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Journal of Sound and Vibration 267 (2003) 955–960
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Sensitivity of the eigenvalues of beams to the change of element correction factors

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Received 10 September 2002; accepted 20 January 2003

1. Introduction

Finite element (FE) analysis is widely used to predict the dynamic responses of mechanical systems and structures subject to dynamic loading. The predicted responses may differ from the experimentally measured ones and there have been active researches on FE model updating so that the predicted responses based on the model agree with the measured ones [1]. One of the approaches to model updating is to use sensitivity analysis. In the approach the sensitivity of the modal parameters such as eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the FE model to changes in the updating parameters are calculated. These updating parameters can be material properties or element dimensions or element correction factors [2] in proportion to which element mass and stiffness matrices are increased or decreased. This paper investigates some characteristics on the sensitivity of eigenvalues of beams with various boundary conditions to the element correction factors.

2. Sensitivity of eigenvalues

2.1. Element correction factors

We can introduce element correction factors to update FE models, P_{mj} and P_{kj}. These are the correction factors for the mass and the stiffness matrix of the jth element, respectively. With the use of these factors, the element mass and stiffness matrices are varied proportionally as follows.

$$[M_{ej}] = (1 + P_{mj})[M_{ej}]_0, \tag{1}$$

$$[K_{ej}] = (1 + P_{kj})[K_{ej}]_0, \tag{2}$$

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doi:10.1016/S0022-460X(03)00124-X

where $[M_{ej}]$ and $[K_{ej}]$ represent the mass and stiffness matrices of the jth element, respectively, and the subscript 0 initial matrices. The element matrices $[M_{ej}]$ and $[K_{ej}]$ have the same sizes as the whole system matrices, with zeros outside the corresponding positions, and rows and columns deleted for fixed boundary conditions. Then the system matrices become

$$[M] = \sum_{j=1}^{N} [M_{ej}] = \sum_{j=1}^{N} (1 + P_{mj})[M_{ej}]_0, \tag{3}$$

$$[K] = \sum_{j=1}^{N} [K_{ej}] = \sum_{j=1}^{N} (1 + P_{kj})[K_{ej}]_0, \tag{4}$$

where N is the number of elements.

2.2. Sensitivity of the eigenvalues for a cantilever beam

It is known that the sensitivity of the eigenvalue (square of the natural frequency) of mode i, λ_i to change in the updating variable θ_j is expressed by Eq. (5)

$$\frac{\partial \lambda_i}{\partial \theta_j} = \phi_i^{\mathrm{T}} \left(\frac{\partial [K]}{\partial \theta_j} - \lambda_i \frac{\partial [M]}{\partial \theta_j} \right) \phi_i, \tag{5}$$

where ϕ_i represents the eigenvector of mode i [3]. If we take element correction factors as updating variables, the sensitivities of eigenvalues become

$$\frac{\partial \lambda_i}{\partial P_{kj}} = \phi_i^{\mathrm{T}} [K_{ej}]_0 \phi_i, \tag{6}$$

$$\frac{\partial \lambda_i}{\partial P_{mj}} = \phi_i^{\mathrm{T}} (-\lambda_i [M_{ej}]_0) \phi_i. \tag{7}$$

Using the FE analysis [4], the above sensitivities of eigenvalues were calculated for a cantilever beam with length 270 mm, width 35 mm, thickness 1.5 mm, Young's modulus 175e9 N/m^2, and density 7850 kg/m^3. The beam is composed of five beam elements with equal length and is shown in Fig. 1. The calculated sensitivities of eigenvalues to the element correction factors are listed in Table 1. Examining the table, it can be found that the sensitivity to the stiffness correction factor of one element is almost equal and opposite to the sensitivity to the mass correction factor of the element in symmetric position. As Fig. 1 shows, elements 1 and 5, and 2 and 4 are in symmetric positions. The above observation can be expressed in equations as follows:

$$\frac{\partial \lambda_i}{\partial P_{k1}} = -\frac{\partial \lambda_i}{\partial P_{m5}}, \tag{8}$$

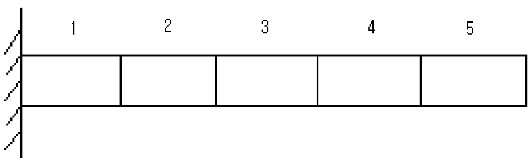

Fig. 1. Cantilever beam composed of five elements.

Table 1
Sensitivity of the eigenvalues to the element correction factors for a cantilever beam (units: rad^2/s^2)

Element I	1	2	3	4	5
$\frac{\partial\lambda_1}{\partial P_{ki}}$	5.8359e3	2.7688e3	0.9421e3	0.1704e3	0.0064e3
$\frac{\partial\lambda_1}{\partial P_{mi}}$	−0.0066e3	−0.1704e3	−0.9421e3	−2.7687e3	−5.8358e3
$\frac{\partial\lambda_2}{\partial P_{ki}}$	1.0742e5	0.3845e5	1.4500e5	0.8502e5	0.0636e5
$\frac{\partial\lambda_2}{\partial P_{mi}}$	−0.0645e5	−0.8496e5	−1.4499e5	−0.3852e5	−1.0734e5
$\frac{\partial\lambda_3}{\partial P_{ki}}$	0.5653e6	0.8227e6	0.2218e6	1.1607e6	0.2450e6
$\frac{\partial\lambda_3}{\partial P_{mi}}$	−0.2446e6	−1.1566e6	−0.2231e6	−0.8243e6	−0.5669e6
$\frac{\partial\lambda_4}{\partial P_{ki}}$	2.3904e6	1.8709e6	3.1323e6	2.3059e6	2.0686e6
$\frac{\partial\lambda_4}{\partial P_{mi}}$	−1.9972e6	−2.3236e6	−3.1128e6	−1.8690e6	−2.4656e6

$$\frac{\partial\lambda_i}{\partial P_{k2}} = -\frac{\partial\lambda_i}{\partial P_{m4}}. \tag{9}$$

In general, these relations can be written as

$$\frac{\partial\lambda_i}{\partial P_{kj}} = -\frac{\partial\lambda_i}{\partial P_{m(N+1-j)}}. \tag{10}$$

The relationship (10) holds true exactly for lower modes, and there are slight differences between the two sensitivities for higher modes. For example, the maximum difference for the fourth mode was 3.45%.

The relationship means that to increase stiffness in one element has the same effects on the eigenvalues as to decrease mass by the same proportion in the symmetrically positioned element. For the above cantilever beam, the eigenvalues were calculated for the case with the stiffness of the second element increased by 20%, that is, $P_{k2} = 0.2$. Next, the eigenvalues were calculated with the mass of the fourth element decreased by 20%, that is, $P_{m4} = -0.2$. These eigenvalues are compared in Table 2 and it can be found that they agree very well.

From the definition of eigenvalues and eigenvectors, we have the following relation

$$[K]\phi_i = \lambda_i[M]\phi_i. \tag{11}$$

Pre-multiplying ϕ_i^{T} on both sides of the above equation,

$$\phi_i^{\mathrm{T}}[K]\phi_i = \phi_i^{\mathrm{T}}(\lambda_i[M])\phi_i. \tag{12}$$

Expressing the mass and stiffness matrices as sums of the element matrices,

$$\begin{aligned}\phi_i^{\mathrm{T}}([K_{e1}] + [K_{e2}] + \cdots + [K_{eN}])\phi_i = \phi_i^{\mathrm{T}}(\lambda_i[M_{e1}] + \lambda_i[M_{e2}] \\ + \cdots + \lambda_i[M_{eN}])\phi_i.\end{aligned} \tag{13}$$

Table 2
Eigenvalues of the cantilever beam for the two cases (units: rad^2/s^2)

Case 1 ($P_{k2} = 0.2$)	Case 2 ($P_{m4} = -0.2$)
0.0161e3	0.0162e3
0.0993e3	0.0994e3
0.2833e3	0.2848e3
0.5538e3	0.5555e3
0.9207e3	0.9239e3

Table 3
Sensitivity of the eigenvaluses to the element correction factors for a simply supported beam (units: rad^2/s^2)

Element I	1	2	3	4	5
$\frac{\partial \lambda_1}{\partial P_{ki}}$	0.3727e4	1.9757e4	2.9664e4	1.9757e4	0.3727e4
$\frac{\partial \lambda_1}{\partial P_{mi}}$	−0.3727e4	−1.9757e4	−2.9664e4	−1.9757e4	−0.3727e4
$\frac{\partial \lambda_2}{\partial P_{ki}}$	1.8852e5	3.9642e5	0.6002e5	3.9642e5	1.8852e5
$\frac{\partial \lambda_2}{\partial P_{mi}}$	−1.8856e5	−3.9630e5	−0.6018e5	−3.9630e5	−1.8856e5
$\frac{\partial \lambda_3}{\partial P_{ki}}$	1.4536e6	0.7565e6	1.8845e6	0.7565e6	1.4536e6
$\frac{\partial \lambda_3}{\partial P_{mi}}$	−1.4518e6	−0.7613e6	−1.8786e6	−0.7613e6	−1.4518e6
$\frac{\partial \lambda_4}{\partial P_{ki}}$	4.6566e6	3.8950e6	3.4243e6	3.8950e6	4.6566e6
$\frac{\partial \lambda_4}{\partial P_{mi}}$	−4.6178e6	−3.9098e6	−3.4723e6	−3.9098e6	−4.6178e6

Substituting Eqs. (6) and (7) into the above equation, we obtain

$$\sum_{j=1}^{N} \frac{\partial \lambda_i}{\partial P_{kj}} = -\sum_{j=1}^{N} \frac{\partial \lambda_i}{\partial P_{mj}}. \tag{14}$$

Therefore, the sum of the sensitivities to P_{kj} is equal and opposite to the sum of the sensitivities to P_{mj}. However, it does not explain the equality between the corresponding terms of the sums, Eq. (10).

2.3. Sensitivity of the eigenvalues for beams with other boundary conditions

The sensitivities of eigenvalues to changes in the element correction factors were calculated for a simply supported beam with the same material properties and dimensions as the previous cantilever beam. This beam is also composed of five elements. Examining the calculated sensitivities in Table 3, it can be found that the same relationship as for a cantilever beam holds

Table 4
Sensitivity of the eigenvalues to the element correction factors for a clamped–simply supported beam (units: rad^2/s^2)

Element I	1	2	3	4	5
$\frac{\partial \lambda_1}{\partial P_{ki}}$	6.3173e4	0.6633e4	4.8904e4	5.5600e4	1.2761e4
$\frac{\partial \lambda_1}{\partial P_{mi}}$	−0.1746e4	−2.8388e4	−7.5234e4	−6.7227e4	−1.4477e4
$\frac{\partial \lambda_2}{\partial P_{ki}}$	3.8962e5	5.2240e5	1.8318e5	5.2571e5	3.5302e5
$\frac{\partial \lambda_2}{\partial P_{mi}}$	−1.2006e5	−7.7120e5	−2.1846e5	−5.1363e5	−3.5059e5
$\frac{\partial \lambda_3}{\partial P_{ki}}$	1.7069e6	1.6369e6	2.2876e6	1.0230e6	2.0775e6
$\frac{\partial \lambda_3}{\partial P_{mi}}$	−1.3053e6	−2.0880e6	−2.2331e6	−1.0320e6	−2.0733e6
$\frac{\partial \lambda_4}{\partial P_{ki}}$	5.9950e6	4.7015e6	4.9469e6	5.2059e6	5.4471e6
$\frac{\partial \lambda_4}{\partial P_{mi}}$	−5.6830e6	−5.0665e6	−4.9549e6	−5.1999e6	−5.3921e6

true. Moreover, because of the symmetry of the simply supported beam, the following relationship also holds

$$\frac{\partial \lambda_i}{\partial P_{kj}} = -\frac{\partial \lambda_i}{\partial P_{mj}}. \tag{15}$$

Explaining in words, the sensitivity of eigenvalues to the stiffness correction factor in one element is equal and opposite to the sensitivity to the mass correction factor in the same element.

The sensitivities of eigenvalues were calculated for beams with other types of boundary conditions: a clamped–simply supported beam and a clamped–clamped beam. However, for these beams the previous relationship (10) does not hold and the author has not obtained the answer for its reason. Table 4 lists the calculated sensitivities for a clamped–simply supported beam.

3. Summary

Some characteristics of the sensitivities of the eigenvalues for beams have been found in the paper. For cantilever beams and simply supported beams, the sensitivities of the eigenvalues to the stiffness correction factor of one element are equal and opposite to the sensitivities to the mass correction factor of the symmetrically positioned element. The relationship means that to increase stiffness in one element has the same effects on the eigenvalues as to decrease mass by the same proportion in the symmetrically positioned element. For beams with other boundary conditions, however, the relationship does not hold.

Acknowledgements

This work was supported by the Brain Korea 21 Project and the Gyeongsang National University.

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