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Eigenfrequency spacing analysis and eigenmode breakdown for semi-stadium-type 2-D fields

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Abstract

This paper describes eigenfrequency statistics including modal patterns and degrees of freedom for a semi-stadium-type 2-D field. The authors numerically investigated sound fields surrounded by 2-D semi-stadium-type boundaries as examples of boundaries where chaotic properties are hidden, in order to understand the characteristics of complex sound fields and to gain new insight into sound-field design. One limit of the semi-stadium boundaries is a rectangular boundary that gives a regular field, while another limit is a stadium boundary in which the chaotic property emerges. The numerical results show that eigenfrequency spacing in all the cases can be expressed as a family of Γ distributions extended to a non-integer degree of freedom. This fractal degree of freedom can be interpreted as the degree of freedom of the sound field. For the regular limit case, i.e., a rectangular case, the distribution is the exponential distribution with the freedom of unity, while in the chaotic case, i.e., the stadium case, it is the Wigner distribution with a degree of freedom of two. The authors, however, analyze the semi-stadium sound fields. The analysis of the fluctuations in the distribution function for the eigenfrequencies showed that the skewness decreases as the boundary approaches the stadium condition. Modal patterns also clearly showed breaks of the regular pattern of nodal lines in a rectangular case as the boundary was deformed from the rectangular to the stadium condition. The breaks of the modal pattern could be also confirmed by decreasing of the skewness for the sound pressure distribution.

1. Introduction

Chaotic properties are hidden in a sound ray propagated in a sound space surrounded by irregularly shaped boundaries. A stadium-type boundary is an example of the type of condition

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where chaotic properties are hidden. This type of chaotic phenomenon was pointed out by Berry [1] who proposed a new research area called quantum chaology. The main aim of quantum chaology is to find a new corresponding principle between the classical and quantum systems by studying the ‘‘fingerprints of chaos’’ that remain in linear-wave fields. The theoretical study of the fingerprints of waves, which are termed leftover in linear-wave theory and are caused by the emergence of chaotic behavior in sound ray propagation, is also important for sound-field analysis and soundspace design for irregularly shaped boundary conditions. Eigenfrequency and modal patterns are significant topics in the study of fingerprints.

This paper analyzes numerically, based on computer simulation, eigenfrequency statistics and the modal patterns in semi-stadium-type 2-D fields as examples of boundaries where chaotic properties are hidden. We define a family of semi-stadium shapes that can be continuously deformed from a rectangular to a chaotic-stadium boundary by changing the curvature of the circular portion of the stadium-boundary. The eigenfrequency spacing distribution for a stadium-boundary condition follows the Wigner distribution. (Gaussian orthogonal ensemble (GOE)/ Gaussian unitary ensemble (GUE) are known by the random matrix theory [2,3], where GOE and GUE correspond to the system of reverse time symmetry and non-symmetry. If the diagonal and upper diagonal elements are independently chosen in a random real symmetric $N \times N$ matrix X, then the ensemble of eigenvalues for the matrix X is GOE. If the diagonal elements (which must be real) and the upper triangular elements are independently chosen in a random Hermitian $N \times N$ matrix X, then the ensemble of eigenvalues for the matrix X is GUE.) Lyon [4] has already shown that this spacing generally follows the Wigner distribution (GOE) instead of the exponential distribution (Poisson), that spacing which follows in a rectangular case. Schroeder [5] introduced the distribution of frequency intervals between adjacent resonance frequencies of irregular cavity into eigenfrequency statistics on sound fields. The authors will show that eigenfrequency spacing in all cases can be expressed as a family of Γ distributions extended to a non-integer (fractal) degree of freedom, which can be interpreted as the degree of freedom of a sound field.

In other study fields, the GOE predictions were studied in various ways for the spectral statistics of the systems. It was discussed from the level spacing distribution for the rectangle membrane with point scatter by Seba [6] and Weaver and Sornette [7] and from the 3-D viewpoint by Shigehara and Cheon [8]. Cheon and Cohen discussed the implications of quantum spectra of 2-D pseudo-integrable billiards on the conjectured relation between classical chaos and quantum level statistics [9]. Weaver confirmed the predictions of the random matrix theory for the level repulsions and spectral rigidities of the higher eigenfrequencies of small aluminum blocks with slit cuts [10].

It is possible to interpret the spectrum analysis by using these studies as the characteristic analysis of complex sound fields in room acoustics. Our approach does not place any scatterer in the membrane and does not treat the pseudo-integrable systems. The authors purely review the shape that becomes ergodic in the classical system from the linear-wave fields. The authors, however, analyze the semi-stadium sound fields in order to understand the characteristics of complex sound fields from the point of view of future application to the design of sound fields. The authors discuss the semi-stadium family from Poisson to GOE.

The accumulated number of eigenfrequencies is also an important subject in the study of the characteristics of sound fields. This paper also reports the statistical analysis of the accumulated process of the number of eigenfrequencies. The authors will show that the skewness becomes lower as the boundary approaches the stadium condition. Moreover, modal patterns are used to show the break of the regular nodal pattern as the boundary is deformed from the rectangular to the stadium condition.

2. Eigenfrequency spacing statistics

The eigenfrequency spacing [3,4] is from the wave-theoretical point of view, a significant issue not only for fingerprint analysis but sound-field control as well. Examples of the spacing distributions for the rectangular and stadium cases are shown in Figs. 1(a) and (b), respectively. Here, the eigenfrequencies from the first to the 300th for odd–odd modes are counted from the theoretical results for the rectangular case (a) and from the numerical results obtained by FEM (b). We used a software package named MSC/emas for FEM and created about 100 000 meshes. The spacing follows the exponential distribution under the rectangular condition

$$P(x) = \frac{1}{\mu} e^{-x/\mu}, \tag{1}$$

while it follows the Wigner distribution

$$P(x) = \frac{4}{\mu^2} x e^{-2x/\mu}, \tag{2}$$

under the stadium condition. Here, the horizontal axis shows $x \equiv \Delta\omega/c$ where c is sound speed. We can see that degeneration occurred quite often for the rectangular case, while it was quite unlikely in the stadium case. We will show a transition of the exponential to the Wigner distribution by comparing examples of the rectangular and stadium boundaries.

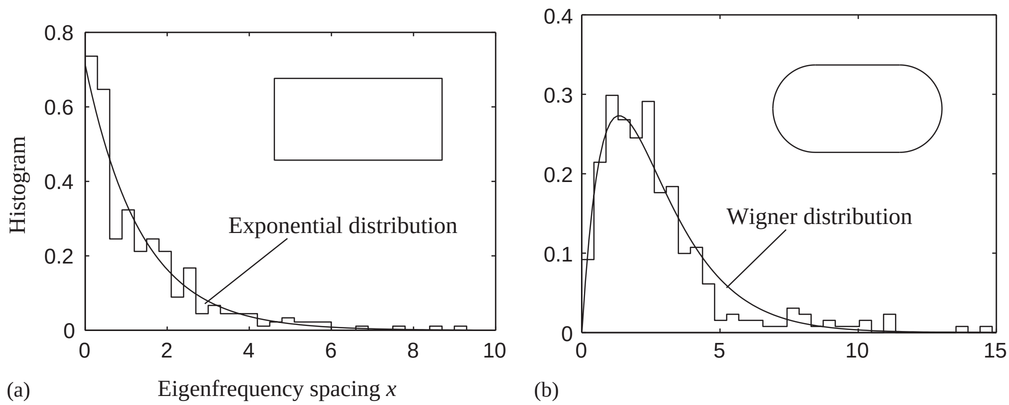

Fig. 1. (a) Eigenfrequency spacing distributions for rectangular and (b) stadium boundaries. $x \equiv \bar{\Delta}\omega/\bar{\Delta}\omega$, $\Delta\omega$ denotes the eigenfrequency spacing, and $\Delta\omega$ is the averaged spacing (exponential distribution: $(1/\mu)\exp(-x/\mu)$, Wigner distribution: $(4/\mu^2)x\exp(-2x/\mu)$).

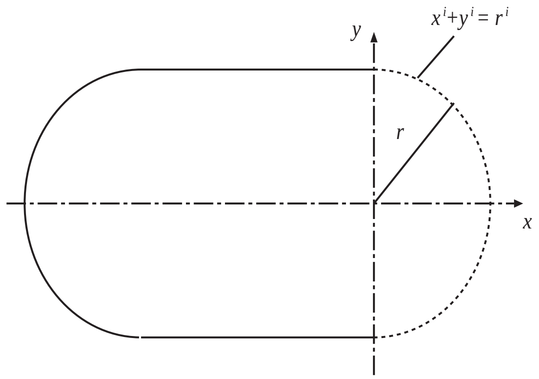

Fig. 2. Definition of semi-stadium-type 2-D boundary.

3. Transition from exponential to Wigner distributions in semi-stadium fields and degree of freedom of sound field

We define our 2-D field examples as a family of semi-stadium shapes which can be continuously deformed from a rectangular- to the chaotic-stadium boundary by changing the curvature of the circular portion of the stadium boundary. That is, the curved portion is expressed as $x^i + y^i = r^i$ (see Fig. 2). The shape of a semi-stadium is changed from a stadium to a rectangle as i is increased. The l^p unit ball [11] is also included in the higher order curvature shape which is changed from a square ($p = 1$) via a circle ($p = 2$) to square ($p \to \infty$) as is increased. Fig. 3 shows the spacing distributions of the eigenfrequencies when i=4,7,10, and 15. We found that the Γ distribution [12]

$$P(x, n) \equiv \frac{1}{\beta^n \Gamma(n)} x^{n-1} e^{-x/\beta}, \; x > 0, \tag{3}$$

($\beta = \mu/n$, x indicates the eigenvalue spacing, and μ indicates the mean of x) could be fitted to the distribution with a fractal degree of freedom n, 1.9, 1.5, 1.3, and 1.0 for each boundary, respectively. Here, the exponential distribution degree of freedom is unity, and the Wigner distribution degree of freedom is two. The authors already have found that the correlation dimension is about two for the history of sound-ray-reflection points in a stadium; however, the dimension stays at unity for the rectangular case [13]. Consequently, the authors surmised that this fractal degree of freedom could be interpreted as the freedom of the 2-D sound fields between the rectangular and irregular cases. This suggests that the almost same behavior (changing the distribution) appeared from the 3-D result [8] in not only the semi-stadium shape but also arbitrary shape.

4. Eigenfrequency distribution fluctuations

4.1. Δ_3 statistics

Eigenfrequency spacing statistics shows the local properties for an eigenfrequency distribution. However, the spectral rigidity or Δ_3 statistics, which were introduced by Dyson and Metha [14]

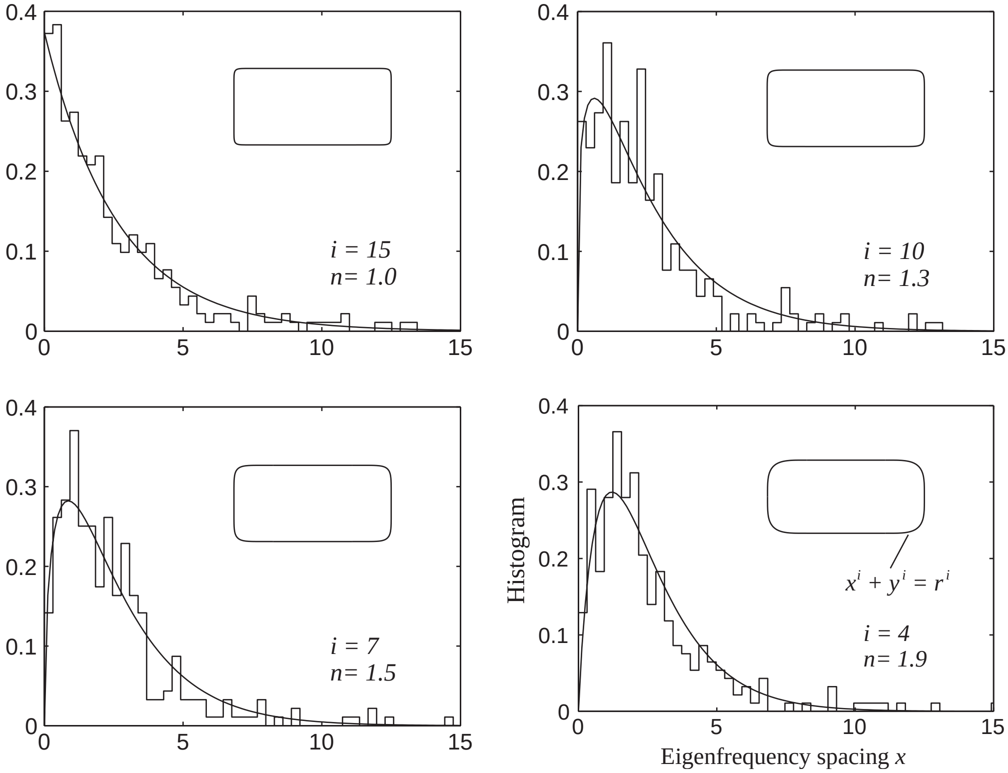

Fig. 3. Eigenfrequency distributions for semi-stadium boundaries. $x \equiv \Delta\omega/\bar{\Delta}\omega$, $\Delta\omega$ denotes the eigenfrequency spacing, and $\bar{\Delta}\omega$ is the averaged spacing (Γ distribution: $P(x,n) \equiv (1/(\beta^n \Gamma(n)))x^{n-1}\exp(-x/\beta)$).

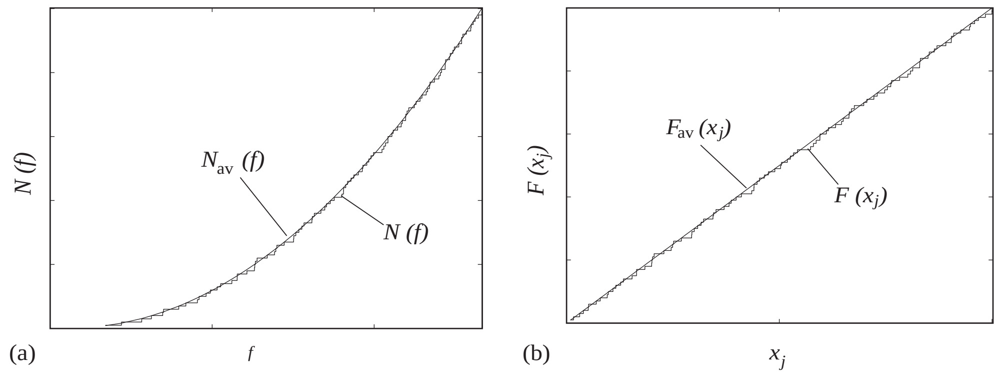

Fig. 4. (a) Accumulated distribution of eigenfrequency and (b) unfolded staircase function.

are useful for investigating the global fluctuation characteristics of the eigenfrequency distributions. We investigated such global characteristics as well as the spacing statistics for the semi-stadium family. Fig. 4(a) shows the accumulated distribution of eigenfrequencies that is illustrated by a staircase function for the stadium example. Here, the staircase function, that is, the accumulated number of eigenfrequencies below the frequency f, $N(f)$, can be expressed as

$$N(f) \equiv N_{av}(f) + N_{fluc}(f), \tag{4}$$

where $N_{av}(f)$ is the average distribution and $N_{fluc}(f)$ indicates the fluctuation term from the average. We transform the accumulated distribution by changing the variable as $x_j = N_{av}(f_j), F(x_j) = N(f_j)$ so that the Δ_3 statistics can be conducted.

The average number of the eigenfrequency becomes a straight line ($F_{av} = x_j$) by this transformation and, consequently, the fluctuation term can be expressed as the deviation from the straight line, as shown in Fig. 4(b). This fluctuation term gives us the spectrum rigidity introduced by Dyson and Metha [14], which shows the degree of the dispersion of the eigenfrequency distribution. The spectrum rigidity can be evaluated using the Δ_3 statistics,

$$\Delta_3(L, \alpha) = \frac{1}{L} \min A, B \int_{\alpha}^{\alpha+L} [F(x) - Ax - B]^2 \, dx, \tag{5}$$

where α shows the starting point for an observation interval and L denotes the length of the interval. The Δ_3 statistics can be interpreted as the least-square error from the linear approximation for the eigenfrequency distribution within an observation interval L on the frequency scale. Fig. 5 shows the averaged Δ_3 for the semi-staduim family, which was given by the average Δ_3 for all possible α's. Metha [2] gave theoretical solutions of the averaged Δ_3 for the regular (Poisson) and the irregular (GOE) systems

$$\bar{\Delta}_{3(Poisson)}(L) \simeq L/15, \tag{6}$$

$$\bar{\Delta}_{3(GOE)}(L) \simeq 1/\pi^2 \ln L - 0.007, \tag{7}$$

respectively, by using the random matrix theory. We can see that the global statistics of the eigenfrequency distributions changed from the irregular to the regular type of behavior as the curved portion of the boundary approach a rectangular shape in the semi-stadium family. We only have results up to $i = 15$, but we emphasize that the results appear to be approaching a Poisson result when i is very large, because in an integrable case, such as the rectangle, Δ_3 statistics follows a Poisson prediction [2] and the results of the number variance for the rectangles without scatterer follows a Poisson prediction shown by Weaver and Sornette [7]. The circle- and Africa-shaped cases of spectral rigidity were reviewed by Berry [15].

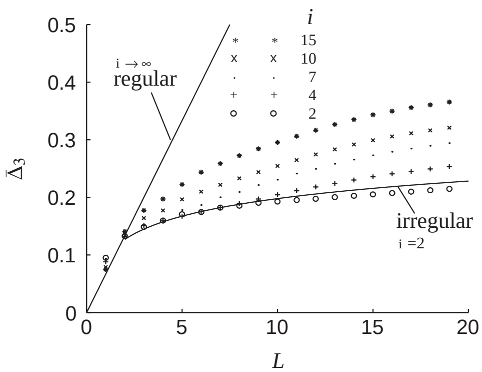

Fig. 5. Averaged Δ_3 statistics for semi-stadium boundaries.

4.2. Skewness of the fluctuations

Fig. 6 shows the fluctuation term for the stadium-type boundary. Aurich et al. [16] showed that the fluctuations for a chaotic system follow the Gaussian distribution. We evaluated the deviation from the Gaussian distribution by studying the skewness. The skewness was obtained by

$$\gamma_1 = \frac{\mu_3}{\sigma^3}, \tag{8}$$

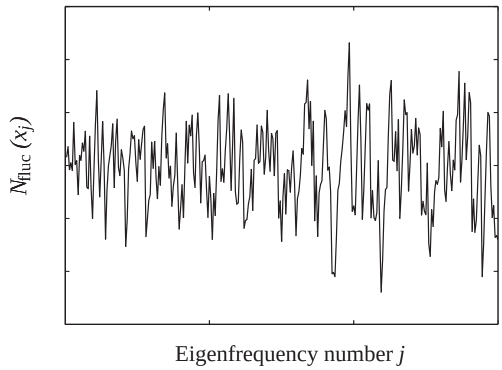

Fig. 6. N_{fluc} for stadium.

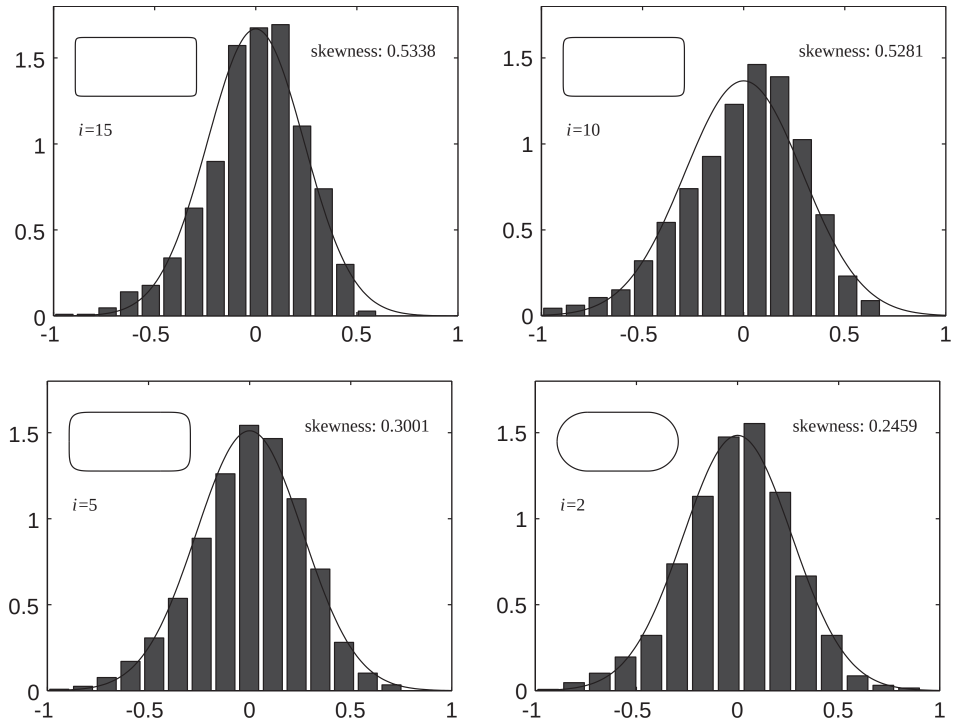

Fig. 7. Histogram of eigenfrequency fluctuation N_{fluc} and skewness.

where μ_v and σ indicate

$$\mu_v = \int_{-\infty}^{\infty} (x - m)^v \, dF(x), \tag{9}$$

and

$$\sigma = \sqrt{\mu_2}, \tag{10}$$

respectively. Fig. 7 shows the histograms of the fluctuations for the semi-stadium family. We can see that the fluctuation distribution changed from regular to irregular behavior (Gaussian distribution) as the curved portion of the boundary approached a stadium shape in the semi-stadium family. The skewness decreased as the boundary approached the stadium condition.

5. Modal patterns for the semi-stadium

Spatial distributions of modal patterns are also significant features used for the analysis of fingerprints and sound fields. The modal patterns of irregularly shaped rooms are of particular interest for evaluating the perturbed effects caused by the boundaries that affect the sound field,

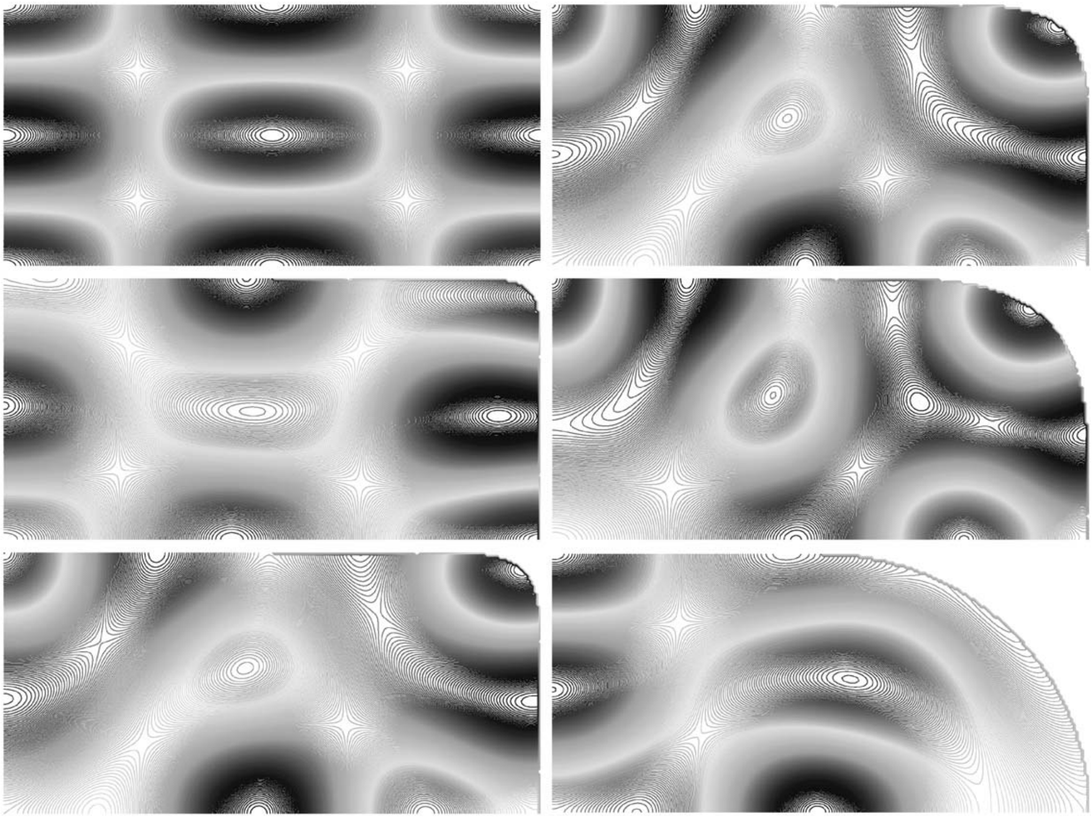

Fig. 8. Corresponding (2,2) rectangle mode for the semi-stadium. The left- and right-hand sides of this figure indicate the rectangle ($i = \infty$), $i = 15$ and $i = 10$, and $i = 7$, $i = 4$ and stadium ($i = 2$), from the top respectively.

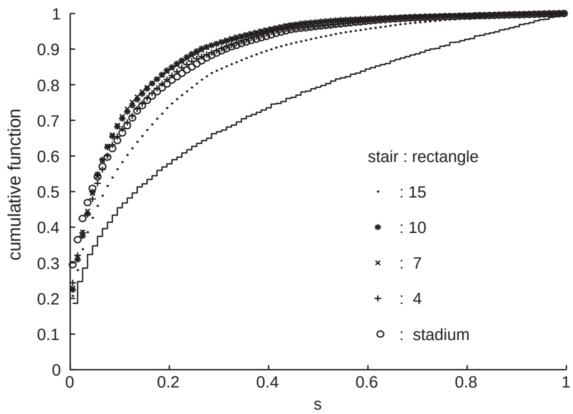

Fig. 9. Cumulative function for each shape.

particularly in regard to the acoustics of a reverberation room. The authors analyzed these modal patterns using the point-matching method for 2-D fields [17] (see Appendix A: this analysis is applied only in this section.) and calculated by using 20 nodes as shown in Fig. 10. Fig. 8 shows examples of lower modes that correspond to (2,2) modes in the rectangular portion. We can clearly see that the regular pattern of nodal lines in the rectangular case became greatly deformed as the room shape was changed to the stadium condition. Fig. 9 shows the accumulated distribution functions for the squared quantities sampled from the modal patterns. The low values of the samples are more likely to be observed in the semi-stadium cases rather than in the rectangular case [18,19].

However, a full evaluation of the perturbed effects from the point of view of the degree of freedom of the sound field has not yet been done.

6. Conclusion

The authors numerically identified that eigenfrequency spacing in 2-D semi-stadium boundaries can be expressed as a family of Γ distributions extended to a non-integer degree of freedom. This fractal degree of freedom can be surmised as the degree of freedom of the sound field. In a rectangular case, the degree of freedom was unity, while the degree of freedom was two in the chaotic (stadium) case. We calculated the Δ_3 statistics and the skewness of the eigenfrequency distribution fluctuation in order to investigate the global characteristics of the eigenfrequency distributions. The Δ_3 statistics clearly showed the change from irregular (GOE) to regular (Poisson) as the boundary was changed from the stadium to the rectangular condition. The fluctuations in the stadium condition followed the Gaussian distribution. It is shown that the degree of freedom can be determined for an arbitrary shaped boundary.

For the modal analysis, the modal patterns clearly showed breaks of the regular pattern of nodal lines seen in a rectangular case as a boundary was deformed. The relationship between the break of the modal pattern and the degree of freedom is a problem to be studied in the future. In future application, this type of modal statistics described in this article can be used for reverberation sound rendering, since only simple parameters such as room volume are required for rendering algorithm.

Appendix A. Point-matching method

In an infinite sound field, N nodes are placed along an imaginary contour, which is the exact same shape as the boundary of the cavity. A representation of the stadium shape is shown in Fig. 10. Then the sound pressure at point $\mathbf{r}$ of the field can be obtained according to the non-dimensional Green's function of the linear addition, if N waves circularly spread from each node are considered. Here J_0 indicates the Bessel function of the first kind of zero order.

$$\Psi(r) = \sum_{s=1}^{N} a_s J_0(k|\mathbf{r} - \mathbf{r}_s|). \tag{A.1}$$

The unknown coefficients $a_1, a_2, \ldots, a_N$ are determined by applying a given boundary condition to the function. The Neumann-boundary condition continuously defined along the boundary is given in the form

$$\frac{\partial \Psi(r_i)}{\partial n_i} = \bar{V}_n(\mathrm{r}_i), \quad i = 1, 2, \ldots, N. \tag{A.2}$$

In Eq. (A.2), n_i denotes the normal vector from the boundary, as it is shown in Fig. 10. The normal vector is determined approximately from the sum total of two vectors, if the contour lacks a smooth geometry when two normal vectors are given at a corner. By substituting Eq. (A.1) into the discrete-boundary condition Eq. (A.2) is given by

$$\sum_{i=1}^{N} a_s \frac{\partial}{\partial n_i} J_0(k|\mathbf{r}_i - \mathbf{r}_s|) = \bar{V}_n(\mathbf{r}_i), \quad i = 1, 2, \ldots, N. \tag{A.3}$$

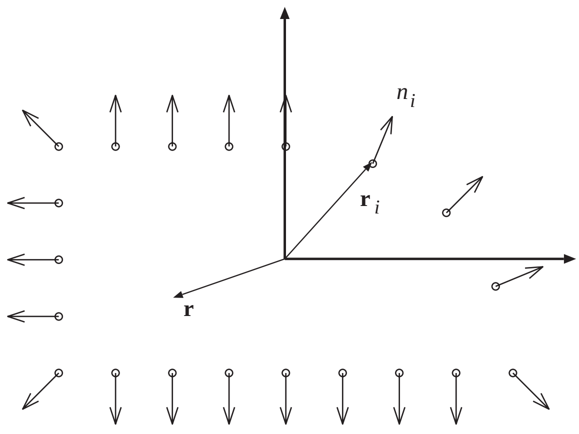

Fig. 10. Nodes and normal vectors for stadium shape.

Eq. (A.3) can be described as having the following matrix form, where the system matrix **SM** (k) of the order $N \times N$ is given by $\mathbf{SM}_{i,s}(k) = (\partial/\partial n_i)\mathrm{J}_0(k|\mathbf{r}_i - \mathbf{r}_s|)$,

$$\mathbf{SM}(k)\mathbf{a} = \mathbf{b}. \tag{A.4}$$

The column vector **b** is given by $b_i = \bar{V}_n(\mathbf{r}_i)$. The unknown coefficients **a** can be calculated as a following equation from Eq. (A.4).

$$\mathbf{a} = \mathbf{SM}(k)^{-1}\mathbf{b}. \tag{A.5}$$

In case of the rigid boundary ($\partial\Psi/\partial n = 0$), all elements of vector **b** become zero, i.e., $\mathbf{b} = 0$. Then Eq. (A.4) becomes the system equation

$$\mathbf{SM}(k)\mathbf{a} = 0. \tag{A.6}$$

By assuming that the place where the determinant of the system matrix **SM** in some k becomes zero is an eigenvalue k, the eigenvalue can be calculated from

$$\det[\mathbf{SM}(k)] = 0. \tag{A.7}$$

Eigenmodes can be displayed from Eq. (A.1) using the eigenvalue k and eigenvector **a**.

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