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Combined parametrical transverse and in-plane harmonic response of an inclined stretched string

Caswita*, A.H.P. Van Der Burgh

Department of Applied Mathematical Analysis, Faculty of Information Technology and Systems, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

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Abstract

In this paper, a two-point boundary value problem for an integrodifferential equation is studied. This equation describes the dynamics of an inclined stretched string suspended between a fixed support and a vibrating support. Due to the inclination, the string will vibrate under combined parametrical and transversal excitation. The attention will be focused on time-periodic solutions consisting of one mode (semi-trivial solution) generated by transverse (external) excitation and two modes (non-trivial solution) generated by combined parametrical and transverse excitation.

1. Introduction

Consider a perfectly flexible string in a stretched situation. At the end $x = 0$ the string is attached to a horizontal plane and at $x = 1$ the other end is fixed to a vertical rigid bar, which is excited in horizontal direction. A sketch of the vibrating system is given in Fig. 1. The case studied is when the system is embedded in an elastic medium where Hooke's law applies. In the literature [1–3], one usually studies vibrating strings positioned along a horizontal plane and excited either vertically or horizontally. Belhaq and Houssni [4] studied the dynamic response of a one-degree-of-freedom system with quadratic non-linearities and subjected to combined parametric and external excitations. The system can serve as a model for the one-mode vibration of a heavy elastic structure suspended between two fixed supports at the same level and excited by a quasi-periodic forcing. Perkins [5] studied modal interactions in the non-linear response of suspended elastic

*Corresponding author. Tel.: +31-15-278-4420; fax: +31-15-278-7209.
E-mail addresses: caswita@dv.twi.tudelft.nl, caswita@its.tudelft.nl (Caswita), a.h.p.vanderburgh@its.tudelft.nl (A.H.P. Van Der Burgh).

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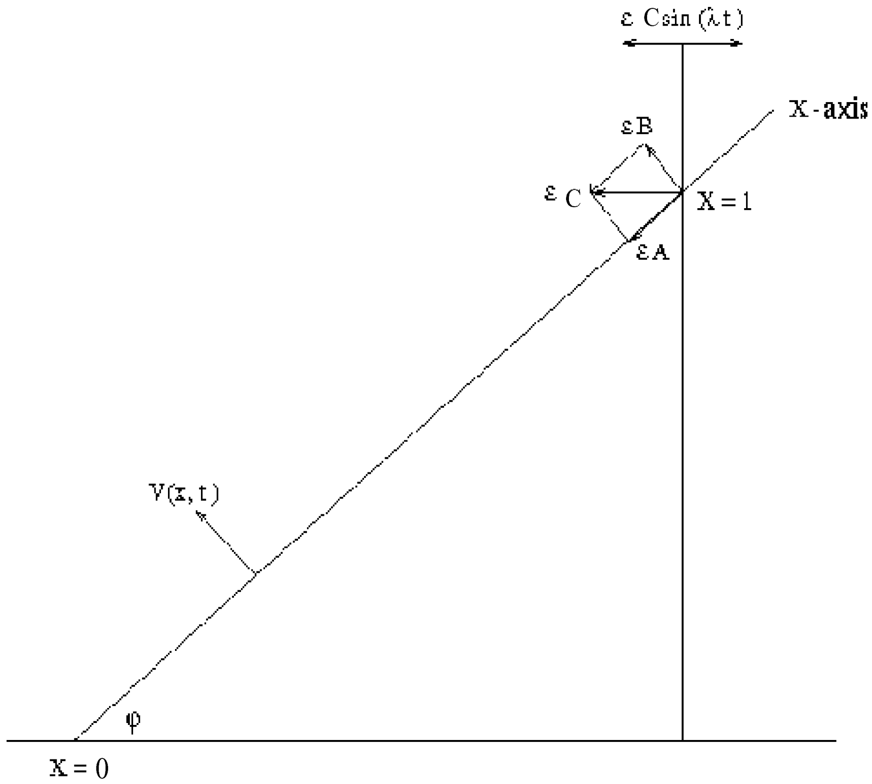

Fig. 1. A simple vibrating system including parametrical and transversal excitation of a stretched inclined string.

cables under parametrical and external excitations with horizontal position of the cable. Zhang and Tang [6] investigated analytically the global dynamic behaviour of an elastic cable under combined parametrical and external excitations. The system describes a model for the coupling between in- and out-of-plane modes of a suspended elastic cable between a fixed support and a vibrating support. Lilien and Pinto da Costa [7] studied pure parametric excitation of an inclined elastic cable with a small sag. He et al. [8] studied the control of seismic excitation of a cable-stayed bridge by means of special dampers.

In a recent paper by Nielsen and Kirkegaard [9], the in- and out-of-plane excitation of an inclined elastic cable with a small sag is investigated. In contrast to the present paper, primary parametric excitation due to longitudinal excitation is not considered by them. As is known from experimental work by, e.g., Melde [10], this primary parametric excitation in stretched strings may lead to a transverse response. In this paper, a system with combined transverse (i.e., perpendicular to the X-axis as indicated in Fig. 1) and longitudinal (i.e., in the direction of the X-axis) excitation due to the angle φ between the string and the horizontal plane will be investigated.

The mathematical model for this system is an extension of the model given in Refs. [1,11]. The extension concerns the term p^2V describing the elasticity of the medium in which the system is embedded and the time-periodic boundary condition at $x = 1$. The mathematical model obtained in this way is as follows:

$$\begin{aligned}
&\bar{V}_{\tau\tau}(x,\tau) - \bar{V}_{xx}(x,\tau) + p^2\bar{V}(x,\tau)\\
&\quad = \varepsilon\bar{V}_{xx}(x,\tau)\left[\frac{1}{2}\int_0^1 \bar{V}_x^2(x,\tau)\,\mathrm{d}x + A\sin(\lambda\tau)\right] - \varepsilon\alpha\bar{V}_\tau(x,\tau), \quad x\in(0,1),\ \ \tau>0,\\
&\bar{V}(0,\tau) = 0, \quad \tau>0,\\
&\bar{V}(1,\tau) = \varepsilon B\sin(\lambda\tau), \quad \tau>0,
\end{aligned} \tag{1}$$

where $0<\varepsilon\ll 1$, $\alpha\geqslant 0$, $p^2\geqslant 0$, $\lambda>0$, $A = C\cos(\varphi)$, $B = C\sin(\varphi)$ and $\bar{V}(x,\tau)$ is the transverse displacement. The magnitudes of A and B are of the same order implying that the angle φ between the string and the horizontal plane is of order 1. In model (1) gravitation is not considered, implying that there is no sag. Hence, the parametric excitation only applies to elastic elongations. Additional elongation due to the presence of the sag due to gravity will be studied in a subsequent paper. This type of elongation is also relevant in a practical situation. Model equation (1) is of particular relevance for shorter stays in cable-stayed bridges. In this paper, formal approximations to the solutions of the boundary value problem (1) will be constructed by using a Fourier mode expansion. Subsequently, the averaging method [12] will be applied and for special combinations of λ and p^2, periodic solutions will be studied. Values of λ and p^2 for which a periodic solution consisting of two modes exist, are determined. Those values will give rise to mode interaction. The interesting cases $p^2 = 0$ and λ near 2π are considered. Especially, the interaction between the first and the second mode is studied.

2. Analysis of the model equation

Consider the two-point boundary value problem for the integrodifferential equation (1) with a small parameter ε. To reduce the problem to a problem with homogeneous boundary values, the following transformation is used:

$$\bar{V}(x,\tau) = \varepsilon Bx\sin(\lambda\tau) + V(x,\tau). \tag{2}$$

Substitution of Eq. (2) into Eq. (1) yields

$$\begin{aligned}
&V_{\tau\tau} - V_{xx} + p^2V\\
&\quad = \varepsilon(\lambda^2 - p^2)Bx\sin(\lambda\tau) + \varepsilon V_{xx}\left[\frac{1}{2}\int_0^1 V_x^2\,\mathrm{d}x + A\sin(\lambda\tau)\right] - \varepsilon\alpha V_\tau + O(\varepsilon^2),\\
&V(0,\tau) = V(1,\tau) = 0.
\end{aligned} \tag{3}$$

The eigenfunctions and the eigenvalues of the Sturm–Liouville problem,

$$\begin{aligned}
&-\frac{\mathrm{d}^2V_n}{\mathrm{d}x^2} + p^2V_n = \mu_n^2V_n,\\
&V_n(0) = V_n(1) = 0,
\end{aligned} \tag{4}$$

related to homogeneous unperturbed system (3) (i.e., $\varepsilon = 0$) are $V_n(x) = q_n\sin(n\pi x)$ and $\mu_n^2 = p^2 + n^2\pi^2$, respectively, where q_n, $n = 1,2,3,\ldots$, are constants. The $O(\varepsilon^2)$ terms are neglected;

exact solutions $V(x,\tau)$ of Eq. (3) exist in the form of a Fourier sine-series (eigenfunction-series):

$$V(x,\tau) = \sum_{n=1}^{\infty} q_n(\tau)\sin(n\pi x). \tag{5}$$

By substituting Eq. (5) into Eq. (3) and using the orthogonal properties of the eigenfunctions, one obtains an infinite-dimensional system for q_n:

$$\ddot{q}_n + \mu_n^2 q_n = -\varepsilon\left\{n^2\pi^2 q_n\left(\sum_{k=1}^{\infty}\frac{1}{4}k^2\pi^2 q_k^2 + A\sin(\lambda\tau)\right) + \alpha\dot{q}_n - (\lambda^2 - p^2)Bc_n\sin(\lambda\tau)\right\}, \tag{6}$$

where $c_n = (-1)^{(n+1)}(2/n\pi)$, $n = 1, 2, \ldots$.

Notice that system (6) can be expected to have solutions whenever the series in the right-hand side converges. It is assumed that $V(x,\tau)$ is a twice continuously differentiable function with respect to x on the open interval $(0,1)$. In the framework of this paper, the assumption about the differentiability is satisfied as follows conditions on initial values as given Ref. [13]. In the actual paper, the attention is focused on periodic solutions implying that no initial values are considered. However, one can associate initial values as defined in Ref. [13] with the periodic solutions studied in this paper. By using integration by parts, it follows that the coefficients q_n of series (5) are of order n^{-2}. Therefore, the series in the right-hand side of Eq. (6) converges if all q_n are sufficiently small.

In system (6), the frequency of the vertical and parametrical excitations is λ. There are two values of λ which are of interest for the study of periodic solutions:

$$\text{(i)}\ \lambda = \mu_m, \quad \text{(ii)}\ \lambda = 2\mu_s,$$

where m and s are certain values of n. As is well known, (i) corresponds to elementary resonance whereas (ii) corresponds to parametrical resonance. If no positive integers m and s exist such that (i) and/or (ii) hold then the effects of the excitation on the solutions are of higher order, i.e., of order ε^2. Both types of resonance can be expected if $\mu_m = 2\mu_s$, i.e.,

$$p^2 + m^2\pi^2 = 4p^2 + 4s^2\pi^2 \quad \text{or} \quad 3p^2 = (m^2 - 4s^2)\pi^2. \tag{7}$$

In other words, by choosing the integers m and s such that $m^2 - 4s^2 \geqslant 0$ and p such that Eq. (7) holds, then λ is defined by $\lambda = \mu_m = 2\mu_s$ and two types of resonance corresponding to finite amplitude oscillations may be expected.

Introduce the transformation $(q_n(\tau),\ \dot{q}_n(\tau)) \rightarrow (A_n(\tau), B_n(\tau))$ as follows:

$$\begin{aligned} q_n(\tau) &= A_n(\tau)\sin(\mu_n\tau) + B_n(\tau)\cos(\mu_n\tau),\\ \dot{q}_n(\tau) &= \mu_n[A_n(\tau)\cos(\mu_n\tau) - B_n(\tau)\sin(\mu_n\tau)]. \end{aligned} \tag{8}$$

Substitute Eq. (8) into Eq. (6) and solve the equations for $\dot{A}_n(\tau)$ and $\dot{B}_n(\tau)$, giving

$$\begin{aligned} \dot{A}_n &= -\varepsilon F_n(\mathbf{A}, \mathbf{B}; \varphi, \alpha, \tau)\cos(\mu_n\tau),\\ \dot{B}_n &= \varepsilon F_n(\mathbf{A}, \mathbf{B}; \varphi, \alpha, \tau)\sin(\mu_n\tau), \end{aligned} \tag{9}$$

where $\mathbf{A} = (A_1, A_2, \ldots, A_n, \ldots)$, $\mathbf{B} = (B_1, B_2, \ldots, B_n, \ldots)$, and

$$F_n = \frac{1}{\mu_n}\Bigg\{(A_n \sin(\mu_n\tau) + B_n\cos(\mu_n\tau))\Bigg[\sum_{k=1}^{\infty}\frac{1}{8}n^2\pi^4k^2((A_k^2 + B_k^2) + (B_k^2 - A_k^2)\cos(2\mu_k\tau) + 2A_kB_k\sin(2\mu_k\tau)) + n^2\pi^2 A\sin(\lambda\tau)\Bigg] + \alpha\mu_n(A_n\cos(\mu_n\tau) + B_n\sin(\mu_n\tau)) - (\lambda^2 - p^2)Bc_n\sin(\lambda\tau)\Bigg\}.$$

As is known from the theory of averaging, the averaged equations have solutions $\bar{A}_n(t)$ and $\bar{B}_n(t)$ which are $O(\varepsilon)$ approximations to $A_n(t)$ and $B_n(t)$, respectively, on a long $1/\varepsilon$ time-scale. Isolated stable critical points of the averaged system correspond with stable (quasi-) periodic solutions of Eq. (9) in case that the systems are finite dimensional. Here, however, Eq. (9) is an infinite-dimensional system. To the knowledge of the authors it seems not to be known whether these stability properties of the averaged system correspond with stability properties of Eq. (9). However, it is assumed that these results for finite-dimensional systems also hold for infinite-dimensional systems.

The terms on the right-hand side of Eq. (9) are periodic functions with respect to τ. This means that one can approximate the functions A_n and B_n, for all n, by using the averaging method. In order to apply this method to system Eq. (9), the value of λ must be determined. For $\lambda \neq \mu_k$ and $\lambda \neq 2\mu_k$, k is an arbitrary positive integer, the averaged equations of Eq. (9) are as follows:

$$\begin{aligned}\dot{\bar{A}}_n &= -\frac{1}{2}\varepsilon\left(\alpha\bar{A}_n + \frac{n^2\pi^4}{8\mu_n}\bar{B}_n\left[\sum_{k=1}^{\infty}k^2(\bar{A}_k^2 + \bar{B}_k^2) + \frac{1}{2}n^2(\bar{A}_n^2 + \bar{B}_n^2)\right]\right),\\ \dot{\bar{B}}_n &= -\frac{1}{2}\varepsilon\left(\alpha\bar{B}_n - \frac{n^2\pi^4}{8\mu_n}\bar{A}_n\left[\sum_{k=1}^{\infty}k^2(\bar{A}_k^2 + \bar{B}_k^2) + \frac{1}{2}n^2(\bar{A}_n^2 + \bar{B}_n^2)\right]\right),\end{aligned} \tag{10}$$

for $n = 1, 2, 3, \ldots$. From Eq. (10), it follows that if $\bar{A}_n(0) = \bar{B}_n(0) = 0$, then $\bar{A}_n(\tau) \equiv \bar{B}_n(\tau) \equiv 0$ for $\forall\tau > 0$. It means that if there is no initial energy in the nth mode, there will be no energy present up to $O(\varepsilon)$ on a time-scale of order ε^{-1}. This allows one to truncate to those modes that have non-zero initial energy. For $\lambda = \mu_m$ and $\lambda \neq 2\mu_s$, where m and s are certain values of n, an extra constant term only occurs in the equation for $\bar{B}_m$ but not in the one for $\bar{A}_m$. For $\lambda = 2\mu_s$ and $\lambda \neq \mu_m$, an extra term multiplying $\bar{A}_s$ and $\bar{B}_s$ in the equation for $\bar{A}_s$ and $\bar{B}_s$ occurs. Thus, if $\lambda = \mu_m = 2\mu_s$ (full resonance) so that p satisfies Eq. (7), extra terms occur in the equations for $\bar{A}_s$, $\bar{B}_s$, and $\bar{B}_m$. After some calculations, the obtained averaged equations for $\bar{A}_n$ and $\bar{B}_n$, $n = s, m$, are as follows:

$$\begin{aligned}\dot{\bar{A}}_m &= -\frac{1}{2}\varepsilon\left(\alpha\bar{A}_m + \frac{m^2\pi^4}{8\mu_m}\bar{B}_m\left[\sum_{k=1}^{\infty}k^2(\bar{A}_k^2 + \bar{B}_k^2) + \frac{1}{2}m^2(\bar{A}_m^2 + \bar{B}_m^2)\right]\right),\\ \dot{\bar{B}}_m &= -\frac{1}{2}\varepsilon\left(\alpha\bar{B}_m - \frac{m^2\pi^4}{8\mu_m}\bar{A}_m\left[\sum_{k=1}^{\infty}k^2(\bar{A}_k^2 + \bar{B}_k^2) + \frac{1}{2}m^2(\bar{A}_m^2 + \bar{B}_m^2)\right] + (-1)^{m+1}B\frac{2m\pi}{\mu_m}\right),\end{aligned}$$

$$\dot{\bar{A}}_s = -\frac{1}{2}\varepsilon\left(\left(\alpha + \frac{s^2\pi^2}{2\mu_s}A\right)\bar{A}_s + \frac{s^2\pi^4}{8\mu_s}\bar{B}_s\left[\sum_{k=1}^{\infty} k^2(\bar{A}_k^2 + \bar{B}_k^2) + \frac{1}{2}s^2(\bar{A}_s^2 + \bar{B}_s^2)\right]\right),$$
$$\dot{\bar{B}}_s = -\frac{1}{2}\varepsilon\left(\left(\alpha - \frac{s^2\pi^2}{2\mu_s}A\right)\bar{B}_s - \frac{s^2\pi^4}{8\mu_s}\bar{A}_s\left[\sum_{k=1}^{\infty} k^2(\bar{A}_k^2 + \bar{B}_k^2) + \frac{1}{2}s^2(\bar{A}_s^2 + \bar{B}_s^2)\right]\right). \quad (11)$$

For $n \neq s$ and $n \neq m$, the equations for $\bar{A}_n$ and $\bar{B}_n$ are given by Eq. (10). Supposing that $\bar{A}_n(0) = \bar{B}_n(0) = 0$ it follows that $\bar{A}_n(\tau) \equiv 0$ and $\bar{B}_n(\tau) \equiv 0$ for $n \neq s$ and $n \neq m$. Substitution of this result into Eq. (11) yields the following system:

$$\dot{\bar{A}}_m = -\frac{1}{2}\varepsilon\left[\alpha\bar{A}_m + \frac{m^2\pi^4}{16\mu_m}\bar{B}_m(2s^2(\bar{A}_s^2 + \bar{B}_s^2) + 3m^2(\bar{A}_m^2 + \bar{B}_m^2))\right],$$
$$\dot{\bar{B}}_m = -\frac{1}{2}\varepsilon\left[\alpha\bar{B}_m - \frac{m^2\pi^4}{16\mu_m}\bar{A}_m\big(2s^2(\bar{A}_s^2 + \bar{B}_s^2) + 3m^2(\bar{A}_m^2 + \bar{B}_m^2)\big) + (-1)^{m+1}B\frac{2m\pi}{\mu_m}\right],$$
$$\dot{\bar{A}}_s = -\frac{1}{2}\varepsilon\left[\left(\alpha + \frac{s^2\pi^2}{2\mu_s}A\right)\bar{A}_s + \frac{s^2\pi^4}{16\mu_s}\bar{B}_s(3s^2(\bar{A}_s^2 + \bar{B}_s^2) + 3m^2(\bar{A}_m^2 + \bar{B}_m^2))\right],$$
$$\dot{\bar{B}}_s = -\frac{1}{2}\varepsilon\left[\left(\alpha - \frac{s^2\pi^2}{2\mu_s}A\right)\bar{B}_s - \frac{s^2\pi^4}{16\mu_s}\bar{A}_s(3s^2(\bar{A}_s^2 + \bar{B}_s^2) + 2m^2(\bar{A}_m^2 + \bar{B}_m^2))\right]. \quad (12)$$

It may be clear that there is a coupling between the modes s and m and if one wants to truncate series (5) for the study of resonance one has to take into account at least m modes, where m is determined by the values of p and λ.

Based on the above results it follows that only specific combinations of λ and p^2 may give periodic solutions. The values of p^2 and λ for which periodic solutions consisting of two modes are found are called critical values and can easily be determined. For these critical values, mode interaction will occur. The critical values of p^2 with the corresponding values of λ are given in Table 1.

3. Model interaction for the specific combinations of values of p^2 and λ

As stated above only the equations for $n = s$ and $n = m$ are considered. An interesting case is $p = 0$ corresponding to $m = 2s$ and $\lambda = 2s\pi$. The model equation as presented here can be used for the study of the dynamics of inclined stay-cables connecting the bridge deck and pylon of a cable-stayed bridge by assuming that the motion of the deck can be ignored. In terms of accuracy of the model equation one could say that the motion of the bridge deck at the endpoint of the stay-cable is assumed to be of $O(\varepsilon^2)$. Substitution of $q_n(\tau) = 0$, $n \neq s$ and $n \neq m$, into Eq. (6), gives the two coupled second order equations:

$$\ddot{q}_m + \mu_m^2 q_m = -\varepsilon(m^2\pi^2 q_m[\tfrac{1}{4}s^2\pi^2 q_s^2 + \tfrac{1}{4}m^2\pi^2 q_m^2 + A\sin(\lambda\tau)] + \alpha\dot{q}_m - (\lambda^2 - p^2)Bc_m\sin(\lambda\tau)),$$
$$\ddot{q}_s + \mu_s^2 q_s = -\varepsilon(s^2\pi^2 q_s[\tfrac{1}{4}s^2\pi^2 q_s^2 + \tfrac{1}{4}m^2\pi^2 q_m^2 + A\sin(\lambda\tau)] + \alpha\dot{q}_s - (\lambda^2 - p^2)Bc_s\sin(\lambda\tau)). \quad (13)$$

It may be clear that if $\lambda \neq \mu_m$ and $\lambda \neq 2\mu_s$, using transformation (8), the averaged system of Eq. (13) consists of four first order equations with a structure similar to Eq. (10) and has as critical point $(0, 0, 0, 0)$ which is stable for $\alpha > 0$. If $\lambda = \mu_m$ but $\lambda \neq 2\mu_s$ then $(0, 0, 0, 0)$ is not a critical point unless $\varphi = 0$ (or $O(\varepsilon)$), corresponding to $B = 0$. In this case, the stable critical point $(0, 0, \bar{A}_m, \bar{B}_m)$ is

Table 1
The critical values of p^2 and λ

$p^2 = \frac{1}{3}\pi^2(m^2 - 4s^2)$	$\lambda^2 = \frac{4}{3}\pi^2(m^2 - s^2)$	Critical points One mode : two modes
$0\ (m = 2s)$	$4\pi^2\ (m = 2)$	$(0, 0, \bar{A}_{20}, \bar{B}_{20}) : (\bar{A}_{12}, \bar{B}_{12}, \bar{A}_{21}, \bar{B}_{21})$
	$16\pi^2\ (m = 4)$	$(0, 0, \bar{A}_{40}, \bar{B}_{40}) : (\bar{A}_{24}, \bar{B}_{24}, \bar{A}_{42}, \bar{B}_{42})$
	$36\pi^2\ (m = 6)$	$(0, 0, \bar{A}_{60}, \bar{B}_{60}) : (\bar{A}_{36}, \bar{B}_{36}, \bar{A}_{63}, \bar{B}_{63})$
	$\vdots$	$\vdots$
	$\lambda = 4s^2\pi^2$	$(0, 0, \bar{A}_{m0}, \bar{B}_{m0}) : (\bar{A}_{sm}, \bar{B}_{sm}, \bar{A}_{ms}, \bar{B}_{ms})$
$\frac{5}{3}\pi^2\ (s = 1,\ m = 3)$	$\frac{32}{3}\pi^2$	$(0, 0, \bar{A}_{30}, \bar{B}_{30}) : (\bar{A}_{13}, \bar{B}_{13}, \bar{A}_{31}, \bar{B}_{31})$
$4\pi^2\ (s = 1,\ m = 4)$	$20\pi^2$	$(0, 0, \bar{A}_{40}, \bar{B}_{40}) : (\bar{A}_{14}, \bar{B}_{14}, \bar{A}_{41}, \bar{B}_{41})$
$3\pi^2\ (s = 2,\ m = 5)$	$28\pi^2$	$(0, 0, \bar{A}_{50}, \bar{B}_{50}) : (\bar{A}_{25}, \bar{B}_{25}, \bar{A}_{52}, \bar{B}_{52})$
$\vdots$	$\vdots$	$\vdots$

found. If $\lambda \neq \mu_m$ and $\lambda = 2\mu_s$ then in the $(\varphi - \alpha)$ parameter plane there is domain defined by $(\cos(\varphi) - 2(\alpha\mu_s/Cs^2\pi^2)) > 0$ where there are two critical points: one, the origin, is unstable; the other, $(\bar{A}_s, \bar{B}_s, 0, 0)$, is stable. In the complement of this domain the only critical point is the stable origin. All cases discussed above concern systems with one stable critical point and hence one mode, implying there is no interaction between modes s and m. Therefore, in what follows only the case $\lambda = \mu_m + O(\varepsilon) = 2\mu_s + O(\varepsilon)$ will be studied. By setting $\lambda\tau = 2t$, where $\lambda = 2(\mu_s + \varepsilon\eta)$, system (13) becomes

$$
\begin{aligned}
q_m''(t) + 4q_m(t) = & -\frac{\varepsilon}{\mu_s}\left[\frac{m^2\pi^2}{\mu_s} q_m(t)\left(\frac{1}{4}s^2\pi^2 q_s^2(t) + \frac{1}{4}m^2\pi^2 q_m^2(t) + A\sin(2t)\right)\right. \\
& \left. + \alpha q_m'(t) - \beta_m \sin(2t) - 8\eta q_m\right] + O(\varepsilon^2), \\
q_s''(t) + q_s(t) = & -\frac{\varepsilon}{\mu_s}\left[\frac{s^2\pi^2}{\mu_s} q_s(t)\left(\frac{1}{4}s^2\pi^2 q_s^2(t) + \frac{1}{4}m^2\pi^2 q_m^2(t) + A\sin(2t)\right)\right. \\
& \left. + \alpha q_s'(t) - \beta_s \sin(2t) - 2\eta q_s(t)\right] + O(\varepsilon^2),
\end{aligned} \tag{14}
$$

where $\beta_m = (-1)^{m+1} 2(m\pi/\mu_s)B$, $\beta_s = (-1)^{s+1} 2(m^2\pi/s\mu_s)B$, and η is detuning coefficient of the frequency of excitation. A prime denotes differentiation with respect to t. For the sake of simplicity, the case λ near 2π ($p = 0$) is considered. This value implies a system describing the interaction between first ($s = 1$) and second ($m = 2$) modes:

$$
\begin{aligned}
q_2'' + 4q_2 &= -\frac{\varepsilon}{\pi}\left[4\pi q_2\left(\frac{1}{4}\pi^2 q_1^2 + \pi^2 q_2^2 + A\sin(2t)\right) + \alpha q_2' + 4B\sin(2t) - 8\eta q_2\right], \\
q_1'' + q_1 &= -\frac{\varepsilon}{\pi}\left[\pi q_1\left(\frac{1}{4}\pi^2 q_1^2 + \pi^2 q_2^2 + A\sin(2t)\right) + \alpha q_1' - 8B\sin(2t) - 2\eta q_1\right].
\end{aligned} \tag{15}
$$

In the first equation of Eq. (15), the excitation term $4B\sin(2t)$ is relevant for having an $O(1)$ amplitude response, whereas in the second equation $\pi q_1 A\sin(2t)$ is the relevant excitation term. Clearly, the first term describes ordinary and the second one, parametric resonance. System (15) can be used for the study of rotor-bearings system as well [14].

By using transformation (8) for $n=1$ and $n=2$, the following averaged system is obtained:

$$\begin{aligned}
\bar{A}_2' &= -\frac{\varepsilon}{2\pi}(\alpha\bar{A}_2 + \bar{B}_2[\gamma_2((\bar{A}_1^2+\bar{B}_1^2)+6(\bar{A}_2^2+\bar{B}_2^2))-4\eta]),\\
\bar{B}_2' &= -\frac{\varepsilon}{2\pi}(\alpha\bar{B}_2 - \bar{A}_2[\gamma_2((\bar{A}_1^2+\bar{B}_1^2)+6(\bar{A}_2^2+\bar{B}_2^2))-4\eta]-2B),\\
\bar{A}_1' &= -\frac{\varepsilon}{2\pi}(\beta_+\bar{A}_1 + \bar{B}_1[\gamma_1(3(\bar{A}_1^2+\bar{B}_1^2)+8(\bar{A}_2^2+\bar{B}_2^2))-2\eta]),\\
\bar{B}_1' &= -\frac{\varepsilon}{2\pi}(\beta_-\bar{B}_1 - \bar{A}_1[\gamma_1(3(\bar{A}_1^2+\bar{B}_1^2)+8(\bar{A}_2^2+\bar{B}_2^2))-2\eta]),
\end{aligned} \tag{16}$$

where $\gamma_1 = \frac{1}{16}\pi^3$, $\gamma_2 = \frac{1}{4}\pi^3$, and $\beta_\pm = \alpha \pm \frac{1}{2}\pi A$.

In what follows, the critical points and their dependence on the parameters α, η, A and B will be investigated. Recall that these parameters are supposed to be $O(1)$ and that A and B are defined by $A = C\cos(\varphi)$ and $B = C\sin(\varphi)$. The following special cases will be studied.

3.1. The case without damping, i.e. $\alpha = 0$

From the first two equations of Eq. (16), it follows that $\bar{B}_2 = 0$. As a consequence, the following system of algebraic equation is obtained:

$$\begin{aligned}
&\bar{A}_2[\gamma_2((\bar{A}_1^2+\bar{B}_1^2)+6\bar{A}_2^2)-4\eta]+2B=0,\\
&\beta\bar{A}_1 + \bar{B}_1[\gamma_1(3(\bar{A}_1^2+\bar{B}_1^2)+8\bar{A}_2^2)-2\eta]=0,\\
&\beta\bar{B}_1 + \bar{A}_1[\gamma_1(3(\bar{A}_1^2+\bar{B}_1^2)+8\bar{A}_2^2)-2\eta]=0,
\end{aligned} \tag{17}$$

where $\beta = \frac{1}{2}\pi A$. From the last two equations, it follows that $\bar{A}_1 = \pm\bar{B}_1$. Clearly, the following type of critical points are found:

$$\begin{aligned}
&\text{type 1:}\quad (\bar{A}_1,\bar{B}_1,\bar{A}_2,\bar{B}_2) = (0,0,\bar{A}_2,0), &&\text{type 1 CP},\\
&\text{type 2:}\quad (\bar{A}_1,\bar{B}_1,\bar{A}_2,\bar{B}_2) = (\bar{A}_1,\bar{A}_1,\bar{A}_2,0), &&\text{type 2 CP},\\
&\text{type 3:}\quad (\bar{A}_1,\bar{B}_1,\bar{A}_2,\bar{B}_2) = (\bar{A}_1,-\bar{A}_1,\bar{A}_2,0), &&\text{type 3 CP}.
\end{aligned} \tag{18}$$

The first type of critical points are on the $\bar{A}_2$-axis and can be studied as solutions of a cubical equation. Particularly, their dependence on η is well known and is depicted in Fig. 2.

In what follows, a (φ,η)-diagram will be constructed which gives an overview of all possible critical points for C fixed. Starting with $\bar{A}_1 = \bar{B}_1$, system (17) can be rewritten as

$$\begin{aligned}
&\bar{A}_2^3 - \frac{8}{5\pi^3}(\eta+\beta)\bar{A}_2 + \frac{12B}{5\pi^3} = 0,\\
&(4\eta - 2\beta) - \pi^3\bar{A}_2^2 > 0,\\
&\bar{A}_1^2 = \frac{8}{3\pi^3}(2\eta-\beta) - \frac{4}{3}\bar{A}_2^2.
\end{aligned} \tag{19}$$

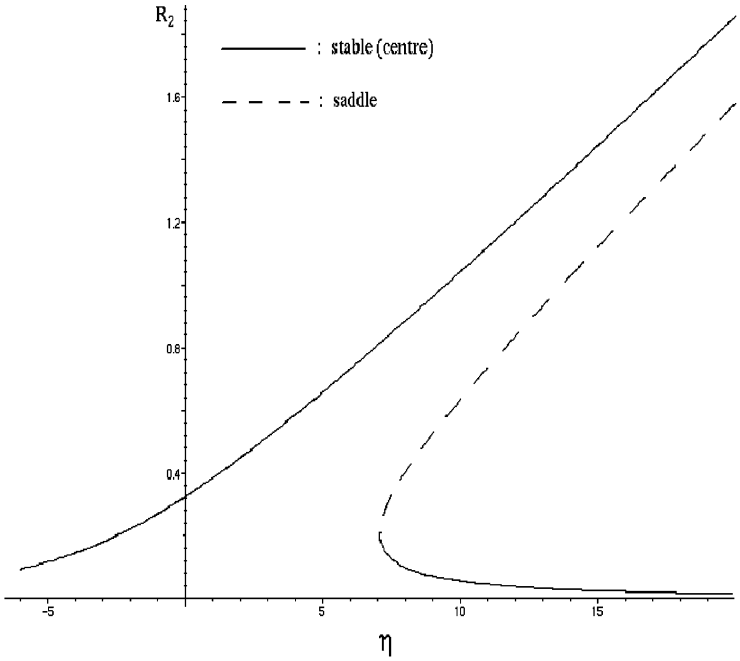

Fig. 2. The frequency–response curve $R_2 = \bar{A}_2^2$ of Eq. (16) with $\varphi = \frac{1}{4}\pi$ and $C = 6$: The stability only applies to the $(\bar{A}_2, \bar{B}_2)$-plane.

The case $4\eta - 2\beta - \pi^3\bar{A}_2^2 = 0$ corresponds with the first type of critical points in Eq. (18). System (19) defines domains in the $(\eta - \varphi)$-plane where there exists one, two, or three real non-zero solutions $(\bar{A}_1, \bar{A}_1, \bar{A}_2, 0)$. These domains are found by determining the boundary curves which follow from $\Delta = 0$, where Δ is the discriminant of the cubic equation in standard form in Eq. (19) and the equality $(4\eta - 2\beta) - \pi^3\bar{A}_2^2 = 0$ holds. In this equality, $\bar{A}_2$ as solution (obtained from the Cardano formulas) depending on η and φ is substituted. As a result one obtains Fig. 3. In this figure, there are four domains I–IV with one, two, or three critical points: I—1, II—2, III—3, IV—1. Here, II—2 means that in domain II there exist two critical points of the type $(\bar{A}_1, \bar{A}_1, \bar{A}_2, 0)$. When one looks separately at the case $\bar{A}_1 = \bar{B}_1 = 0$ corresponding to the so-called semi-trivial solution one obtains the following cubical equation for $\bar{A}_2$:

$$\frac{3}{2}\pi^3\bar{A}_2^3 - 4\eta\bar{A}_2 + 2B = 0. \tag{20}$$

This equation differs from the first equation in Eq. (19). By setting the discriminant $\Delta = 0$ one obtains a curve indicated $P_7{-}P_9$ in Fig. 4 on which there are two critical points of the type $(0, 0, \bar{A}_2, 0)$. On the left-hand side, there is one critical point and on the right-hand side there are three critical points. In the domains in Fig. 4, the notation (n, m) means that there are n critical points of type $(0, 0, \bar{A}_2, 0)$ and m of type $(\bar{A}_1, \bar{A}_1, \bar{A}_2, 0)$.

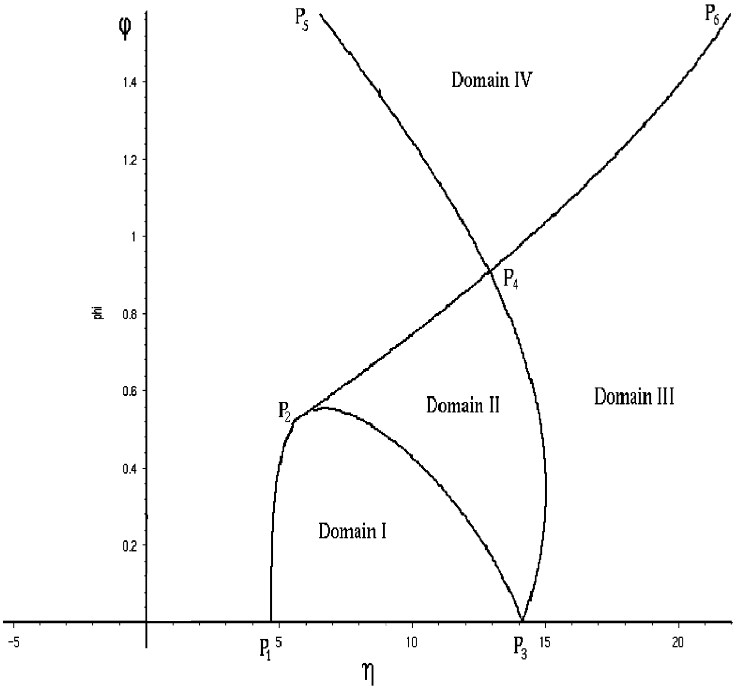

Fig. 3. Four domains with real solutions of system (19) for $C = 6$.

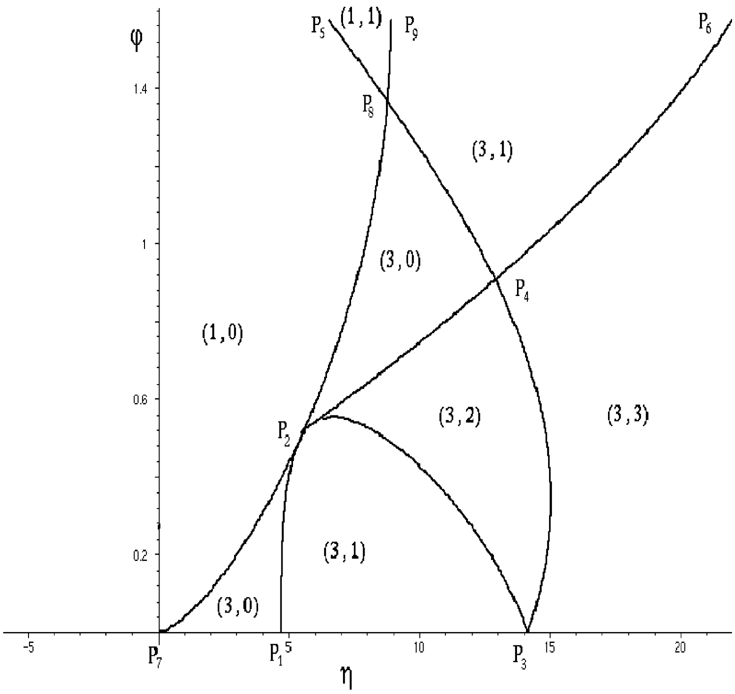

Fig. 4. Eight domains with real solutions of types 1 and 2 of system (17) for $C = 6$.

In a similar way, the critical points of type $(\bar{A}_1, -\bar{A}_1, \bar{A}_2, 0)$ are analyzed. They are found from the system

$$\bar{A}_2^3 - \frac{8}{5\pi^3}(\eta - \beta)\bar{A}_2 + \frac{12B}{5\pi^3} = 0,$$
$$(4\eta + 2\beta) - \pi^3 \bar{A}_2^2 > 0,$$
$$\bar{A}_1^2 = \frac{8}{3\pi^3}(2\eta + \beta) - \frac{4}{3}\bar{A}_2^2. \tag{21}$$

The resulting boundary curves are additionally presented in Fig. 5 which follows from Fig. 4. Clearly, new curves $P_{10}-P_5$ and $P_{11}-P_6$ are found defined in the 12 domains. The type and the number of critical points in these domains and on the boundary curves are given in Table 2.

It is of interest to look what happens if one chooses φ fixed, for instance $\varphi = \frac{1}{4}\pi$, and starts in domain I and then increases η. In particular, it is of interest to compute explicitly $R_1 = \bar{A}_1^2 + \bar{B}_1^2$ and $R_2 = \bar{A}_2^2$ as functions of η. The results are presented in Figs. 6 and 7. Clearly, in domain I there is one real solution R_2 as sketched in Fig. 6. In Q_1, this solution bifurcates into a stable (in the sense Lyapunov) and an unstable one. When one arrives in Q_2 two new solutions appear, a stable and an unstable one. Analogously, in Q_3 two new unstable ones appear whereas in Q_4 one new unstable, and in Q_5 one stable and one unstable solution appear leading totally to nine critical points in domain XII. All bifurcation points Q_i, $i = 1, 2, \ldots, 5$, are indicated in Figs. 5–7.

The semi-trivial solution as indicated in Fig. 6 corresponds with the response curve in Fig. 2. The difference however is that in Fig. 6, a part of the curve with stable solutions has become unstable due to the interaction with the first mode. Apparently, this part is unstable in the

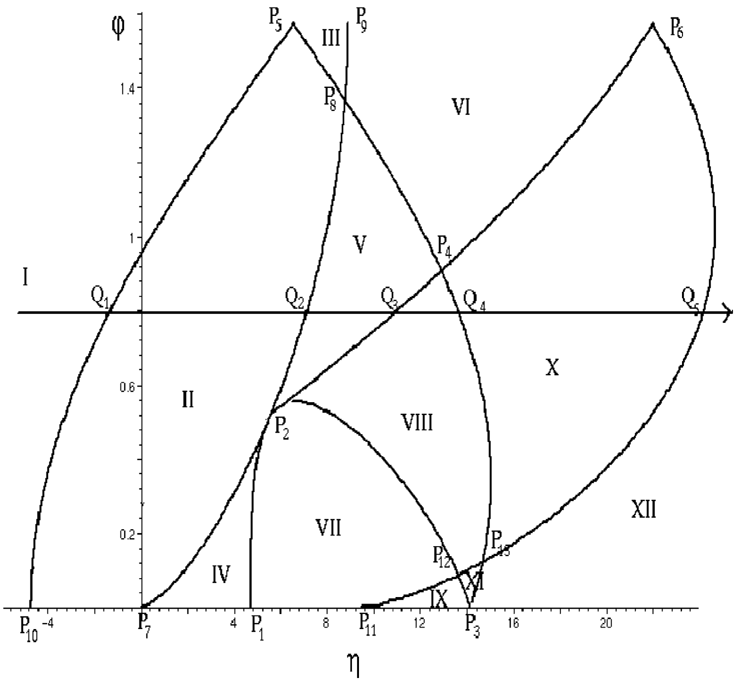

Fig. 5. Twelve domains with real solutions of types 1–3 of Eq. (17) for $C = 6$.

Table 2
The number of critical points of Eq. (16) describing Fig. 5

Domains/curves/points	The number of critical points of Eq. (16)			Total
	$(0, 0, A_2, 0)$	$(A_{11}, A_{11}, A_{21}, 0)$	$(A_{12}, -A_{12}, A_{22}, 0)$	
I	1	0	0	1
II	1	0	1	2
III	1	1	1	3
IV and V	3	0	1	4
VI and VII	3	1	1	5
VIII	3	2	1	6
IX	3	1	3	7
X	3	3	1	7
XI	3	2	3	8
XII	3	3	3	9
P_5P_{10}	1	0	0	1
P_5P_8	1	0	1	2
P_7P_8	2	0	1	3
P_1P_2 and P_4P_8	3	0	1	4
P_8P_9	2	1	1	4
P_2P_4 and P_2P_{12}	3	1	1	5
P_4P_6 and P_4P_{13}	3	2	1	6
$P_{11}P_{12}$	3	1	2	6
P_3P_{12}	3	1	3	7
$P_{12}P_{13}$	3	2	2	7
P_3P_{13}	3	2	3	8
$P_6\bar{P}_{13}$	3	3	2	8
P_5	1	0	0	1
P_2 and P_8	2	0	1	3
P_9	2	1	1	4
P_4	3	1	1	5
P_{12}	3	1	2	6
P_6 and P_{13}	3	2	2	7

four-dimensional phase space. The most interesting solutions are the non-trivial ones in the four-dimensional phase space of type 3 in Eq. (18). Starting from a large value of $\eta < 36$ and subsequently decreasing η one observes at Q_5 two jumps, R_1 decreases whereas R_2 increases. A remarkable result is that the amplitude of the parametrical induced mode at Q_5 before the jump is much larger (i.e., a factor 30) than the amplitude of the transverse excited mode.

3.2. The case with (positive) damping

When there is no damping system (16) has both unstable and stable critical points of type 1. The eigenvalues of the stable critical points of type 1 have zero real part implying that no conclusions can be drawn about the stability of the corresponding periodic solutions of the original system. The curves on which these critical points are located are given in Figs. 2 and 6 and indicated with semi-trivial solutions.

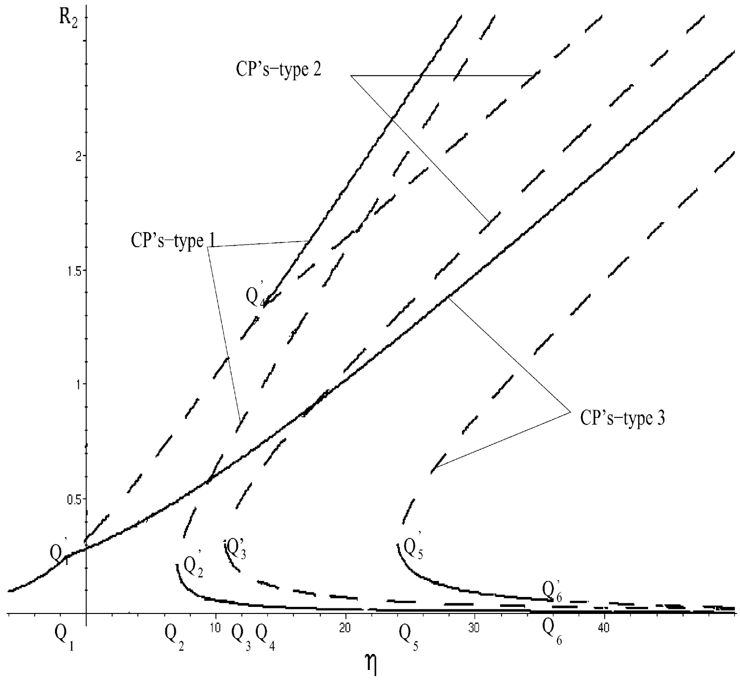

Fig. 6. Stability response curves of the second mode with respect to the detuning of Eq. (16) with $\varphi = \frac{1}{4}\pi$ and $C = 6$.

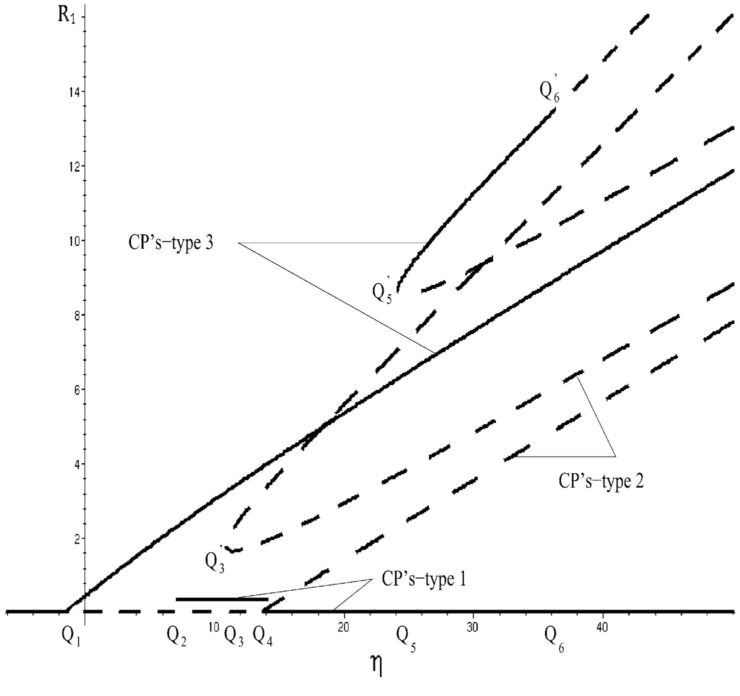

Fig. 7. Stability response curves of the first mode with respect to the detuning of Eq. (16) with $\varphi = \frac{1}{4}\pi$ and $C = 6$.

When one considers however small positive damping, the centre points in the $(\bar{A}_2, \bar{B}_2)$-plane as indicated in Fig. 2 now become positive attractors. In what follows, the number of critical points of system (16) with $\alpha > 0$ and their stability will be studied in more detail. The critical points of Eq. (16) are solutions of

$$\begin{aligned} &\alpha\bar{A}_2 + \bar{B}_2[\gamma_2((\bar{A}_1^2 + \bar{B}_1^2) + 6(\bar{A}_2^2 + \bar{B}_2^2)) - 4\eta] = 0, \\ &\alpha\bar{B}_2 - \bar{A}_2[\gamma_2((\bar{A}_1^2 + \bar{B}_1^2) + 6(\bar{A}_2^2 + \bar{B}_2^2)) - 4\eta] - 2B = 0, \\ &\beta_+\bar{A}_1 + \bar{B}_1[\gamma_1(3(\bar{A}_1^2 + \bar{B}_1^2) + 8(\bar{A}_2^2 + \bar{B}_2^2)) - 2\eta] = 0, \\ &\beta_-\bar{B}_1 - \bar{A}_1[\gamma_1(3(\bar{A}_1^2 + \bar{B}_1^2) + 8(\bar{A}_2^2 + \bar{B}_2^2)) - 2\eta] = 0, \end{aligned} \tag{22}$$

In order to have non-zero solutions $\bar{A}_1$ and $\bar{B}_1$ from the last two equations of Eq. (22) it follows that the condition $A^2 - 4(\alpha^2/\pi^2) \geqslant 0$ should hold. The last two equations of Eq. (22) can be reduced to

$$(\gamma_1(3(\bar{A}_1^2 + \bar{B}_1^2) + 8(\bar{A}_2^2 + \bar{B}_2^2)) - 2\eta)^2 = \beta^2 - \alpha^2. \tag{23}$$

Eq. (23) implies the possibilities:

$$\begin{aligned} &\text{(a)} \quad \gamma_1(3(\bar{A}_1^2 + \bar{B}_1^2) + 8(\bar{A}_2^2 + \bar{B}_2^2)) - 2\eta = -\sqrt{\beta^2 - \alpha^2}, \\ &\text{(b)} \quad \gamma_1(3(\bar{A}_1^2 + \bar{B}_1^2) + 8(\bar{A}_2^2 + \bar{B}_2^2)) - 2\eta = \sqrt{\beta^2 - \alpha^2}. \end{aligned} \tag{24}$$

From the last two possibilities and the case $\bar{A}_1 = \bar{B}_1 = 0$, it follows that the co-ordinates of the critical points can be classified in three types:

$$\begin{aligned} &\text{type 1:} \quad (\bar{A}_1, \bar{B}_1, \bar{A}_2, \bar{B}_2) = (0, 0, \bar{A}_2, \bar{B}_2), \quad \text{type 1 CP}, \\ &\text{type 2:} \quad (\bar{A}_1, \bar{B}_1, \bar{A}_2, \bar{B}_2) = \left(\bar{A}_1, \sqrt{\frac{\beta_+}{|\beta_-|}}\bar{A}_1, \bar{A}_2, \bar{B}_2\right), \quad \text{type 2 CP}, \\ &\text{type 3:} \quad (\bar{A}_1, \bar{B}_1, \bar{A}_2, \bar{B}_2) = \left(\bar{A}_1, -\sqrt{\frac{\beta_+}{|\beta_-|}}\bar{A}_1, \bar{A}_2, \bar{B}_2\right), \quad \text{type 3 CP}. \end{aligned} \tag{25}$$

Substitution of the first equation (24) in the first and second equations of Eq. (22) gives

$$\begin{aligned} &R_2[\gamma_2(R_1 + 6R_2) - 4\eta]^2 + \alpha^2 R_2 - 4B^2 = 0, \\ &(2\eta - \sqrt{\beta^2 - \alpha^2}) - 8\gamma_1 R_2 > 0, \\ &R_1 = \frac{1}{3\gamma_1}(2\eta - \sqrt{\beta^2 - \alpha^2}) - \frac{8}{3}R_2, \end{aligned} \tag{26}$$

where $R_1 = \bar{A}_1^2 + \bar{B}_1^2$ and $R_2 = \bar{A}_2^2 + \bar{B}_2^2$. Now substitution of the third into the first equation of Eq. (26) and then introducing the new variable $R_2 = Y + \frac{4}{15\gamma_2}(\eta + \sqrt{\beta^2 - \alpha^2})$, yields

$$\begin{aligned} &Y^3 + \kappa_{11}Y + \delta_{11} = 0, \\ &\tfrac{2}{15}(22\eta - 23\sqrt{\beta^2 - \alpha^2}) - \pi^3 Y > 0, \end{aligned} \tag{27}$$

where

$$\kappa_{11} = -\frac{4}{25\pi^6}\left[\frac{16}{3}(\eta + \sqrt{\beta^2 - \alpha^2})^2 - 9\alpha^2\right],$$
$$\delta_{11} = -\frac{16}{25\pi^6}\left[9B^2 - \frac{4}{5\pi^3}(\eta + \sqrt{\beta^2 - \alpha^2})\left(\frac{16}{27}(\eta + \sqrt{\beta^2 - \alpha^2})^2 + 3\alpha^2\right)\right].$$

In similar way, but by using now the second equation of Eq. (24), one obtains a second system of equations with solutions of type 3:

$$Z^3 + \kappa_{12} Z + \delta_{12} = 0,$$
$$\tfrac{2}{15}(22\eta + 23\sqrt{\beta^2 - \alpha^2}) - \pi^3 Z > 0, \tag{28}$$

where

$$\kappa_{12} = -\frac{4}{25\pi^6}\left[\frac{16}{3}(\eta - \sqrt{\beta^2 - \alpha^2})^2 - 9\alpha^2\right],$$
$$\delta_{12} = -\frac{16}{25\pi^6}\left[9B^2 - \frac{4}{5\pi^3}(\eta - \sqrt{\beta^2 - \alpha^2})\left(\frac{16}{27}(\eta - \sqrt{\beta^2 - \alpha^2})^2 + 3\alpha^2\right)\right],$$
$$R_2 = Z + \frac{16}{15\pi^3}(\eta - \sqrt{\beta^2 - \alpha^2}),$$
$$R_1 = \frac{1}{3\gamma_1}\left(2\eta + \sqrt{\beta^2 - \alpha^2}\right) - \frac{8}{3}R_2.$$

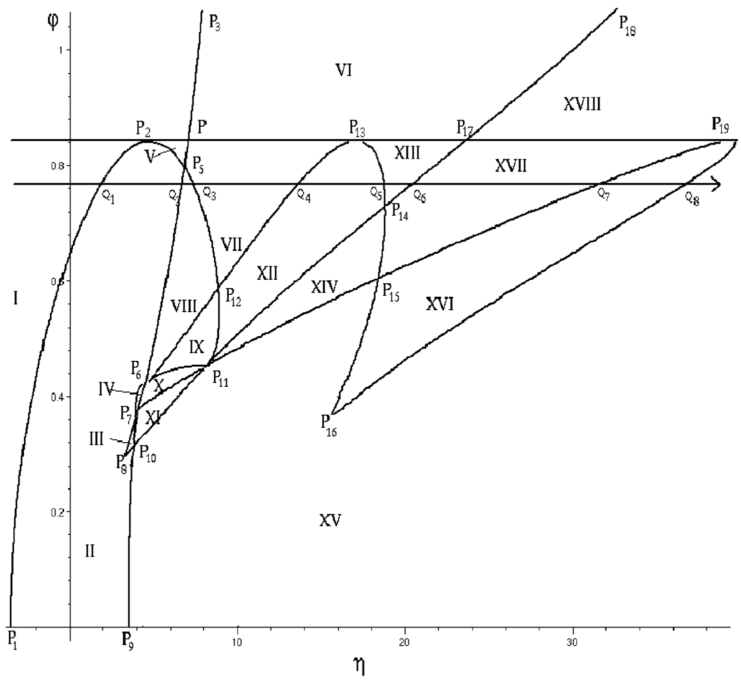

Fig. 8. Eighteen domains with real solutions of types 1, 2, and 3 of Eq. (22) for $\alpha = 2\pi$ and $C = 6$.

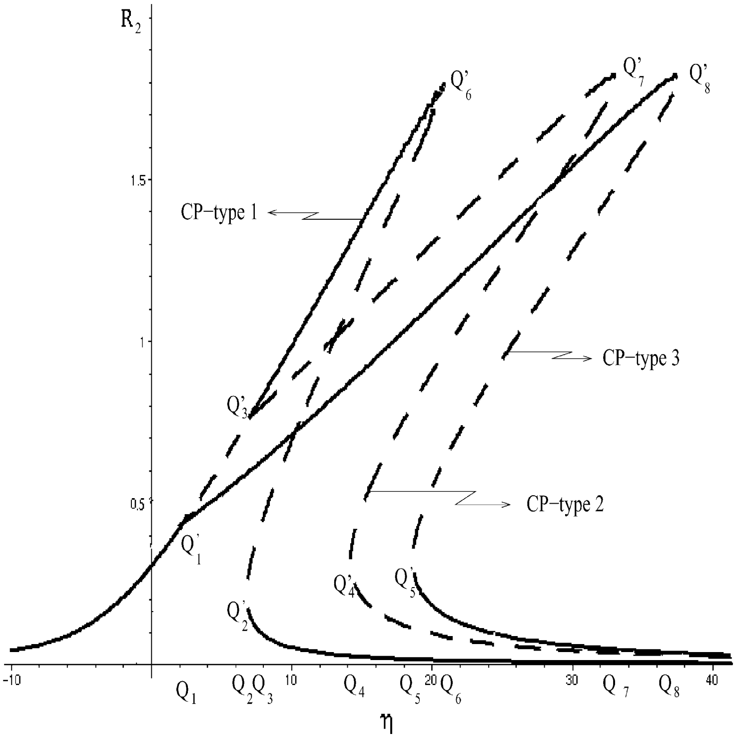

Fig. 9. Stability response curves of the second mode with respect to the detuning of Eq. (16) with $C = 6$, $\varphi = \frac{1}{4}\pi$ and $\alpha = 2\pi$.

As indicated in Section 3.1, an overview of the number of real solutions of types 1–3 of Eq. (22) and their dependence on η and φ for certain values of α and C can be given in the diagram in Fig. 8. Apparently, in this figure 18 domains can be distinguished. The boundary curves separating the domains are defined as solutions of system (22). In Figs. 9 and 10, the solutions $R_2 = \bar{A}_2^2 + \bar{B}_2^2$ and $R_1 = \bar{A}_1^2 + \bar{B}_1^2$ are given as a function of η for $C = 6$, $\varphi = \frac{1}{4}\pi$ and $\alpha = 2\pi$.

Let us suppose that η is increased while φ is held constant. This process is represented by the line through the points $Q_1, Q_2, \ldots, Q_8$ in Fig. 8. For $\eta < Q_1$ only the CPs of type 1 exist. Between Q_1 and Q_2 there are CPs of type 1 and CPs of type 3, etc. An overview of the domains and the number of critical points is given in Table 3.

In Figs. 9 and 10, R_2 and R_1 are plotted as a function of η. The presence of jump phenomena can be observed in these figures. These phenomena are due to the non-linearities and excitations. To explain this one starts in domain I in Fig. 8 for $\varphi = \frac{1}{4}\pi$ and follows the line indicated and parallel to the η-axis. At the point Q_1 one enters domain II, at Q_2 one enters domain VIII, etc. All points $Q_1, Q_2, \ldots, Q_8$ are also indicated in Figs. 9 and 10. In these figures, however, the η co-ordinate of Q_i' is indicated Q_i, $i = 1, 2, \ldots, 8$. At points Q_1 in Fig. 9 one clearly leaves the CPs of type 1 (corresponding with the transversally excited mode) because the parametrically excited mode comes in. By increasing η one arrives at Q_8 and a jump occurs. Then by decreasing η again a jump occurs at Q_5 where now the amplitude increases, while simultaneously in Fig. 10 at Q_5 the amplitude decreases with a jump. It is of interest to note that additionally two types of stable

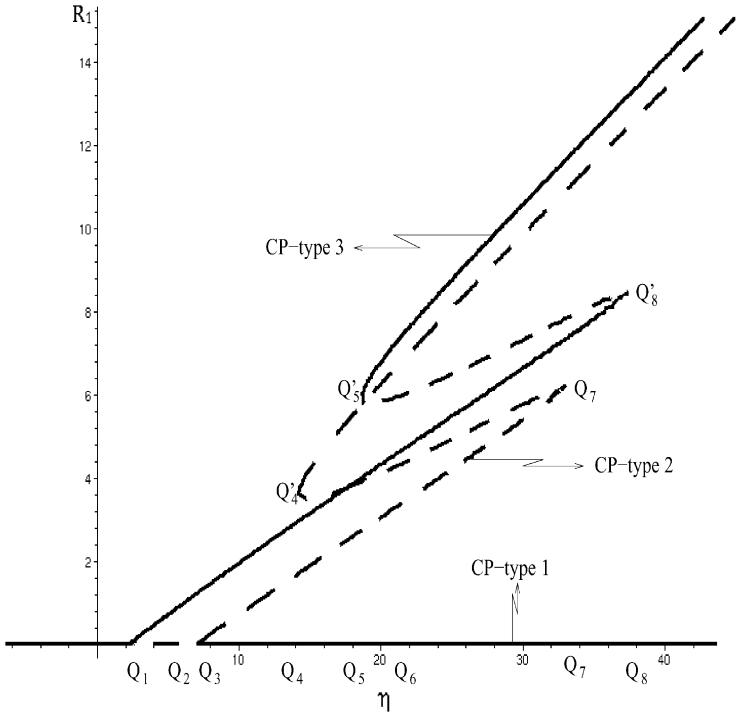

Fig. 10. Stability response curves of the first mode with respect to the detuning of Eq. (16) with $C = 6$, $\varphi = \frac{1}{4}\pi$ and $\alpha = 2\pi$.

critical points are present: the one on the upper $Q'_3Q'_6$ curve and the one on the lower curve starting at Q'_2 all belonging to the CPs of type 1. When one follows the solutions along these curve apparently at the ends only a jump downward in Fig. 9, and a jump upward in Fig. 10 are possible. At the left end of the Q_3Q_6 curve corresponding to the CPs of type 1 the parametric excited mode comes in; hence, a jump to the lower ST curve, starting at Q'_2 is not possible and the jump should end at the $Q'_1Q'_8$ curve. At the right end of the $Q'_3Q'_6$ curve, no parametric excitation comes in and hence a jump to the lower curve of CPs of type 1 takes place. As in the case without damping, the difference in order of magnitude of the parametrically excited mode can easily be a factor 10 greater than the amplitude of the transversally excited mode. For $\varphi = \frac{1}{4}\pi$, the excitation amplitudes A and B are equal. This does not, however, imply that the excitation energy in both directions is equal. On the other hand, there is a non-linear interaction between the modes implying that energy transfer between the two modes is possible.

4. Conclusion

In this paper, the simultaneous small amplitude excitation in horizontal and vertical directions at an endpoint of an inclined stretched string is studied. As the attention is focused to transverse standing wave modes, a modified Kirchhoff model is used implying that acceleration of horizontal elements of the string are neglected. The mechanisms of mode generation are combining classical resonance with parametric resonance. The conditions to have this combination of resonances are

Table 3
The number of critical points of Eq. (16) describing Fig. 8

Domains/curves/points	The number of critical points of Eq. (16)			Total
	$(0, 0, A_2, B_2)$	$(A_1, \sqrt{\frac{\beta_+}{\lvert\beta_-\rvert}}A_1, A_2, B_2)$	$(A_1, -\sqrt{\frac{\beta_+}{\lvert\beta_-\rvert}}A_1, A_2, B_2)$	
I and XVIII	1	0	0	1
II	1	0	1	2
IV , V and XV	1	1	1	3
VI	3	0	0	3
III and VIII	3	0	1	4
VII , X and XI	3	1	1	5
XIV	1	3	1	5
XVI	1	1	3	5
IX	3	2	1	6
XII	3	3	1	7
XVII	1	3	3	7
XIII	3	3	3	9
P_1P_2, P_2P_4, and $P_{17}P_{19}$	1	0	0	1
P_2P_5, P_6P_7, and P_9P_{10}	1	0	1	2
P_3P_4 and $P_{17}P_{18}$	2	0	0	2
P_4P_{17}	3	0	0	3
P_5P_6, P_7P_8, and P_8P_{10}	2	0	1	3
P_4P_5 and $P_{10}P_{11}$	2	1	1	4
P_5P_{12} and P_7P_{10}	3	0	1	4
$P_{11}P_{15}$	1	2	1	4
$P_{15}P_{16}$ and $P_{16}P_{19}$	1	1	2	4
P_6P_{11}, P_6P_{12}, and P_7P_{11}	3	1	1	5
$P_{11}P_{12}$ and $P_{12}P_{13}$	3	2	1	6
$P_{11}P_{14}$	2	3	1	6
$P_{14}P_{15}$	1	3	2	6
$P_{15}P_{19}$	1	2	3	6
$P_{13}P_{14}$	3	3	2	8
$P_{14}P_{17}$	2	3	3	8
P_2 and P_{19}	1	0	0	1
P_3, P_4, P_{17}, and P_{18}	2	0	0	2
P_8	1	0	1	2
P_5, P_6, P_7, and P_{10}	2	0	1	3
P_{13}	3	0	0	3
P_{11}	2	1	1	4
P_{16}	1	1	2	4
P_{12}	3	1	1	5
P_{15}	1	2	2	5
P_{14}	2	3	2	7

given by $p^2 = \frac{1}{3}(m^2 - 4s^2)\pi^2$ and $\lambda^2 = \frac{4}{3}(m^2 - s^2)\pi^2$, where $p \geqslant 0$ describes the linear elastic behaviour of the medium in which the string is embedded, $p = 0$ corresponds with a model for a vibrating string in air under normal conditions, λ is the excitation frequency and m and s are integers representing mode numbers. An interesting case is $p = 0$, λ near 2π, and $m = 2$ and $s = 1$, describing the interaction of the first and second modes. Three parameters, i.e., the damping coefficient (α), the angle of inclination (φ), and the detuning coefficient (η) are relevant to describe this interaction. Equations are derived for the time-varying behaviour of the mode amplitudes. Fix points of these equations corresponding to time-periodic solutions, i.e., modes with constant amplitudes, are analyzed. A classification of all critical points and their stability are given. Of particular interest are the critical points in $\mathbf{R}^4$, corresponding to non-trivial solutions (CPs of types 2 and 3) and describing mode interaction. A number of amplitude jumps are found with saddle-node bifurcation as underlying mechanism. A remarkable result is that when the horizontal and vertical excitation amplitudes are equal (corresponding with an inclination of $\frac{1}{4}\pi$) the order of magnitude of the mode response may be quite different: modes generated by parametric excitation may easily have amplitudes ten times larger than the amplitudes of the transversally excited modes. The model equation as presented here can be used for the study of the dynamics of inclined stay-cables connecting the bridge deck and pylon of a cable-stayed bridge. Due to the inclination of the stay cables, an aerodynamically unstable pylon will simultaneously induce horizontal and vertical motions of the cable.

Acknowledgements

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