\mu^2)a^2 + 10(1+\mu^2)^2a^4 - (1+\mu^2)^3a^6},\\
d_{22} &= \frac{2a^2[-8 - 4(-2+\mu)a^2 + \mu(3+\mu^2)a^4]}{-80 + 56(1+\mu^2)a^2 - 10(1+\mu^2)^2a^4 + (1+\mu^2)^3a^6},\\
c_{21} &= d_{20} = d_{21} = 0.
\end{aligned} \tag{5.5}$$

The first and second derivatives of x_1 with respect to t are

$$\dot{x}_1 = -a\sin\varphi + \lambda(A_1\cos\varphi - a\Phi_1\sin\varphi) + O(\lambda^2), \tag{5.6a}$$

$$\ddot{x}_1 = -a\cos\varphi - \lambda\left[2A_1\sin\varphi + a\Phi_1\cos\varphi + a\frac{\partial}{\partial\varphi}(\Phi_1\sin\varphi)\right] + O(\lambda^2). \tag{5.6b}$$

Substituting Eqs. (5.4), (5.6) into Eq. (5.3) and equating coefficient of order λ, we obtain

$$\begin{aligned}
&2A_1\sin\varphi + a\Phi_1\cos\varphi + a\frac{\partial}{\partial\varphi}(\Phi_1\sin\varphi) - X_1\\
&\quad = a\left\{\mu\sin\varphi + (\sin\varphi + \mu\cos\varphi)\left[\mu - \sum_{j=0}^{M}(c_{2j}\cos j\varphi + d_{2j}\sin j\varphi)\right]\right\}
\end{aligned}$$

which implies

$$a\Phi_1 \sin^2\varphi = a \int_0^{\varphi} (\sin\varphi + \mu\cos\varphi)\left[\mu - \sum_{j=0}^{M}(c_{2j}\cos j\varphi + d_{2j}\sin j\varphi)\right]\sin\varphi \, \mathrm{d}\varphi$$
$$+ (a\mu - 2A_1)\int_0^{\varphi} \sin^2\varphi \mathrm{d}\varphi + X_1 \int_0^{\varphi} \sin\varphi \, \mathrm{d}\varphi. \quad (5.7)$$

Letting $\varphi = 2\pi$ in Eq. (5.7), we have

$$A_1 = \frac{a}{4}[\mu(4 - d_{22}) - 2c_{20} + c_{22}]. \quad (5.8)$$

For a steady state periodic solution, $\mathrm{d}a/\mathrm{d}t = 0$. Since c_{20}, c_{22} and d_{22} are functions of a, periodic solution with non-zero amplitude can be found by substituting these coefficients into the following equation:

$$\mu(4 - d_{22}) - 2c_{20} + c_{22} = 0. \quad (5.9)$$

Letting $\varphi = \pi$ in Eq. (5.7), we have

$$X_1 = \frac{a}{2}\left[\sum_{j=0}^{M} \int_0^{\pi} (c_{2j}\cos j\varphi + d_{2j}\sin j\varphi)(\sin\varphi + \mu\cos\varphi)\sin\varphi \, \mathrm{d}\varphi - \pi\mu\right]. \quad (5.10)$$

In particular, for $M = 2$,

$$X_1 = \frac{a\pi}{8}(2c_{20} - c_{22} - 4\mu + \mu d_{22}) = 0. \quad (5.11)$$

Then, from Eq. (5.7), Φ_1 can be expressed as

$$\Phi_1(\varphi) = \frac{1}{\sin^2\varphi}\left[\mu\left(\varphi - \frac{\sin 2\varphi}{2}\right) + \frac{\mu^2}{4}(1 - \cos 2\varphi) + \frac{X_1}{a}(1 - \cos\varphi)\right.$$
$$\left. - \sum_{j=0}^{M} \int_0^{\varphi} (c_{2j}\cos j\varphi + d_{2j}\sin j\varphi)(\sin\varphi + \mu\cos\varphi)\sin\varphi \; d\varphi\right] \quad (5.12)$$

where a can be solved from Eq. (5.9) and X_1 is given in Eq. (5.10). Hence, the first order analytical approximation to a periodic solution of Eqs. (5.3) and (5.2b) in the form of Eq. (5.4) is obtained. We compare this approximate solution with the corresponding periodic solution of the original system (5.1). As an illustration, let $\lambda = 1,\ \mu = 0.1$ and $M = 2$. From Eqs. (5.5), (5.9), (5.11) and (5.12), we have

$$a = 0.5948, \ \ X_1 = 0,$$
$$\Phi_1 = 0.9792 + 0.0144\cos 2\varphi - 0.0068\sin 2\varphi. \quad (5.13a)$$

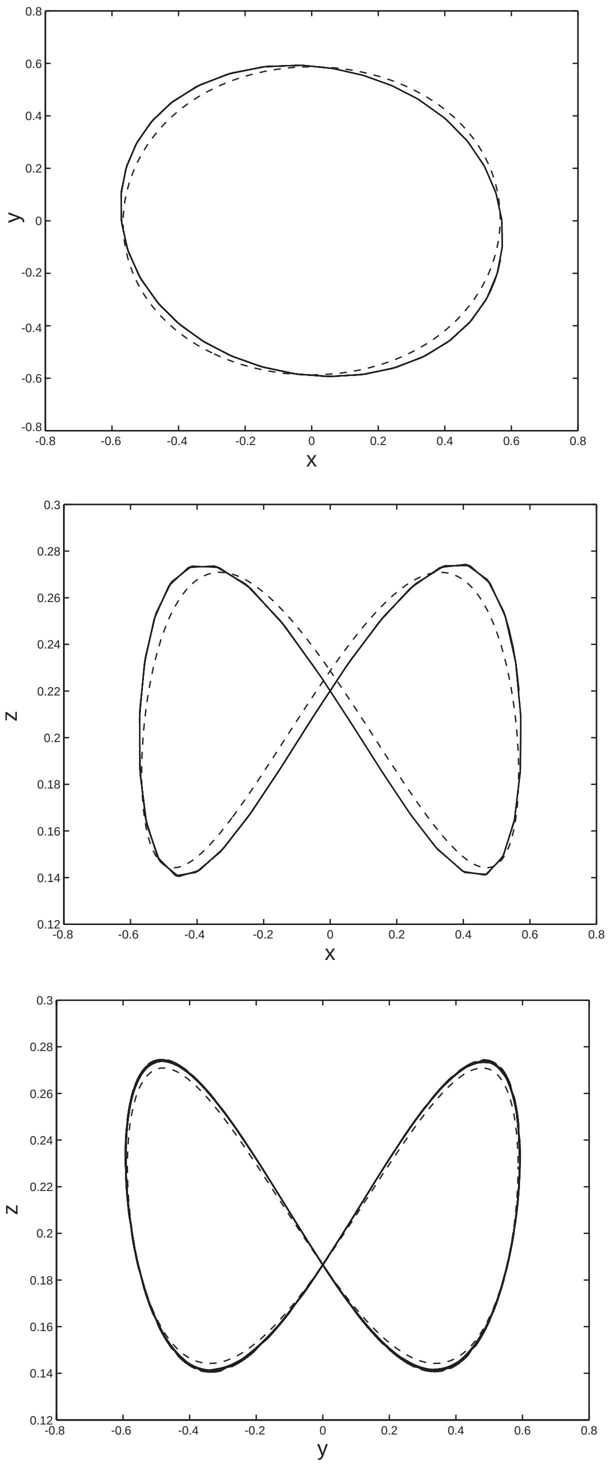


Fig. 6. Different projections of the limit cycle of Eq. (5.1) with $\mu = 0.1$. (—) Runge–Kutta method, (- - -) perturbation method.

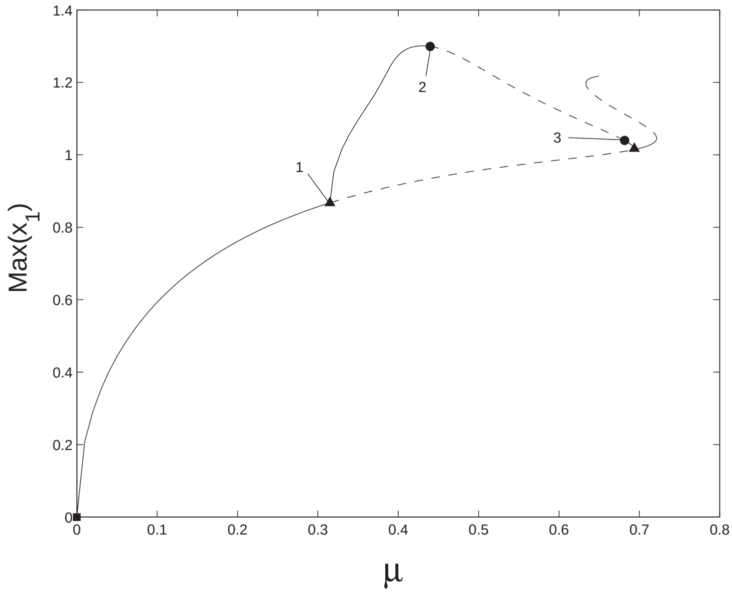


Fig. 7. Max(x_1) versus parameter μ of the limit cycles of Eq. (5.2). (—) Stable limit cycle, (- - -) unstable limit cycle, (■) Hopf bifurcation, (●) period-doubling bifurcation, (▲) symmetry-breaking bifurcation.

Since $(y, z) = (x_1, x_2)$, it follows from Eqs. (5.1b), (5.4) and (5.5) that

$$\begin{aligned} x =& -a\sin\varphi - \lambda a\Phi_1 \sin\varphi - \mu a\cos\varphi \\ =& \,0.0020\cos\varphi - 0.0020\cos 3\varphi - 0.5704\sin\varphi - 0.0042\sin 3\varphi, \end{aligned} \tag{5.13b}$$

$$y = 0.5869\cos\varphi, \tag{5.13c}$$

$$z = 0.2076 + 0.0211\cos 2\varphi + 0.0597\sin 2\varphi. \tag{5.13d}$$

Different projections of the limit cycle generated using Eq. (5.13) are shown in Fig. 6 and are compared to the result of the numerical integration obtained by using the fourth order Runge–Kutta method. It can be seen that the approximate solution agrees with the numerical solution.

We next consider the incremental step of the PI method. In this example, μ is taken as the control parameter. The linearized incremental equations of Eq. (5.2) are given as

$$\begin{aligned} &\left(2\Phi x_1'' + \Phi' x_1' - \frac{\partial f_1}{\partial \Phi}\right)\Delta\Phi + x_1'\Phi\Delta\Phi' - \sum_{j=1}^{2}\frac{\partial f_1}{\partial x_j}\Delta x_j \\ &- \sum_{j=1}^{2}\frac{\partial f_1}{\partial x_j'}\Delta x_j' + \Delta x_1 + \Phi\Phi'\Delta x_1' + \Phi^2\Delta x_1'' - \frac{\partial f_1}{\partial \mu}\Delta\mu = -\Phi^2 x_1'' - \Phi\Phi' x_1' - x_1 + f_1 \end{aligned}$$

Table 3
Amplitude and bias of the limit cycle at $\mu = 0.4400$ (label 2 of Fig. 7) which is near a period-doubling bifurcation, and Fourier coefficients of x_2 and $\Phi(\varphi)$ for Eq. (3.1)

$\mu = 0.4400,\ a = 1.0234,\ b = 0.2758$				
j	c_{2j}	d_{2j}	p_j	q_j
0	0.8414	0	0.7845	0
1	0.4784	0.3300	−0.0025	−0.2694
2	0.0141	0.3739	0.1027	−0.0827
3	−0.1095	0.0708	0.0468	0.0080
4	−0.0661	−0.0247	0.0112	0.0218
5	−0.0153	−0.0349	−0.0054	0.0122
6	0.0111	−0.0186	−0.0074	0.0014
7	0.0128	−0.0009	−0.0030	−0.0031
8	0.0048	0.0061	0.0006	−0.0025
9	−0.0015	0.0045	0.0015	−0.0005
10	−0.0029	0.0007	0.0008	0.0006

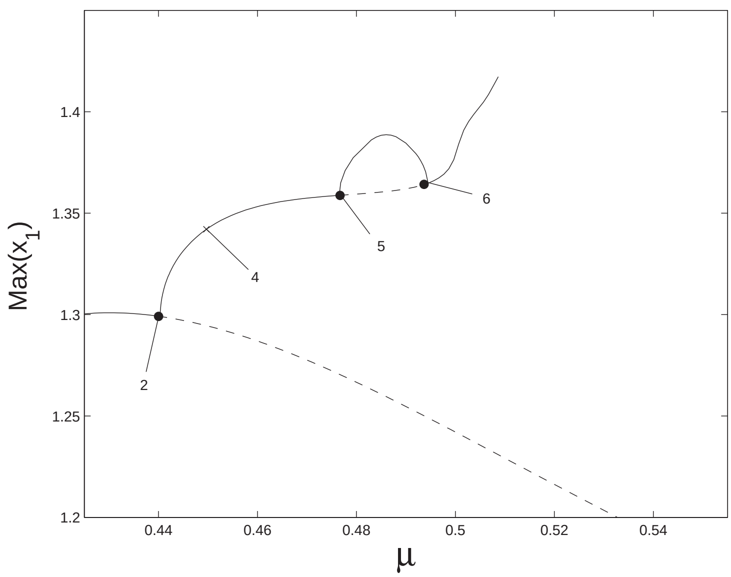


Fig. 8. Emanating branches from the first and second period-doubling bifurcations of Fig. 7. (—) Stable limit cycle, (- - -) unstable limit cycle, (•) period-doubling bifurcation.

and

$$\left(x_2' - \frac{\partial f_2}{\partial \Phi}\right)\Delta\Phi - \sum_{j=1}^{2} \frac{\partial f_2}{\partial x_j}\Delta x_j - \sum_{j=1}^{2} \frac{\partial f_2}{\partial x_j'}\Delta x_j' + \Delta x_2 + \Phi\Delta x_2' - \frac{\partial f_2}{\partial \mu}\Delta\mu = -\Phi x_2' - x_2 + f_2,$$

Table 4
Fourier coefficients of x_1, x_2 and $\Phi(\varphi)$ for the period-2 limit cycle at $\mu = 0.4497$ (label 4 of Fig. 8)

$M = 8,\ \mu = 0.4497,\ b_1 = 0.0271$
$a_2 = 1.031,\ a_1 = -0.0432,\ a_0 = 0.2683$

j	c_{2j}	d_{2j}	p_j	q_j
0	0.8593	0	0.3901	0
1	−0.0325	−0.0322	0.0082	0.0051
2	0.4779	0.3133	−0.0066	−0.1337
3	−0.0552	−0.0161	0.0075	0.0135
4	0.0167	0.3762	0.0499	−0.0432
5	−0.0218	−0.0195	−0.0008	0.0090
6	−0.1023	0.0784	0.0239	0.0012
7	0.0055	−0.0255	−0.0056	0.0031
8	−0.0659	−0.0163	0.0074	0.0097
9	0.0178	−0.0109	−0.0042	−0.0017
10	−0.0213	−0.0303	−0.0011	0.0067
11	0.0140	0.0057	−0.0008	−0.0030
12	0.0050	−0.0202	−0.0033	0.0020
13	0.0027	0.0106	0.0014	−0.0017
14	0.0110	−0.0056	−0.0021	−0.0007
15	−0.0050	0.0060	0.0015	0.0001
16	0.0070	0.0028	−0.0005	−0.0013

where $f_1 = \mu\Phi x_1' + (\Phi x_1' - \mu x_1)(\mu - x_2)$ and $f_2 = x_1^2 + (\Phi x_1' - \mu x_1)^2 x_2$. The incremental process proceeds mainly by adding $\Delta\mu$ increment to the converged value of μ, using previous solution as initial approximation until a new converged solution is obtained. Stability of a limit cycle can be determined by calculating the Floquet multipliers of the following Jacobian matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ \mu(x_2 - \mu) - 1 & 2\mu - x_2 & \mu x_1 - y_1 \\ 2[x_1 - \mu x_2(y_1 - \mu x_1)] & 2x_2(y_1 - \mu x_1) & (y_1 - \mu x_1)^2 - 1 \end{pmatrix},$$

where $y_1 = \dot{x}_1$.

Fig. 7 shows the continuation of the limit cycles of Eq. (5.2) with $\max(x_1) = a + b$ as the ordinate. Starting from the approximate solution (5.13) with μ increasing from 0.1, the symmetric limit cycle is stable until we reach the critical value $\mu^* \simeq 0.3150$ (label 1). At this critical point, the symmetric periodic solution is non-hyperbolic with two of the associated Floquet multipliers at $+1$. For values of $\mu > \mu^*$, the stable symmetric limit cycle becomes unstable and two other stable asymmetric limit cycles appear. Hence, a symmetry-breaking bifurcation occurs at μ^*.

As μ is increased beyond μ^*, the periodic solution obtained by the PI method switches automatically to the bifurcated branch containing labels 2 and 3. (If we want to keep to the original curve with unstable symmetric limit cycles, we have to exclude the bias b in $x_1 = a\cos\varphi + b$ in the incremental step as $b = 0$ for a symmetric limit cycle.) The first period-doubling bifurcation occurs at $\mu^{**} \simeq 0.4400$ (label 2) where one of the Floquet multipliers leaves the unit

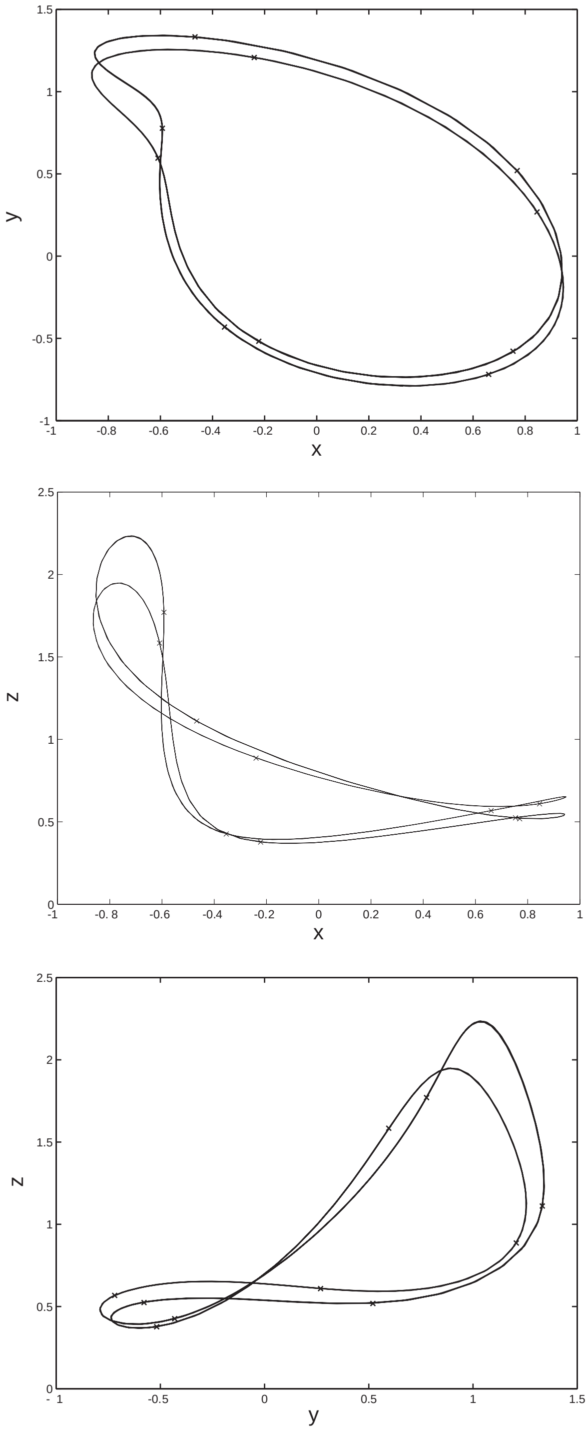


Fig. 9. Limit cycle of period 2 at $\mu = 0.4497$ of Eq. (5.2). (—) Runge–Kutta method, ($\times$) perturbation-incremental method.

circle through -1. Information of the limit cycle at μ^{**} is given in Table 3. To switch to the emanating branch, this limit cycle is rewritten in the form of Eq. (3.2) and is used as an initial solution for the continuation of the new branch. Fig. 8 shows the emanating branches from the first and second period-doubling bifurcations with $\max[x_1(\varphi)]$ as the ordinate. For the emanating branch from the kth period-doubling bifurcation, $\max[x_1(\varphi)] = \sum_{j=0}^{2^k} a_j \cos j\varphi + \sum_{j=1}^{2^k-1} b_j \sin j\varphi$ for all φ. (If amplitude is used as the ordinate instead of $\max[x_1(\varphi)]$ in Fig. 8, the emanating branches are almost indistinguishable from the original curve.) Second period-doubling bifurcations occur at $\mu \simeq 0.4767$ (label 5) and $\mu \simeq 0.4936$ (label 6). In this case, both the PI method and AUTO 97 fail to detect third period-doubling bifurcation.

Information of the period-2 limit cycle at $\mu = 0.4497$ (label 4) is given in Table 4. Phase portrait is shown in Fig. 9 and is compared to the result of the numerical integration obtained by using the fourth order Runge–Kutta method. It can be seen that they are in good agreement.

6. Conclusion

A simple but efficient algorithm for branch switching of period-doubling bifurcations of strongly non-linear autonomous oscillators with many degrees of freedom is described by using the perturbation-incremental method. When a period-doubling bifurcation is detected, the periodic solution at that bifurcation point is extended from M harmonics to $2M$ harmonics. A parameter b_1 is then turned on from zero to positive in order to obtain a solution on the emanating branch for subsequent continuation. Therefore, the calculation of tangent of the emanating branch and the second derivatives which is quite involved can all be avoided. Period-doubling bifurcation value can also be determined accurately by decreasing b_1 gradually to zero. Compared with the results obtained by using the bifurcation package AUTO 97, it is found that even higher period-doubling bifurcations can be obtained by using the PI method. Limit cycles obtained by using the PI method are compared with those from the Runge–Kutta method and they are in good agreement. The advantage of the PI method lies in its simplicity and ease of application.

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