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On the transversal vibrations of a conveyor belt with a low and time-varying velocity. Part II: the beam-like case

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Abstract

In this paper, initial–boundary-value problems for a beam equation (with string effect) are considered. These problems can be used as simple models to describe the vertical vibrations of a conveyor belt, for which the velocity is small with respect to the wave speed. In this paper, the belt is assumed to move with a time-varying velocity $V(t) = \varepsilon(V_0 + \alpha\sin(\Omega t))$. Formal asymptotic approximations of the solutions are constructed to show the complicated dynamical behaviour of the belt. Complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or difference of any two natural frequencies of the system with zero belt velocity. For special values of the belt parameters these sum type and difference type of internal resonances coincide giving rise to even more complicated dynamical behaviour. Some examples (including detuning cases) have been studied in detail.

1. Introduction

Axially moving systems are present in a wide class of engineering problems which arise in industrial, civil, aerospatial, mechanical, electronic and automotive applications. Aerial cables, tram-ways, oil pipelines, magnetic tapes, power transmission belts, paper sheet and web processes, fibre winding and band-saw blades are examples of cases where an axial transport of mass can be associated with transverse vibrations.

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see Refs. [1–4] for a recent overview) and is still of interest today. The

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vibrations can be classified into two types, i.e., whether it is of a string-like type or of beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibration of a belt system (with constant belt velocity V) can be modelled as

string-like by

$$v_{tt} + 2Vv_{xt} + (\kappa V^2 - c^2)v_{xx} = 0, \tag{1}$$

and

beam-like (*with a string effect*) *by*

$$v_{tt} + 2Vv_{xt} + (\kappa V^2 - c^2)v_{xx} + \frac{E_b I_y}{\rho A} v_{xxxx} = 0, \tag{2}$$

where $V, \kappa, c, E_b, I_y, \rho$, and A are constants which are described in Section 2.

The main purpose of studying the dynamic behaviour of a belt system is to determine the *natural frequencies* of the vibrations. By knowing these natural frequencies, the so-called resonance-free belt system can be designed (see Ref. [3]). Resonances that can cause severe vibrations can be initiated by some parts of the belt system, such as the varying belt speed, the roll eccentricities, and other belt imperfections. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support components, and the increase of energy consumption of the system.

In this paper, vibrations of a moving belt which is modelled by a beam equation with a string effect will be studied. The belt speed is considered to be time-varying and to be small compared to the wave speed. In Ref. [5], a string-like model for a similar belt system has been studied. It will turn out that the beam-like model and the string-like model give rise to different behaviour of the solutions. It is assumed that the low and time-varying belt speed $V(t)$ is given by $\varepsilon(V_0 + \alpha \sin(\Omega t))$, where ε, V_0, α, and Ω are constants with $0 < \varepsilon \ll 1$ and $V_0 > |\alpha|$. It should be observed that the velocity changes periodically such that the belt moves in one direction. In fact the small parameter ε indicates that the belt speed $V(t)$ is small compared to the wave speed c. Recently, the authors of Ref. [6] also studied the vibrations of an axially moving beam with a time-dependent velocity. As has been pointed out in Ref. [5] their application of the truncation method does not give approximations which are valid on long time-scales of order ε^{-1}. More results on axially moving strings and beams can also be found in Refs. [7,8,14]. The variation in $V(t)$ can be considered as some kind of excitation. In relation to excitations, some results in this area have been obtained by Sack [9] and Archibald and Emslie [10]. Sack considered the problem of a string moving with a constant velocity at which one of its end (i.e., $x = L$) is subjected to a harmonic excitation. In Ref. [9] the vibrations of the string at $x = L$ is forced to be $v(x,t) = v_0 \cos(\Omega t)$. Archibald and Emslie also studied the case where one end of the moving string is subjected to a harmonic excitation to represent the case of a belt travelling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied by Mahalingham [11]. A moving string model to study the transverse vibrations of power transmission chains has been used in Ref. [11]. In all of these works, the belt movement is assumed to be constant.

This paper is organized as follows. In Section 2, the equation of motion describing the dynamic behaviour of a belt moving with a non-constant velocity is derived. The belt is assumed to be simply supported in the vertical direction. Then in Section 3, the two time-scales perturbation method is used to find approximations of the solution of the problem. It will turn out in Section 3 that complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or the difference of any two natural frequencies ω_k and ω_n of the zero belt–velocity system. For special values of the belt parameters these sum type and difference type of internal resonances can coincide giving rise to even more complicated dynamical behaviour. In Section 4, the (difference type) case $\Omega = \omega_2 - \omega_1$ and the detuned case $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$ with ϕ of order one will be studied. While in Section 5, the (sum type) cases $\Omega = \omega_2 + \omega_1, \Omega = 2\omega_1$ and $\Omega = \omega_3 + \omega_2$ will be considered. For some special values of the beam parameters the case (including detuning) for which a sum type and a difference type of internal resonance coincide will also be studied: that is, the case $\Omega = \omega_3 + \omega_2 = \omega_5 - \omega_2$. Finally in Section 6, some remarks and some conclusions will be drawn.

2. A beam model

If the belt speed V is not constant but a function of t, then Eq. (2) becomes

$$v_{tt} + (\kappa V^2 - c^2)v_{xx} + 2Vv_{xt} + V_t v_x + \frac{E_b I_y}{\rho A} v_{xxxx} = 0, \tag{3}$$

for $0<x<L$, $t>0$ and where $v(x,t)$ is the displacement of the belt in the y (vertical) direction, $V(t)$ the time-varying belt speed, c the wave speed, E_b Young's modulus, I_y the moment of inertia with respect to the x (horizontal) axis, ρ the mass density of the belt, A the area of the cross-section of the belt, κ is a constant representing the relative stiffness of the belt, its value is in $[0,1]$. x the co-ordinate in horizontal direction, t the time and L the distance between the pulleys.

Since the beam is assumed to be simply supported, it will follow that the boundary conditions are

$$v(0,t) = v(L,t) = v_{xx}(0,t) = v_{xx}(L,t) = 0. \tag{4}$$

The initial values are given by

$$v(x,0) = f(x), \qquad v_t(x,0) = g(x), \tag{5}$$

where f is the initial displacement of the beam, and where g is the initial velocity of the beam. Considering the case where $V(t) = \varepsilon(V_0 + \alpha\sin(\Omega t))$, in which V_0 and α are constants and $V_0 > |\alpha|$, Eq. (3) becomes

$$\begin{aligned} v_{tt} - c^2 v_{xx} + \frac{E_b I_y}{\rho A} v_{xxxx} = & -\varepsilon\Omega\alpha\cos(\Omega t)v_x - 2\varepsilon(V_0 + \alpha\sin(\Omega t))v_{xt} \\ & - \varepsilon^2\kappa(V_0 + \alpha\sin(\Omega t))^2 v_{xx}. \end{aligned} \tag{6}$$

It should be noticed that Eq. (6) is a subcase of a problem which has been studied by Öz and Pakdemirli [6].

Solutions of the form $v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin(n\pi x/L)$ certainly satisfy the boundary conditions. There are two equivalent methods to determine what equations $v_n(t)$ for $n = 1,2,3,\ldots$ have to satisfy. The first method is based on the principle of reflections. Using this method the initial–boundary-value problem (3–6) is extended to an initial-value problem. Special attention has to be paid to the terms v_x and v_{xt} on the right side of Eq. (6) when this method is applied. Since this method has already been applied in Ref. [5] (and for instance in Refs. [12,13]) one now applies the other method which is based on the orthogonality properties of the set of functions $\sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$ on $0<x<L$. The following should be observed:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0 & \text{for } n\neq k, \\ \frac{1}{2}L & \text{for } n = k, \end{cases} \tag{7}$$

and

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0 & n\pm k \text{ even}, \\ \dfrac{-2Lk}{(n^2-k^2)\pi} & n\pm k \text{ odd}. \end{cases} \tag{8}$$

Substituting $v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin(n\pi x/L)$ into Eq. (6) gives

$$\sum_{n=1}^{\infty}\left[\ddot{v}_n + \left\{\left(\frac{cn\pi}{L}\right)^2 + \delta\left(\frac{n\pi}{L}\right)^4\right\}v_n\right]\sin\left(\frac{n\pi x}{L}\right)$$
$$= -\varepsilon\sum_{n=1}^{\infty}\frac{n\pi}{L}[\alpha\Omega\cos(\Omega t)v_n + 2(V_0 + \alpha\sin(\Omega t))\dot{v}_n]\cos\left(\frac{n\pi x}{L}\right) + \mathcal{O}(\varepsilon^2), \tag{9}$$

where $\delta = E_b I_y/\rho A$. Multiplying both sides of Eq. (9) with $\sin(k\pi x/l)$, and then integrating with respect to x from $x = 0$ to L gives (using Eqs. (7) and (8))

$$\ddot{v}_k + \left\{\left(\frac{ck\pi}{L}\right)^2 + \delta\left(\frac{k\pi}{L}\right)^4\right\}v_k$$
$$= \varepsilon\sum_{n=1}^{\infty}{}^{*}\frac{nk}{(n^2-k^2)L}[4\alpha\Omega\cos(\Omega t)v_n + 8(V_0 + \alpha\sin(\Omega t))\dot{v}_n] + \mathcal{O}(\varepsilon^2), \tag{10}$$

where the * in $\sum_{n=1}^{\infty}{}^{*}$ indicates that the summation is only carried out for $n\pm k$ is odd. For $t = 0, v_k(t)$ satisfies: $v_k(0) = (2/L)\int_0^L f(x)\sin(k\pi x/L)\,\mathrm{d}x$, and $\dot{v}_k(0) = (2/L)\int_0^L g(x)\sin(k\pi x/L)\,\mathrm{d}x$.

It should be observed that in order to obtain Eq. (10) the terms v_x and v_{xt} in Eq. (6) are in fact expanded in eigenfunctions (i.e., in $\sin(n\pi x/L)$) of the boundary-value problem (4), (6) with $\varepsilon = 0$. In Ref. [6] these terms were not expanded accordingly (see also Appendix C). In the next sections approximations of the solutions of Eq. (10) will be constructed for different Ω-values.

3. Application of the two time-scales perturbation method

Due to occurrence of the so-called secular terms a straightforward perturbation method cannot be used to solve Eq. (10) approximately. For that reason a two-time-scales perturbation method (with time scales $t_0 = t$ and $t_1 = \varepsilon t$) is used. The introduction of these two time scales define the transformations

$$v_k(t;\varepsilon) = w_k(t_0, t_1;\varepsilon), \qquad \frac{\mathrm{d}v_k}{\mathrm{d}t} = \frac{\partial w_k}{\partial t_0} + \varepsilon\frac{\partial w_k}{\partial t_1},$$
$$\frac{\mathrm{d}^2 v_k}{\mathrm{d}t^2} = \frac{\partial^2 w_k}{\partial t_0^2} + 2\varepsilon\frac{\partial^2 w_k}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 w_k}{\partial t_1^2}. \tag{11}$$

Substitution of Eq. (11) into Eq. (10) yields

$$\frac{\partial^2 w_k}{\partial t_0^2} + 2\varepsilon\frac{\partial^2 w_k}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 w_k}{\partial t_1^2} + \left\{\left(\frac{ck\pi}{L}\right)^2 + \delta\left(\frac{k\pi}{L}\right)^4\right\} w_k$$
$$= \varepsilon \sum_{n=1}^{\infty}{}^{*} \frac{nk}{(n^2-k^2)L}\left[4\alpha\Omega w_n \cos(\Omega t_0) + 8(V_0 + \alpha\sin(\Omega t_0))\frac{\partial w_n}{\partial t_0}\right] + \mathcal{O}(\varepsilon^2). \tag{12}$$

Assuming that $w_k(t_0, t_1;\varepsilon) = w_{k0}(t_0, t_1) + \varepsilon w_{k1}(t_0, t_1) + \varepsilon^2 w_{k2}(t_0, t_1) + \cdots$ then Eq. (12) becomes

$$\left[\frac{\partial^2 w_{k0}}{\partial t_0^2} + \varepsilon\frac{\partial^2 w_{k1}}{\partial t_0^2} + \mathcal{O}(\varepsilon^2)\right] + 2\varepsilon\left[\frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} + \varepsilon\frac{\partial^2 w_{k1}}{\partial t_0 \partial t_1} + \mathcal{O}(\varepsilon^2)\right]$$
$$+\ \mathcal{O}(\varepsilon^2) + \left\{\left(ck\pi/L\right)^2 + \delta\left(k\pi/L\right)^4\right\}(w_{k0} + \varepsilon w_{k1} + \mathcal{O}(\varepsilon^2))$$
$$= \varepsilon \sum_{n=1}^{\infty}{}^{*} \frac{nk}{(n^2-k^2)L}\left[4\alpha\Omega w_{n0} \cos(\Omega t_0) + 8(V_0 + \alpha\sin(\Omega t_0))\frac{\partial w_{n0}}{\partial t_0}\right] + \mathcal{O}(\varepsilon^2). \tag{13}$$

By combining terms of equal powers in ε from Eq. (13) the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations will be obtained as

$$\mathcal{O}(1): \frac{\partial^2 w_{k0}}{\partial t_0^2} + \left[\left(\frac{ck\pi}{L}\right)^2 + \delta\left(\frac{k\pi}{L}\right)^4\right] w_{k0} = 0,$$
$$\mathcal{O}(\varepsilon): \frac{\partial^2 w_{k1}}{\partial t_0^2} + 2\frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} + \left[\left(\frac{ck\pi}{L}\right)^2 + \delta\left(\frac{k\pi}{L}\right)^4\right] w_{k1}$$
$$= \sum_{n=1}^{\infty}{}^{*} \frac{nk}{(n^2-k^2)L}\left[4\alpha\Omega w_{n0} \cos(\Omega t_0) + 8(V_0 + \alpha\sin(\Omega t_0))\frac{\partial w_{n0}}{\partial t_0}\right]. \tag{14}$$

The $\mathcal{O}(1)$ equation can be easily solved, yielding

$$w_{k0} = A_{k0}(t_1)\sin(\omega_k t_0) + B_{k0}(t_1)\cos(\omega_k t_0), \tag{15}$$

where

$$\omega_k^2 = \left(\frac{ck\pi}{L}\right)^2 + \delta\left(\frac{k\pi}{L}\right)^4, \qquad B_{k0}(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x,$$

and

$$A_{k0}(0) = \frac{2}{\omega_k L}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x. \tag{16}$$

The $A_{k0}(t_1)$ and $B_{k0}(t_1)$ in Eq. (15) are still arbitrary and can be used to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$ equation (14).

The $\mathcal{O}(\varepsilon)$ equation now becomes

$$\begin{aligned}\frac{\partial^2 w_{k1}}{\partial t_0^2} + \omega_k^2 w_{k1} = &-2\omega_k[\dot{A}_{k0}(t_1)\cos(\omega_k t_0) - \dot{B}_{k0}(t_1)\sin(\omega_k t_0)] \\ &+ \sum_{n=1}^{\infty}{}^{*} \frac{nk}{(n^2-k^2)L}(4\alpha\Omega\cos(\Omega t_0)[A_{n0}\sin(\omega_n t_0) + B_{n0}\cos(\omega_n t_0)] \\ &+ 8(V_0 + \alpha\sin(\Omega t_0))\omega_n[A_{n0}\cos(\omega_n t_0) - B_{n0}\sin(\omega_n t_0)]).\end{aligned} \tag{17}$$

From Eq. (17) it can readily be seen that there are infinitely many values of Ω that can cause internal resonances. In fact these values are (in an $\mathcal{O}(\varepsilon)$ neighbourhood of) $\omega_n + \omega_k$, $\omega_n - \omega_k$, $\omega_k - \omega_n$, and $-(\omega_n + \omega_k,)$, where $k = n - 2j - 1$, or $k = 2j + 1 + n$ or $k = 2j + 1 - n$ for $j = 0, 1, 2, \ldots$ (see also the summation in Eq. (17)). To show how the secular terms can be eliminated and how the belt system can behave, the (difference type) case $\Omega = \omega_2 - \omega_1$ and its detuned case $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$ with ϕ of order one will be studied in Section 4 while the (sum type) cases $\Omega = \omega_2 + \omega_1$, $\Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$ will be considered in Section 5. For some special values of the beam parameters the case (including detuning) for which a sum type and a difference type of internal resonance coincide will also be studied; that is the case $\Omega = \omega_3 + \omega_2 = \omega_5 - \omega_2$.

4. The case $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$

In this section, the case $\Omega = \omega_2 - \omega_1$, and the case $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$ with ϕ of order one will be studied.

4.1. *The case $\Omega = \omega_2 - \omega_1$*

It is shown in Appendix A that for $\Omega = \omega_2 - \omega_1$ the equation $\Omega \pm \omega_n = \pm\omega_k$ only has the rather trivial solutions $n = 2$ and $k = 1$ if $\Omega - \omega_n = -\omega_k$, and $n = 1$ and $k = 2$ if $\Omega + \omega_n = \omega_k$. Then, by separating those terms in the right side of the $\mathcal{O}(\varepsilon)$ equation (17) that cause secular terms in $w_{k1}(t_0, t_1)$, it is found that A_{k0} and B_{k0} have to satisfy

$$\begin{aligned}\dot{A}_{10} &= \frac{-2\alpha(\omega_1+\omega_2)}{3\omega_1 L}B_{20}, \qquad & \dot{B}_{10} &= \frac{2\alpha(\omega_1+\omega_2)}{3\omega_1 L}A_{20}, \\ \dot{A}_{20} &= \frac{-2\alpha(\omega_1+\omega_2)}{3\omega_2 L}B_{10}, \qquad & \dot{B}_{20} &= \frac{2\alpha(\omega_1+\omega_2)}{3\omega_2 L}A_{10},\end{aligned} \tag{18}$$

and for $k \geqslant 3$,

$$\dot{A}_{k0} = \dot{B}_{k0} = 0.$$

System (18) can readily be solved, yielding

$$A_{10}(t_1) = -\sqrt{\frac{\omega_2}{\omega_1}}\,B_{20}(0)\sin(\gamma t_1) + A_{10}(0)\cos(\gamma t_1),$$
$$B_{10}(t_1) = \sqrt{\frac{\omega_2}{\omega_1}}\,A_{20}(0)\sin(\gamma t_1) + B_{10}(0)\cos(\gamma t_1),$$
$$A_{20}(t_1) = A_{20}(0)\cos(\gamma t_1) - \sqrt{\frac{\omega_1}{\omega_2}}\,B_{10}(0)\sin(\gamma t_1),$$
$$B_{20}(t_1) = B_{20}(0)\cos(\gamma t_1) + \sqrt{\frac{\omega_1}{\omega_2}}\,A_{10}(0)\sin(\gamma t_1), \tag{19}$$

where $\gamma = 2\alpha(\omega_1+\omega_2)/3L\sqrt{\omega_1\omega_2}$ and for $k \geqslant 3$,

$$A_{k0}(t_1) = A_{k0}(0) \quad \text{and} \quad B_{k0}(t_1) = B_{k0}(0).$$

For $n \geqslant 1, A_{n0}(0)$ and $B_{n0}(0)$ can be determined from Eq. (16). From Eq. (17) a solution $w_{k1}(t_0, t_1)$ can now be obtained without unbounded terms (that is without secular terms). So, a formal approximation $w_{k0}(t_0,t_1) + \varepsilon w_{k1}(t_0,t_1)$ of $v_k(t;\varepsilon)$ has been constructed. And finally, an approximation $\sum_{k=1}^{\infty} w_{k0}(t_0,t_1)\sin(k\pi x/L) + \mathcal{O}(\varepsilon)$ of the solution $v(x,t)$ of the initial–boundary-value problem (3–5) with $\Omega = \omega_2 - \omega_1$ is obtained.

4.2. The detuning case $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$

If the frequency of the belt velocity fluctuation is detuned by taking $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$ with $\phi = \mathcal{O}(1)$, then the $\mathcal{O}(\varepsilon)$ equation (17) becomes

$$\begin{aligned}\frac{\partial^2 w_{k1}}{\partial t_0^2} &+ \omega_k^2 w_{k1} \\ = & -2\omega_k[\dot{A}_{k0}\cos(\omega_k t_0) - \dot{B}_{k0}\sin(\omega_k t_0)] \\ & + \sum_{n=1}^{\infty}{}^{*}\frac{nk}{(n^2-k^2)L}(4\alpha\Omega_0\cos(\Omega t_0)[A_{n0}\sin(\omega_n t_0) + B_{n0}\cos(\omega_n t_0)] \\ & + 8(V_0 + \alpha\sin(\Omega t_0))\omega_n[A_{n0}\cos(\omega_n t_0) - B_{n0}\sin(\omega_n t_0)]),\end{aligned} \tag{20}$$

where $\Omega_0 = \omega_2 - \omega_1$. Now, it should be observed that in Eq. (20)

$$\cos(\Omega t) = \cos((\omega_2-\omega_1)t_0 + \phi t_1) = \cos(\Omega_0 t_0)\cos(\phi t_1) - \sin(\Omega_0 t_0)\sin(\phi t_1),$$

and

$$\sin(\Omega t) = \sin((\omega_2-\omega_1)t_0 + \phi t_1) = \sin(\Omega_0 t_0)\cos(\phi t_1) + \cos(\Omega_0 t_0)\sin(\phi t_1).$$

Then, by separating those terms on the right side of Eq. (20) that give rise to secular terms in $w_{k1}(t_0, t_1)$, it is found that in order to avoid secular terms, $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$\dot{A}_{10} = -p\sin(\phi t_1)A_{20} + p\cos(\phi t_1)B_{20}, \qquad \dot{B}_{10} = -p\cos(\phi t_1)A_{20} - p\sin(\phi t_1)B_{20},$$
$$\dot{A}_{20} = -q\sin(\phi t_1)A_{10} - q\cos(\phi t_1)B_{10}, \qquad \dot{B}_{20} = q\cos(\phi t_1)A_{10} - q\sin(\phi t_1)B_{10}, \tag{21}$$

and for $k \geqslant 3$

$$\dot{A}_{k0} = 0 \quad \text{and} \quad \dot{B}_{k0} = 0, \tag{22}$$

where

$$p = \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_1 L}, \qquad q = \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2 L}. \tag{23}$$

Notice that for $\phi = 0$, Eq. (21) is reduced to Eq. (18). In Appendix B, the solution of system (21) has been derived. It turns out that the solution of system (21) is

$$\begin{aligned}
A_{10}(t_1) &= K_1 \sin(\beta_1 t_1) + K_2 \cos(\beta_1 t_1) + K_3 \sin(\beta_2 t_1) + K_4 \cos(\beta_2 t_1),\\
B_{10}(t_1) &= \frac{1}{pq\phi}[A_{10}^{(3)} + (\phi^2 - pq)\dot{A}_{10}],\\
A_{20}(t_1) &= \frac{-1}{p}[\dot{A}_{10} \sin(\phi t_1) + \dot{B}_{10} \cos(\phi t_1)],\\
B_{20}(t_1) &= \frac{1}{p}[\dot{A}_{10} \cos(\phi t_1) - \dot{B}_{10} \sin(\phi t_1)],
\end{aligned} \tag{24}$$

and

$$A_{k0} = A_{k0}(0), \quad B_{k0} = B_{k0}(0) \quad \text{for } k \geqslant 3.$$

Note also that in Eq. (24) $\phi \neq 0$. In Eq. (24), K_1, K_2, K_3, and K_4 are constants of integration, p, q are given by Eq. (23),

$$\beta_1 = \sqrt{\tfrac{1}{2}[\phi^2 - 2pq - \sqrt{\phi^4 - 4pq\phi^2}]}, \quad \text{and} \quad \beta_2 = \sqrt{\tfrac{1}{2}[\phi^2 - 2pq + \sqrt{\phi^4 - 4pq\phi^2}]}.$$

As in Section 4.1, an approximation $\sum_{k=1}^{\infty} w_{k0}(t_0, t_1) \sin(k\pi x/L) + \mathcal{O}(\varepsilon)$ of the solution $v(x, t)$ of the initial–boundary-value problem (3–5) with $\Omega = \omega_2 - \omega_1 + \varepsilon\phi$ and $\phi = \mathcal{O}(1)$ has now been constructed.

For Ω in a neighbourhood of $\omega_2 - \omega_1$, now can be concluded (see also Eq. (24)) that no instabilities for the belt system will occur. A similar analysis as given in this section can be applied to other cases where Ω is of the difference type (that is $\Omega = \omega_m - \omega_n$ for some m and n). However, in some of these cases instabilities can occur as will be explained in the next section.

5. Ω is a sum of two natural frequencies

It has been shown in Section 3 that in order to remove secular terms, one has to solve the equation $\Omega \pm \omega_n = \pm\omega_k$, where $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $j = 0, 1, 2, \ldots$. In the case $\Omega = \omega_2 - \omega_1$, it has been shown in Section 4 that the only solutions of the problem $(\omega_2 - \omega_1) \pm \omega_n = \pm\omega_k$ (for an arbitrary value of $\mu^2 = \delta\pi^2/c^2L^2$) are the trivial solutions $k = 1, n = 2$ and symmetrically $k = 2, n = 1$. For other values of Ω and for certain specific values of μ^2 solutions other than the trivial ones may occur. Cases $\Omega = \omega_2 + \omega_1$, $\Omega = 2\omega_1$ and $\Omega = \omega_3 + \omega_2$ will be considered in this section, other cases can be investigated similarly.

5.1. The case $\Omega = \omega_2 + \omega_1$

First, in order to solve the equation $\Omega \pm \omega_n = \pm\omega_k$, three cases have to be considered:

(i) $-\omega_n - \omega_k = \Omega$, which obviously has no solution since the right side is positive while the left side is negative,

(ii) $\omega_n + \omega_k = \Omega$, which obviously has only the trivial solution $k = 2, n = 1$ or $k = 1, n = 2$,

(iii) $\omega_k - \omega_n = \Omega$ (or equivalently $\omega_n - \omega_k = \Omega$) which may or may not have solutions depending on the value of μ^2. From $\omega_k - \omega_n = \Omega$, it follows that $k\sqrt{1+k^2\mu^2} = n\sqrt{1+n^2\mu^2} + 2\sqrt{1+4\mu^2} + \sqrt{1+\mu^2}$. Since $f(k) = k\sqrt{1+\mu^2k^2}$ is an increasing function it then follows from $k\sqrt{1+\mu^2k^2} > n\sqrt{1+\mu^2n^2}$ that $k > n \geqslant 1$. Then from $1 \leqslant n < k$ it follows that

$$\begin{aligned} k\sqrt{1+k^2\mu^2} &= n\sqrt{1+n^2\mu^2} + 2\sqrt{1+4\mu^2} + \sqrt{1+\mu^2} \\ &< n\sqrt{1+k^2\mu^2} + 2\sqrt{1+k^2\mu^2} + \sqrt{1+k^2\mu^2} \\ &\Rightarrow n < k < n+3, \quad \Rightarrow k = n+1 \quad \text{or } k = n+2. \end{aligned}$$

Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it follows that k can only be equal to $k = n + 1$. So, $\omega_k - \omega_n = \Omega = \omega_2 + \omega_1$ can only have solutions for $k = n + 1$. The possibility of having solutions turns out to be dependent on the values of μ^2. In Table 1, some of these solutions are given.

Assuming that $\Omega \pm \omega_n = \pm\omega_k$ only has the trivial solutions ($k = 2$ and $n = 1$, and $k = 1$ and $n = 2$) it turns out that no secular terms occur in the solution of Eq. (17) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_1} B_{20}, \qquad & \dot{B}_{10} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_1} A_{20}, \\ \dot{A}_{20} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_2} B_{10}, \qquad & \dot{B}_{20} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_2} A_{10}, \end{aligned} \tag{25}$$

Table 1
Values of k, n, and μ^2 for which $\omega_k = \omega_n + \Omega$ has solutions

$\Omega = \omega_2 + \omega_1$			$\Omega = 2\omega_1$			$\Omega = \omega_3 + \omega_2$					
k	n	μ^2	k	n	μ^2	k	n	μ^2	k	n	μ^2
3	2	—	2	1	0.7143	6	5	—	4	1	0.3851
4	3	0.3851	3	2	0.1664	7	6	—	5	2	0.0732
5	4	0.1607	4	3	0.0773	8	7	0.4588	6	3	0.0349
6	5	0.0926	5	4	0.0451	9	8	0.2124	7	4	0.0211
7	6	0.0613	6	5	0.0297	10	9	0.1321	8	5	0.0143
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

and for $k \geqslant 3$,

$$\dot{A}_{k0} = \dot{B}_{k0} = 0.$$

The solution of Eq. (25) can readily be determined, yielding

$$\begin{aligned}
A_{10}(t_1) &= A_{10}(0)\cosh(r_1 t_1) - \sqrt{\frac{\omega_2}{\omega_1}} B_{20}(0)\sinh(r_1 t_1),\\
A_{20}(t_1) &= A_{20}(0)\cosh(r_1 t_1) - \sqrt{\frac{\omega_1}{\omega_2}} B_{10}(0)\sinh(r_1 t_1),\\
B_{10}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_1}} A_{20}(0)\sinh(r_1 t_1) + B_{10}(0)\cosh(r_1 t_1),\\
B_{20}(t_1) &= -\sqrt{\frac{\omega_1}{\omega_2}} A_{10}(0)\sinh(r_1 t_1) + B_{20}(0)\cosh(r_1 t_1),
\end{aligned} \tag{26}$$

where $r_1 = 2\alpha(\omega_2 - \omega_1)/3L\sqrt{\omega_1\omega_2}$, and for $k \geqslant 3$, $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. It is obvious from Eq. (26) that instabilities for the belt system will occur. When for instance, $\mu^2 \approx 0.3851$ it turns out that $\Omega \pm \omega_n = \pm\omega_k$ also has other solutions than the trivial ones (see Table 1). To avoid secular terms in the solution of Eq. (17) it then turns out that A_{10}, B_{10}, A_{20}, and B_{20} still have to satisfy Eq. (25), and that

$$\begin{aligned}
\dot{A}_{30} &= \frac{-12\alpha}{7L\omega_3}(\omega_3 + \omega_4)B_{40}, & \dot{B}_{30} &= \frac{12\alpha}{7L\omega_3}(\omega_3 + \omega_4)A_{40},\\
\dot{A}_{40} &= \frac{-12\alpha}{7L\omega_4}(\omega_3 + \omega_4)B_{30}, & \dot{B}_{40} &= \frac{12\alpha}{7L\omega_4}(\omega_3 + \omega_4)A_{30},
\end{aligned} \tag{27}$$

and that for $k \geqslant 5$, $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of Eq. (27) can readily be determined, yielding

$$\begin{aligned}
A_{30}(t_1) &= -\sqrt{\frac{\omega_4}{\omega_3}} B_{40}(0)\sin(\beta t_1) + A_{30}(0)\cos(\beta t_1),\\
A_{40}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_4}} B_{30}(0)\sin(\beta t_1) + A_{40}(0)\cos(\beta t_1),\\
B_{30}(t_1) &= \sqrt{\frac{\omega_4}{\omega_3}} A_{40}(0)\sin(\beta t_1) + B_{30}(0)\cos(\beta t_1),\\
B_{40}(t_1) &= \sqrt{\frac{\omega_3}{\omega_4}} A_{30}(0)\sin(\beta t_1) + B_{40}(0)\cos(\beta t_1),
\end{aligned} \tag{28}$$

where $\beta = 12\alpha(\omega_3 + \omega_4)/7L\sqrt{\omega_3\omega_4}$ and for $k \geqslant 5$, $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. Also for $\mu^2 \approx 0.3851$ it is obvious that instabilities for the belt system will occur. It should be observed that $\Omega = \omega_2 + \omega_1 = \omega_4 - \omega_3$ for $\mu^2 \approx 0.3851$. So, for special values of the beam parameters, also frequency Ω of difference type can lead to instabilities.

5.1.1. The detuning case $\Omega = \omega_2 + \omega_1 + \varepsilon\phi$

In the detuning case of $\Omega = \omega_2 + \omega_1 + \varepsilon\phi$ secular terms will not occur if

$$\begin{aligned}
\dot{A}_{10} &= \frac{2\alpha}{3L\omega_1}(\omega_2 - \omega_1)[A_{20}\sin(\phi t_1) - B_{20}\cos(\phi t_1)],\\
\dot{B}_{10} &= -\frac{2\alpha}{3L\omega_1}(\omega_2 - \omega_1)[A_{20}\cos(\phi t_1) + B_{20}\sin(\phi t_1)],\\
\dot{A}_{20} &= \frac{2\alpha}{3L\omega_2}(\omega_2 - \omega_1)[A_{10}\sin(\phi t_1) - B_{10}\cos(\phi t_1)],\\
\dot{B}_{20} &= -\frac{2\alpha}{3L\omega_2}(\omega_2 - \omega_1)[A_{10}\cos(\phi t_1) + B_{10}\sin(\phi t_1)],
\end{aligned} \tag{29}$$

and for $k \geqslant 3$,

$$\dot{A}_{k0} = 0 \quad \text{and} \quad \dot{B}_{k0} = 0.$$

In the following it will be assumed that $\alpha > 0$ (for $\alpha < 0$ the procedure is the same). By putting $p = (2\alpha/3L\omega_1)(\omega_2 - \omega_1)$ and $q = (2\alpha/3L\omega_2)(\omega_2 - \omega_1)$, and by differentiating $\dot{A}_{10}$ once more it follows that

$$\ddot{A}_{10} = pqA_{10} - \phi\dot{B}_{10}. \tag{30}$$

By differentiating $\ddot{A}_{10}$ twice and by making use of Eq. (30) it then follows that

$$A_{10}^{(4)} + (\phi^2 - 2pq)\ddot{A}_{10} + p^2q^2A_{10} = 0, \tag{31}$$

where $A_{10}^{(4)}$ is the fourth order derivative of A_{10} with respect to t_1. This fourth order differential equation can be solved elementarily, with the result that:

- for $\phi^2 > 4pq$ the solutions of Eq. (29) will be stable,
- for $\phi^2 = 4pq$ the solutions of Eq. (29) will be unstable (due to the linear term in t_1), and finally
- for $\phi^2 < 4pq$ the solutions of Eq. (29) will increase exponentially.

In Fig. 1, the stability regions for system (29) in (α, ϕ)-plane for positive values of α have been given. The bifurcation lines are given by $\phi^2 = 4pq = k^2(L,\mu)\alpha^2$, where $k^2(L,\mu) = 8(2\sqrt{1+4\mu^2} - \sqrt{1+\mu^2})^2/9L^2\sqrt{1+4\mu^2}\sqrt{1+\mu^2}$. From $\phi^2 = k^2\alpha^2$ it follows that $(\phi - k\alpha)(\phi + k\alpha) = 0$. The slope k is a function of L and μ and for fixed L it can be shown that $8/9L^2 < k^2 < 4/L^2$. Values of α and ϕ located in regions I and IV lead to stable solutions of system (29) while values of α and ϕ in regions II and III (including the lines $\phi^2 = k^2\alpha^2$) lead to unstable solutions of system (29).

5.2. The case $\Omega = 2\omega_1$

As in Section 5.1, in order to solve the equation $\Omega \pm \omega_n = \pm\omega_k$, again three cases have to be considered:

(i) $-\omega_n - \omega_k = \Omega = 2\omega_1$, which obviously has no solutions.
(ii) $\omega_n + \omega_k = \Omega = 2\omega_1$, which obviously only can be satisfied for $k = n = 1$. But since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$, it follows that there is in this case no solution.

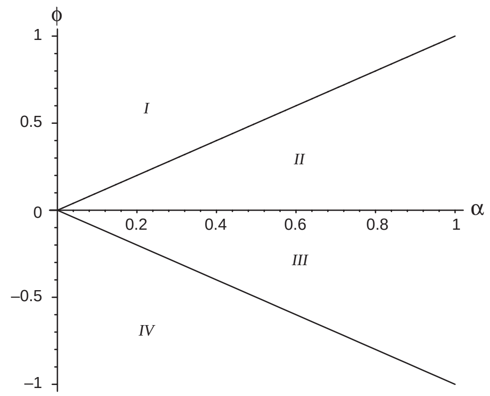

Fig. 1. Areas of stability of system (29) for a specific value of $k = 1$.

(iii) $\omega_k - \omega_n = \Omega = 2\omega_1$ (or equivalently $\omega_n - \omega_k = \Omega$), which may or may not have solutions depending on the value of μ^2. From $\omega_k - \omega_n = 2\omega_1$ it follows that $k\sqrt{1+\mu^2k^2} = n\sqrt{1+\mu^2n^2} + 2\sqrt{1+\mu^2}$. Since $f(k) = k\sqrt{1+\mu^2k^2}$ is an increasing function it then follows from $k\sqrt{1+\mu^2k^2} > n\sqrt{1+\mu^2n^2}$ that $k > n \geqslant 1$. Then, from $1 \leqslant n < k$ it follows that

$$\begin{aligned} k\sqrt{1+\mu^2k^2} &= n\sqrt{1+\mu^2n^2} + 2\sqrt{1+\mu^2} \\ &< n\sqrt{1+\mu^2k^2} + 2\sqrt{1+\mu^2k^2} \\ &\Rightarrow n < k < n+2 \Rightarrow k = n+1. \end{aligned}$$

So, $\omega_k - \omega_n = 2\omega_1$ can only have solutions for $k = n+1$. The possibility of having solutions turns out to be dependent on the values of μ^2. In Table 1, some of these solutions are given.

When μ^2 is not (in neighbourhood of) a value as listed in Table 1 then it easily follows from Eq. (17) that no secular terms occur in the solution if $\dot{A}_{k0} = \dot{B}_{k0} = 0$ for all $k \geqslant 1$. When for instance $\mu^2 \approx 0.7143$ it turns out that no secular terms occur in the solution of Eq. (17) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= \frac{-4\alpha}{3L\omega_1}(\omega_2 - \omega_1)B_{20}, & \dot{B}_{10} &= \frac{4\alpha}{3L\omega_1}(\omega_2 - \omega_1)A_{20}, \\ \dot{A}_{20} &= \frac{-8\alpha\omega_1}{3L\omega_2}B_{10}, & \dot{B}_{20} &= \frac{8\alpha\omega_1}{3L\omega_2}A_{10}, \end{aligned} \tag{32}$$

and for $k \geqslant 3\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of system (32) can readily be determined, yielding:

$$\begin{aligned} A_{10}(t_1) &= -CB_{20}(0)\sin(\theta t_1) + A_{10}(0)\cos(\theta t_1), \\ A_{20}(t_1) &= -\frac{1}{C}B_{10}(0)\sin(\theta t_1) + A_{20}(0)\cos(\theta t_1), \\ B_{10}(t_1) &= CA_{20}(0)\sin(\theta t_1) + B_{10}(0)\cos(\theta t_1), \\ B_{20}(t_1) &= \frac{1}{C}A_{10}(0)\sin(\theta t_1) + B_{20}(0)\cos(\theta t_1), \end{aligned} \tag{33}$$

where $C^2 = \omega_2(\omega_2 - \omega_1)/2\omega_1^2$, $\theta = (4\sqrt{2}\alpha/3L)\sqrt{(\omega_2 - \omega_1)/\omega_2}$ and for $k \geqslant 3$,

$$A_{k0}(t_1) = A_{k0}(0), \quad B_{k0}(t_1) = B_{k0}(0).$$

Obviously, no instabilities for the belt system will occur when $\mu^2 \approx 0.7143$ or when μ^2 is not (in neighbourhood of) a value as listed in Table 1. It should be remarked that a similar analysis can be performed if $\Omega = 2\omega_N$ for some fixed $N > 1$.

5.3. The case $\Omega = \omega_3 + \omega_2$

As in the previous two Sections 5.1 and 5.2, again the following three cases have to be considered in order to solve the equation $\Omega \pm \omega_n = \pm\omega_k$. Those cases are:

(i) $-\omega_n - \omega_k = \Omega = \omega_2 + \omega_3$, which obviously has no solution;
(ii) $\omega_n + \omega_k = \Omega = \omega_2 + \omega_3$, which obviously has the trivial solutions $k = 2$ and $n = 3$, or $k = 3$ and $n = 2$. In this case additional solutions can only occur if $\omega_k + \omega_1 = \Omega = \omega_2 + \omega_3$ has a solution. In Appendix A (see the case $\omega_k = \omega_n + \omega_2 - \omega_1$) it has been shown that this is not possible;
(iii) $\omega_k - \omega_n = \Omega = \omega_2 + \omega_3$ (or equivalently $\omega_n - \omega_k = \Omega$), which may or may not have solutions depending on the value of μ^2. From $\omega_k - \omega_n = \Omega$ it follows that $k\sqrt{1 + \mu^2 k^2} = n\sqrt{1 + \mu^2 n^2} + 3\sqrt{1 + 9\mu^2} + 2\sqrt{1 + 4\mu^2}$. Since $f(k) = k\sqrt{1 + \mu^2 k^2}$ is an increasing function it then follows that $k > n$ and $k > 3$. Then, from $k > n$ and $k > 3$ it follows that

$$\begin{aligned} k\sqrt{1 + \mu^2 k^2} &= n\sqrt{1 + \mu^2 n^2} + 3\sqrt{1 + 9\mu^2} + 2\sqrt{1 + 4\mu^2} \\ &< n\sqrt{1 + \mu^2 k^2} + 3\sqrt{1 + \mu^2 k^2} + 2\sqrt{1 + \mu^2 k^2} \\ &\Rightarrow n < k < n + 3 + 2 \\ &\Rightarrow k = n + 1 \text{ or } k = n + 2 \quad \text{or } k = n + 3 \text{ or } k = n + 4. \end{aligned}$$

Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it follows that k can only be equal to $n + 1$ or $n + 3$. The possibility to have solutions turns out to be dependent on the values of μ^2. In Table 1, some of these solutions are given.

Assuming that $\Omega \pm \omega_n = \pm\omega_k$ only has the trivial solutions ($k = 2$ and $n = 3$, and $k = 3$ and $n = 2$) it turns out that no secular terms occur in the solution of Eq. (17) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)B_{30}, & \dot{B}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)A_{30}, \\ \dot{A}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)B_{20}, & \dot{B}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)A_{20}, \end{aligned} \tag{34}$$

and for $k = 1, 4, 5, 6, \ldots,\ \dot{A}_{k0} = \dot{B}_{k0} = 0$. The solutions of system (34) can readily be determined, yielding:

$$\begin{aligned}
A_{20}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_2}}\, B_{30}(0)\sinh(s_1 t_1) + A_{20}(0)\cosh(s_1 t_1),\\
A_{30}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_3}}\, B_{20}(0)\sinh(s_1 t_1) + A_{30}(0)\cosh(s_1 t_1),\\
B_{20}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_2}}\, A_{30}(0)\sinh(s_1 t_1) + B_{20}(0)\cosh(s_1 t_1),\\
B_{30}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_3}}\, A_{20}(0)\sinh(s_1 t_1) + B_{30}(0)\cosh(s_1 t_1),
\end{aligned} \tag{35}$$

where $s_1 = 6\alpha(\omega_3 - \omega_2)/5L\sqrt{\omega_3\omega_2}$, and for $k = 1, 4, 5, 6, \ldots,$

$$A_{k0} = A_{k0}(0) \quad \text{and} \quad B_{k0}(t_1) = B_{k0}(0).$$

From Eq. (35) it is obvious that instabilities for the belt system will occur. When for instance $\mu^2 = 0.0732$ it turns out that $\Omega \pm \omega_n = \pm\omega_k$ also has other solutions than the trivial ones (see Table 1). To avoid secular terms in the solution of Eq. (17) it now turns out that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\begin{aligned}
\dot{A}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)B_{30} - \frac{10\alpha}{21L\omega_2}(\omega_5 + \omega_2)B_{50},\\
\dot{B}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)A_{30} + \frac{10\alpha}{21L\omega_2}(\omega_5 + \omega_2)A_{50},\\
\dot{A}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)B_{20}, \qquad \dot{B}_{30} = -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)A_{20},\\
\dot{A}_{50} &= -\frac{10\alpha}{21L\omega_5}(\omega_5 + \omega_2)B_{20}, \qquad \dot{B}_{50} = \frac{10\alpha}{21L\omega_5}(\omega_5 + \omega_2)A_{20},
\end{aligned} \tag{36}$$

and for $k = 1, 4, 6, 7, 8, \ldots,$

$$\dot{A}_{k0} = \dot{B}_{k0} = 0.$$

The solution of Eq. (36) can readily be determined, yielding

$$\begin{aligned}
A_{20}(t_1) &= K_1\sin(s_2 t_1) + A_{20}(0)\cos(s_2 t_1),\\
B_{20}(t_1) &= K_2\sin(s_2 t_1) + B_{20}(0)\cos(s_2 t_1),\\
A_{30}(t_1) &= \frac{d_1 K_2}{s_2\omega_3}\cos(s_2 t_1) - \frac{d_1 B_{20}(0)}{s_2\omega_3}\sin(s_2 t_1) + \left(A_{30}(0) - \frac{d_1 K_2}{s_2\omega_3}\right),\\
B_{30}(t_1) &= \frac{d_1 K_1}{s_2\omega_3}\cos(s_2 t_1) - \frac{d_1 A_{20}(0)}{s_2\omega_3}\sin(s_2 t_1) + \left(B_{30}(0) - \frac{d_1 K_1}{s_2\omega_3}\right),\\
A_{50}(t_1) &= \frac{d_2 K_2}{s_2\omega_5}\cos(s_2 t_1) - \frac{d_2 B_{20}(0)}{s_2\omega_5}\sin(s_2 t_1) + \left(A_{50}(0) - \frac{d_2 K_2}{s_2\omega_5}\right),\\
B_{50}(t_1) &= \frac{-d_2 K_1}{s_2\omega_5}\cos(s_2 t_1) + \frac{d_2 A_{20}(0)}{s_2\omega_5}\sin(s_2 t_1) + \left(B_{50}(0) + \frac{d_2 K_1}{s_2\omega_5}\right),
\end{aligned} \tag{37}$$

where

$$s_2 = \left[\left(\frac{10}{21}\right)^2 \frac{(\omega_2+\omega_5)^2}{\omega_2\omega_5} - \left(\frac{6}{5}\right)^2 \frac{(\omega_3-\omega_2)^2}{\omega_3\omega_2} \right],$$
$$K_1 = \frac{-1}{s_2}\left[\frac{d_1}{\omega_2} B_{30}(0) + \frac{d_2}{\omega_2} B_{50}(0)\right],$$
$$K_2 = \frac{1}{s_2}\left[\frac{-d_1}{\omega_2} B_{30}(0) + \frac{d_2}{\omega_2} B_{50}(0)\right],$$
$$d_1 = \frac{6\alpha(\omega_3-\omega_2)}{5L},$$
$$d_2 = \frac{10\alpha(\omega_5+\omega_2)}{21L},$$

and for $k = 1, 4, 6, 7, 8, \ldots,$

$$A_{k0}(t_1) = A_{k0}(0) \quad \text{and} \quad B_{k0}(t_1) = B_{k0}(0).$$

From Eq. (37) it is obvious that now no instabilities for the belt system will occur. It should be observed that $\Omega = \omega_2 + \omega_3 = \omega_5 - \omega_2$ for $\mu^2 \approx 0.0732$. So, for special values of the beam parameters also frequencies Ω of sum type can lead to stable behaviour. To obtain more insight in the complicated dynamical behaviour of the belt system, in the next section the beam parameter μ^2 will be detuned (keeping Ω fixed).

5.3.1. *The detuned case $\mu^2 \approx 0.0732$*

In the previous Section 5.3 it has been shown that if $\Omega = \omega_2 + \omega_3$ (and μ^2 is not in the neighbourhood of a value as listed in Table 1) then the belt system is unstable. For $\mu^2 \approx 0.0732$, however, the belt system is stable. To obtain more insight in this different behaviour μ^2 in a neighbourhood of 0.0732 will be detuned. By observing that $\mu^2 = \delta\pi^2/c^2L^2$ with $\delta = E_bI_y/\rho A$, detuning of μ^2 can be achieved by detuning δ in the original PDE (6). For that reason it will be considered that $\mu^2 = \mu_{cr}^2 + \varepsilon\psi$ with $\mu_{cr}^2 = 0.0732$ and ψ an arbitrary constant of $\mathcal{O}(1)$, and $\delta = \delta_{cr} + \varepsilon v$ with $\psi = (\pi^2/c^2L^2)v$ and $\mu_{cr}^2 = (\pi^2/c^2L^2)\delta_{cr}$. The frequency Ω is kept fixed, that is, $\Omega = \sqrt{(2c\pi/L)^2 + \delta_{cr}(2\pi/L)^4} + \sqrt{(3c\pi/L)^2 + \delta_{cr}(3\pi/L)^4}$. It should be observed that this type of detuning is different from the one studied in Section 4. By replacing δ in Eq. (6) by $\delta_{cr} + \varepsilon v$ the same analysis as presented in Sections 2 and 3 can be repeated. To avoid secular terms in the approximation it turns out that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ now have to satisfy ($\phi = \pi^3 v/cL^3$, and $\bar\omega_k = k\sqrt{1+\mu_{cr}^2k^2}$ for $k = 2, 3$, and 5)

$$\dot A_{20} = -\frac{6\alpha}{5L\bar\omega_2}(\bar\omega_3 - \bar\omega_2)B_{30} - \frac{10\alpha}{21L\bar\omega_2}(\bar\omega_5 + \bar\omega_2)B_{50} - \frac{8\pi^4\phi}{\bar\omega_2 L^4}B_{20},$$
$$\dot B_{20} = -\frac{6\alpha}{5L\bar\omega_2}(\bar\omega_3 - \bar\omega_2)A_{30} + \frac{10\alpha}{21L\bar\omega_2}(\bar\omega_5 + \bar\omega_2)A_{50} + \frac{8\pi^4\phi}{\bar\omega_2 L^4}A_{20},$$
$$\dot A_{30} = -\frac{6\alpha}{5L\bar\omega_3}(\bar\omega_3 - \bar\omega_2)B_{20} - \frac{81\pi^4\phi}{2\bar\omega_3 L^4}B_{30},$$

$$\dot{B}_{30} = -\frac{6\alpha}{5L\bar{\omega}_3}(\bar{\omega}_3 - \bar{\omega}_2)A_{20} + \frac{81\pi^4\phi}{2\bar{\omega}_3L^4}A_{30},$$

$$\dot{A}_{50} = -\frac{10\alpha}{21L\bar{\omega}_5}(\bar{\omega}_5 + \bar{\omega}_2)B_{20} - \frac{625\pi^4\phi}{2\bar{\omega}_5L^4}B_{50},$$

$$\dot{B}_{50} = \frac{10\alpha}{21L\bar{\omega}_5}(\bar{\omega}_5 + \bar{\omega}_2)A_{20} + \frac{625\pi^4\phi}{2\bar{\omega}_5L^4}A_{50}, \tag{38}$$

and for $k = 1, 4, 6, 7, 8, \ldots,$

$$\dot{A}_{k0} = -\frac{v}{2\bar{\omega}_k}\left(\frac{k\pi}{L}\right)^4 B_{k0} \quad \text{and} \quad \dot{B}_{k0} = \frac{v}{2\bar{\omega}_k}\left(\frac{k\pi}{L}\right)^4 A_{k0}. \tag{39}$$

Obviously, system (39) has a bounded solution. The characteristic equation of system (38) is

$$\lambda^6 + (1502.1631\phi^2 + 1.8787\eta^2)\lambda^4 + (0.8824\eta^4 - 1230.2972\eta^2\phi^2 + 170023.5061\phi^4)\lambda^2 + 874.7894\eta^4\phi^2 + 0.1876\,10^7\phi^6 - 81019.0927\eta^2\phi^4 = 0, \tag{40}$$

where $\eta = \alpha/L$. By putting $\lambda^2 = a$ in Eq. (40) the following cubic equation for a is obtained

$$a^3 + (1502.1631\phi^2 + 1.8787\eta^2)a^2 + (0.8824\eta^4 - 1230.2972\eta^2\phi^2 + 170023.5061\phi^4)a + 874.7894\eta^4\phi^2 + 0.1876\,10^7\phi^6 - 81019.0927\eta^2\phi^4 = 0. \tag{41}$$

Eq. (41) can be solved by using the Cardano's formula. The *radicand* R (of the reduced form of the cubic equation (41)) plays an important role in the solution structure. When the radicand R is positive the reduced cubic equation of Eq. (41) will have one real, and two complex conjugate solutions. Since $a = \lambda^2$ it follows that at least two roots of the characteristic equation (40) will have a positive real part. Consequently, the solution of system (38) will be unstable. For $R < 0$ the cubic equation (41) will have three distinct real roots, and for $R = 0$ there are three real roots of which two coincide. For $R \leqslant 0$ it requires an additional analysis to determine whether system (38) is stable or not.

In Fig. 2 the bifurcation values of R as a function of ϕ and η have been given. In this figure it has been assumed that η and so α (the amplitude of the speed fluctuation) are positive. Similar results can be found for $\eta < 0$. When ϕ and η are in the areas II and V then the solutions of Eq. (41) are positive, leading to the unstable solutions for Eq. (38), whereas when ϕ and η are in the areas I, III, IV or VI the solutions of Eq. (41) will be negative leading to stable solutions for Eq. (38). When ϕ and η are exactly on the curves the solutions of Eq. (41) will be also negative which leads to stable solutions of Eq. (38).

6. Conclusions and remarks

In this paper, initial–boundary-value problems for a beam equation have been studied. The equations can be used as simple models to describe the vertical vibrations of a conveyor belt for which the time-varying belt velocity is small with respect to the wave speed. It is assumed that the belt velocity $V(t) = \varepsilon(V_0 + \alpha\sin(\Omega t))$ where ε, V_0, α, and Ω are constants with $0 < \varepsilon \ll 1$ and $|\alpha| < V_0$. Complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or

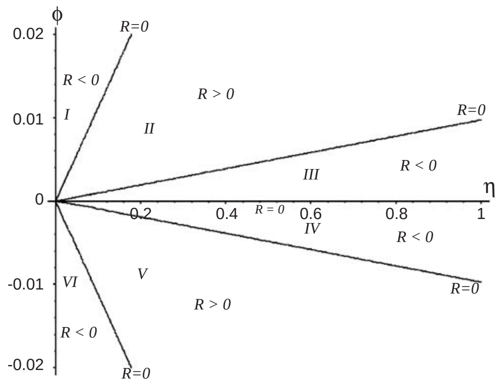

Fig. 2. Bifurcation values of R as a function of η and ϕ.

difference of any two natural frequencies of the system for which the belt velocity is equal to zero. For special values of the belt parameters these sum type and difference type of internal resonances can coincide giving rise to even more complicated dynamical behaviour. For both sum type and difference type of internal resonances instabilities for the belt system can occur.

In this paper, the following cases have been studied in detail with the following results:

(i) $\Omega = \omega_2 - \omega_1$; interaction between the first and the second vibration modes; no instabilities for the belt system (also for the detuned case).
(ii) $\Omega = \omega_2 + \omega_1$; interactions between the first and the second vibration modes, and for special values of the beam parameters (see Table 1) additional interactions; there will always be unstable behaviour of the belt system.
(iii) The detuned case $\Omega = \omega_2 + \omega_1 + \varepsilon\phi$; interactions occur between the first and the second vibration modes. Solutions will be unstable if $\phi^2 \leqslant 4pq$, while for $\phi^2 > 4pq$ the solutions are stable ($p = (2\alpha/3L\omega_1)(\omega_2 - \omega_1)$ and $q = (2\alpha/3L\omega_2)(\omega_2 - \omega_1)$).
(iv) $\Omega = 2\omega_1$; only for special values of the beam parameters (see Table 1) there will be an interaction between two different vibration modes; there are no instabilities for the belt system.
(v) $\Omega = \omega_2 + \omega_3$; interaction between the second and the third vibration modes, and for special values of beam parameters (see Table 1) there are additional interactions; in general there will be instabilities for the belt system. However, for special values of the beam parameters there can be stable behaviour of the belt system. When some of these beam parameters are detuned unstable behaviour can occur again (see Section 5.3.1 where $\Omega = \omega_2 + \omega_3 = \omega_5 - \omega_2$ for $\mu^2 = E_b I_y \pi^2/\rho A c^2 L^2 \approx 0.0732$).

It is expected that for other values of Ω, the same techniques (as presented in this paper) can be applied to determine the stability properties of the belt system.

Appendix A

In this appendix it will be shown that the equation $\Omega \pm \omega_n = \pm\omega_k$ with $\Omega = \omega_2 - \omega_1$ only has as solutions $n = 2$ and $k = 1$ if $\Omega - \omega_n = -\omega_k$, and $n = 1$ and $k = 2$ if $\Omega + \omega_n = \omega_k$. To prove this, the following four cases have to be considered: $\omega_k = \omega_n + \omega_2 - \omega_1$, $\omega_k = -\omega_n + \omega_2 - \omega_1$, $-\omega_k = \omega_n + \omega_2 - \omega_1$, and $-\omega_k = -\omega_n + \omega_2 - \omega_1$. Note that $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$.

A.1. The case $\omega_k = \omega_n + \omega_2 - \omega_1$

Since $\omega_k^2 = (ck\pi/L)^2 + \delta(k\pi/L)^4$, it follows from $\omega_k = \omega_n + \omega_2 - \omega_1$ that

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}} - 1, \tag{A.1}$$

where $\mu^2 = \delta\pi^2/c^2L^2$. It can easily be shown that $f(k) = k\sqrt{1+\mu^2k^2}/\sqrt{1+\mu^2}$ is an increasing function in k, and that $k \leqslant f(k) < k^2$. Then it follows from Eq. (A.1) that

$$\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} < \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}}. \tag{A.2}$$

Since $f(k)$ is increasing in k it follows from the first inequality in Eq. (A.2) that $1 \leqslant n < k$. From the second inequality in Eq. (A.2) it then follows that

$$\begin{aligned} &\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} \\ &\Rightarrow k < n+2. \end{aligned} \tag{A.3}$$

Consequently, $k = n + 1$, and Eq. (A.1) becomes

$$\frac{(n+1)\sqrt{1+\mu^2(n+1)^2}}{\sqrt{1+\mu^2}} - \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} = \frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}} - \frac{\sqrt{1+\mu^2}}{\sqrt{1+\mu^2}}.$$

Denoting the left side of the last equation by $g(n)$ then the right side of the equation is just $g(1)$. It is not too difficult to show that $g(n)$ is an increasing function, so the last equation can only be satisfied if $n = 1$. Since $k = n + 1$ it follows that the only solution in this case is $k = 2$ and $n = 1$.

A.2. The case $\omega_k = -\omega_n + \omega_2 - \omega_1$

In this case it follows from $\omega_k = -\omega_n + \omega_2 - \omega_1$ that

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = -\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}} - 1. \tag{A.4}$$

The only candidate for a solution of this equation is $n = 1$ since the left side is always positive while the right side is negative for $n \geqslant 2$. Accordingly, by substituting $n = 1$ into Eq. (A.4) it will follow that:

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}} - 2. \tag{A.5}$$

Now it should be observed that the left side of Eq. (A.5) is between k and k^2, and that the right side is between 0 and 2. So, the only candidate for a solution is $k = 1$ (and $n = 1$). Since $k =$

$n-2j-1$, or $k=n+2j+1$, or $k=2j+1-n$ with $k,n\in\mathbb{N}^+$ and $j\in\mathbb{N}$ it easily follows that there are no solutions in this case.

A.3. The case $-\omega_k=\omega_n+\omega_2-\omega_1$

In this case it follows from $-\omega_k=\omega_n+\omega_2-\omega_1$ that

$$-\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}}=\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}}+\frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}}-1.$$

Now the left side is always negative while the right side is always positive. So, there are no solutions in this case.

A.4. The case $-\omega_k=-\omega_n+\omega_2-\omega_1$

In this case it follows from $-\omega_k=-\omega_n+\omega_2-\omega_1$ that

$$\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}}=\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}}+\frac{2\sqrt{1+\mu^2 2^2}}{\sqrt{1+\mu^2}}-1.$$

By interchanging n and k, this case becomes the first case. So, the only solution in this case in $k=1$ and $n=2$.

This completes the proof of the statement at the beginning of this appendix.

Appendix B

In this appendix the solutions of Eq. (21) will be determined, that is, the solutions of:

$$\begin{aligned}
\dot{A}_{10}&=-p\sin(\phi t_1)A_{20}+p\cos(\phi t_1)B_{20},\\
\dot{B}_{10}&=-p\cos(\phi t_1)A_{20}-p\sin(\phi t_1)B_{20},\\
\dot{A}_{20}&=-q\sin(\phi t_1)A_{10}-q\cos(\phi t_1)B_{10},\\
\dot{B}_{20}&=q\cos(\phi t_1)A_{10}-q\sin(\phi t_1)B_{10},
\end{aligned}\tag{B.1}$$

where p and q are given by Eq. (23).

By differentiating the first and the second equation in Eq. (B.1) it follows that

$$\begin{aligned}
\ddot{A}_{10}=&-p\phi\cos(\phi t_1)A_{20}-p\sin(\phi t_1)\dot{A}_{20}-p\phi\sin(\phi t_1)B_{20}+p\cos(\phi t_1)\dot{B}_{20}\\
=&\,\phi[-p\cos(\phi t_1)A_{20}-p\sin(\phi t_1)B_{20}]-p\sin(\phi t_1)[-q\sin(\phi t_1)A_{10}\\
&-q\cos(\phi t_1)B_{10}]+p\cos(\phi t_1)[q\cos(\phi t_1)A_{10}-q\sin(\phi t-1)B_{10}]\\
=&\,\phi\dot{B}_{10}+pqA_{10},
\end{aligned}\tag{B.2}$$

$$\ddot{B}_{10}=-\phi\dot{A}_{10}+pqB_{10}.\tag{B.3}$$

Differentiating Eq. (B.2) and using Eq. (B.3), results in

$$A_{10}^{(3)}-pq\dot{A}_{10}=\phi\ddot{B}_{10}=-\phi^2\dot{A}_{10}+pq\phi B_{10},\tag{B.4}$$

and finally by differentiating (B.4) and using (B.2) it follows that

$$A_{10}^{(4)} + (\phi^2 - 2pq)\ddot{A}_{10} + (pq)^2 A_{10} = 0. \tag{B.5}$$

The characteristic equation corresponding to Eq. (B.5) is $r^4 + (\phi^2 - 2pq)r^2 + (pq)^2 = 0$, with solutions $r_1 = \sqrt{\frac{1}{2}[2pq - \phi^2 + \sqrt{D}]}$, $r_2 = \sqrt{\frac{1}{2}[2pq - \phi^2 - \sqrt{D}]}$, $r_3 = -r_1$, and $r_4 = -r_2$ and where $D = \phi^4 - 4pq\phi^2$. Since p and q are of opposite sign it follows that $\phi^4 - 4pq\phi^2 > 0$ and $2pq - \phi^2 < 0$. Therefore, r_2 and r_4 are purely imaginary. And, since $\phi^2 - 2pq = \sqrt{(\phi^2 - 2pq)^2} = \sqrt{\phi^4 - 4pq\phi^2 + 4p^2q^2} > \sqrt{\phi^4 - 4pq\phi^2}$ it follows that $|2pq - \phi^2| > \sqrt{\phi^4 - 4pq\phi^2}$. Accordingly r_1 and r_3 are also purely imaginary. So, all the solutions of the characteristic equation can be written in the form

$$r_1 = \beta_1 \mathrm{i}, \quad r_2 = \beta_2 \mathrm{i}, \quad r_3 = -r_1, \quad \text{and} \quad r_4 = -r_2,$$

where

$$\beta_1 = \sqrt{\tfrac{1}{2}[\phi^2 - 2pq - \sqrt{\phi^4 - 4pq\phi^2}]} \quad \text{and} \quad \beta_2 = \sqrt{\tfrac{1}{2}[\phi^2 - 2pq + \sqrt{\phi^4 - 4pq\phi^2}]}.$$

The solution of Eq. (B.5) now becomes

$$A_{10}(t_1) = K_1 \sin(\beta_1 t_1) + K_2 \cos(\beta_1 t_1) + K_3 \sin(\beta_2 t_1) + K_4 \cos(\beta_2 t_1),$$

where K_1, K_2, K_3, and K_4 are constants of integration.

From Eq. (B.4) $B_{10}(t_1)$ can be derived, yielding

$$B_{10}(t_1) = \frac{1}{pq\phi}[A_{10}^{(3)} + (\phi^2 - pq)\dot{A}_{10}], \quad \phi \neq 0.$$

From the first two equations in Eq. (B.1), A_{20} and B_{20} can now readily be determined, yielding

$$A_{20}(t_1) = \frac{-1}{p}[\dot{A}_{10} \sin(\phi t_1) + \dot{B}_{10} \cos(\phi t_1)],$$

$$B_{20}(t_1) = \frac{1}{p}[\dot{A}_{10} \cos(\phi t_1) - \dot{B}_{10} \sin(\phi t_1)].$$

So, the solutions of Eq. (21) have been derived.

References

[1] A.G. Ulsoy, C.D. Mote Jr., R. Szymani, Principal development in band saw vibration and stability research, Holz als Roh und Werkstoff 36 (1978) 273–280.

[2] J.A. Wickert, C.D. Mote Jr., Linear transverse vibration of an axially moving string-particle system, Journal of the Acoustic Society of America 84 (1988) 963–969.

[3] G. Lodewijks, Dynamics of Belt Systems, Ph.D. Thesis, Delft Universiteitsdrukkerij TU Delft, Delft, 1996.

[4] F. Pellicano, F. Vestroni, Nonlinear dynamics and bifurcations of an axially moving beam, Journal of Vibration and Acoustics 122 (2000) 21–30.

[5] G. Suweken, W.T. van Horssen, On the transversal vibrations of a conveyor belt with low and time-varying velocity, Part I: the string-like case, Journal of Sound and Vibration 264 (2003) 117–133.

[6] H.R. Öz, M. Pakdemirli, Vibrations of an axially moving beam with time-dependent velocity, Journal of Sound and Vibration 227 (2) (1999) 239–257.
[7] H.R. Öz, H. Boyaci, Transverse vibrations of tensioned pipes conveying fluid with time-dependent velocity, Journal of Sound and Vibration 236 (2) (2000) 259–276.
[8] M. Pakdemirli, A.G. Ulsoy, A. Ceranoglu, Transverse vibration of an axially accelerating string, Journal of Sound and Vibration 169 (2) (1994) 179–196.
[9] R.A. Sack, Transverse oscillation in traveling string, British Applied Physics 5 (1954) 224–226.
[10] F.R. Archibald, A.G. Emslie, The vibrations of a string having a uniform motion along its length, Applied Mechanics 25 (1958) 347–348.
[11] S. Mahalingam, Transverse vibration of power transmission chains, British Journal of Applied Physics 8 (1957) 145–148.
[12] G.J. Boertjens, W.T. van Horssen, An asymptotic theory for a beam equation with a quadratic perturbation, SIAM Journal on Applied Mathematics 60 (2000) 602–632.
[13] W.T. van Horssen, Asymptotics for a class of semi-linear hyperbolic equations with an application to a problem with a quadratic nonlinearity, Nonlinear Analysis 19 (1992) 501–530.
[14] M. Pakdemirli, A.G. Ulsoy, Stability analysis of an axially accelerating string, Journal of Sound and Vibration 203 (5) (1997) 815–832.