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Letter to the Editor

Asymptotical behaviour of a system with damping and high power-form non-linearity

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1. Introduction

Asymptotical approaches devoted to the analysis of strongly non-linear dynamical systems still need to be developed. A special attention of this research is focused on an analysis of low dimensional systems, which is motivated by the following observations:

1. The fundamental behaviour of high order dimensional systems can be adequately modelled by systems of low dimension [1–3].
2. A concept of non-linear normal modes very often allows one to reduce a high dimensional system to one with a few degree of freedom [2–6].
3. Recent results show that approximate analytical solutions of strongly non-linear systems can be obtained with an emphasis paid to strong non-linearity [2,5,7–9]. As a zero order approximation a so-called vibro-impact system is used, and thereafter either a method of iteration [2,7] or asymptotic techniques [5,8,9] are applied. However, during this procedure the following drawback occurs: non-smooth solutions with a constant period of oscillations appear.

Contrary to that approach, in this paper, another construction of smooth solutions is proposed using an asymptotical approach. Proposed asymptotics use a concept of generalized functions [10–12].

Consider the following Cauchy problem:

$$\ddot{x} + 2k\dot{x} + x + cx^n = 0, \quad n = 2p + 1, \quad p = 0, 1, 2, \ldots, \tag{1}$$

$$x(0) = 1, \quad \dot{x}(0) = 0. \tag{2}$$

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The asymptotical problem of Eqs. (1) and (2) has been considered in Ref. [7]. A saw-tooth time transformation [8, 9] has been applied, as well as the asymptotics generated by an exact solution for a certain parameter set of the Eq. (1) has been proposed [5].

2. Introduction of a small parameter

In the first step, the damping term is excluded from Eq. (1) using the following standard change of variables [6]:

$$x = y\exp(-kt).$$

The result gives the following Cauchy problem:

$$\ddot{y} + (1 - k^2 + k)y + \exp(-k(n-1)t)y^n = 0, \tag{3}$$

$$y(0) = 1, \quad \dot{y}(0) = k. \tag{4}$$

As it has been shown in many references [2,5,8–12], a construction of asymptotics to the systems with high power form non-linearity x^n for $n \to \infty$ is very important, and can be effectively applied to the analysis of non-linear dynamical systems. In Eq. (3) two parameters occur, k and n. The traditional asymptotical methods are oriented rather to the use of the parameter k, and then both cases, i.e., small values ($k \to 0$) [1] as well as large values ($k \to \infty$) [13] of the k parameter are considered.

In this paper another approach is proposed: for $k \equiv \text{const}$ the case $n \to \infty$ is analyzed.

In Eq. (3) there is the coefficient $\exp(-k(n-1)t)$, and the problem stated reduces to that of an explicit isolation of the large parameter n. In order to realize this approach one can use the formula presented in Ref. [14, p. 410]:

$$H(t)\exp(-\lambda t) \sim \sum_{K=0}^{\infty} \frac{(-1)^k \delta^{(k)}(t)}{\lambda^{k+1}} \tag{5}$$

for $\lambda \to \infty$ valid in the space $\mathscr{P}'(\mathbb{R})$, where δ is the Dirac delta function. This means that if $\phi \in \mathscr{P}$, the following expansions

$$\phi(\lambda) = \int_0^{\infty} \exp(-\lambda t)\phi(t)\,\mathrm{d}t \sim \frac{\phi(0)}{\lambda} + \frac{\phi'(0)}{\lambda^2} + \frac{\phi''(0)}{\lambda^3} + \cdots$$

of the Laplace transform $\phi(\lambda)$ holds as $\lambda \to \infty$.

The space $\mathscr{P}'(\mathbb{R}^n)$ consists of those smooth functions $\phi(x)$ that satisfy $\lim_{t\to\infty} \mathrm{e}^{-\gamma|t|} D^{\beta}\phi(t) = 0$ for each $\gamma > 0$ and each $\beta \in N^n$ with seminorms

$$|\phi|_{\gamma,\beta} = \sup_{R^n} \left|\exp(-\gamma|t| D^{\beta}\phi(t)\right|.$$

The dual space $\mathscr{P}'(\mathbb{R}^n)$ consists of distributions of exponential decay at infinity. So, formula (5) is rigorously mathematically valid in the framework of the asymptotic expansions of generalized functions of a rapid decay [14,15].

A construction of the formula (5) can be described in the following way. First, the Laplace transformation [16] is applied to the function $\exp(-\lambda t)$. Secondly, the obtained transformation is developed into an analytical series with respect to $1/n$. Third, after a successive return to the original one gets the formula (5).

For the case considered one gets

$$\exp(-k(n-1)t) = \frac{\delta(t)}{k(n-1)} - \frac{\delta'(t)}{k^2(n-1)^2} + \frac{\delta''(t)}{k^3(n-1)^3} + \cdots, \tag{6}$$

where $\delta(t), \delta'(t), \delta''(t), \ldots$ are the Dirac's delta functions and their derivatives.

3. Asymptotics construction

At the first stage, substituting the series (6) into both Eq. (3) and the initial conditions (4), one gets to a first order approximation (for $n \to \infty$):

$$\ddot{y}_0 + a^2 y_0 = 0, \tag{7}$$

$$y_0(0) = 1; \quad \dot{y}_0(0) = k, \tag{8}$$

where $a^2 = 1 - k + k^2$.

As a result, a solution to the Cauchy problem Eqs. (7) and (8) has the form

$$y_0 = \cos(at) + \frac{k}{a}\sin(at). \tag{9}$$

However, a construction of higher approximations does not belong to a trivial process. Some of the necessary descriptions needed for further considerations are introduced using the fundamentals given in Ref. [16].

Consider the following non-linear equation

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + \varepsilon x^n = 0, \quad n = 3, 5, 7, \ldots . \tag{10}$$

For $|\varepsilon| \ll 1$ and relatively small values of n ($n = 3$, 5) a solution to Eq. (10) can be obtained using either the standard Lindstedt–Poincaré or averaging methods [1]. For the large values of n the standard approach seems not to lead to correct results. Suppose that one is going to find a solution to Eq. (10) in the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{11}$$

then the non-linear term is approximated by the formula

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^n = x_0^n + \varepsilon n x_0^{n-1} x_1 + \cdots .$$

Thus, a role of the real "small" parameter plays εn instead of ε. It seems that this change is not so important for $n = 3$ and $n = 5$, however it is expected to play a crucial role for large values of n (especially for $n \to \infty$).

In this contribution the following transformation is introduced:

$$x = x_0 \sqrt[n]{1 + \varepsilon n \frac{x_1}{x_0} + \cdots} . \tag{12}$$

For small n values the expressions (11) and (12) are equivalent. For $n \to \infty$, a high order of singularity caused by the nth root seems to play an important role.

The term x_1 can be obtained from the equation in a typical quasi-linear approach

$$\ddot{x}_1 + \gamma \dot{x}_1 + \omega^2 x_1 = -x_0^n.$$

In other words, known quasi-linear solutions [1] can be transformed to formula (12).

In the present case, a solution to the Cauchy problem Eqs. (3) and (4) has the form

$$y = y_0 \sqrt[n]{1 + \frac{y_1}{y_0} + \frac{y_2}{y_0} + \cdots}\,. \tag{13}$$

Therefore, one obtains

$$y = y_0 + \frac{y_1}{n} + \cdots,, \tag{14}$$

$$y^n = y_0^n + \frac{y_0^{n-1} y_1}{n} + \cdots\,. \tag{15}$$

Observe that in formulas (14), (15) '$n-1$' is approximated by 'n' for large n.

In the first order approximation, the following Cauchy problem is obtained

$$\ddot{y}_1 + a^2 y_1 = -\frac{1}{k}\delta(t) y_0^n(0), \tag{16}$$

$$y_1(0) = \dot{y}_1(0) = 0. \tag{17}$$

In addition, the problem of satisfying the initial conditions does not belong to the trivial problems. As it has been pointed out in Ref. [17] the following approach can be used. In zero order approximations some of the parameters are left undefined, and they are estimated in the next successive approximations (a request to satisfy the initial conditions). In this case one takes

$$y_0 = C_1 \cos(at) + C_2 \sin(at), \tag{18}$$

where C_1, C_2 are the constants going to be estimated further. Eq. (16) is transformed to the form

$$\ddot{y}_1 + a^2 y_1 = -\frac{C_1^n}{k}\delta(t). \tag{19}$$

A solution to the Cauchy problem Eqs. (18) and (19) can be found using the Laplace transformation [18]

$$y_1 = \frac{kC_1^n}{a}\sin(at). \tag{20}$$

Substituting Eqs. (18) and (20) into Eq. (15) and satisfying the conditions (4) gives

$$C_1 = 1; \quad C_2 = \frac{kn}{a(n+1)}$$

and finally

$$x = \exp(-kt)\left\{\left[\cos(at) + \frac{kn}{a(n+1)}\sin(at)\right]^n + \frac{k}{a}\left[\cos(at) + \frac{kn}{a(n+1)}\sin(at)\right]^n\right\}^{1/n}.. \tag{21}$$

4. Results and discussion

In order to estimate the approximation given by formula (21), the results have been compared with the numerical integration of the Eq. (1) using the 4th order Runge–Kutta method for some values of k and n.

For a given damping coefficient value k and for fixed $c = 1$ a comparison with formula (21) has been carried out for different n values. The results are shown in Fig. 1, where the dashed curves correspond to the numerical solutions.

In all cases considered, i.e., for $k = 0.01$; 0.1; 0.5 the analytical approximations increase with an increase of n. In addition, the proposed analytical approximations work better for smaller values of the damping coefficients k.

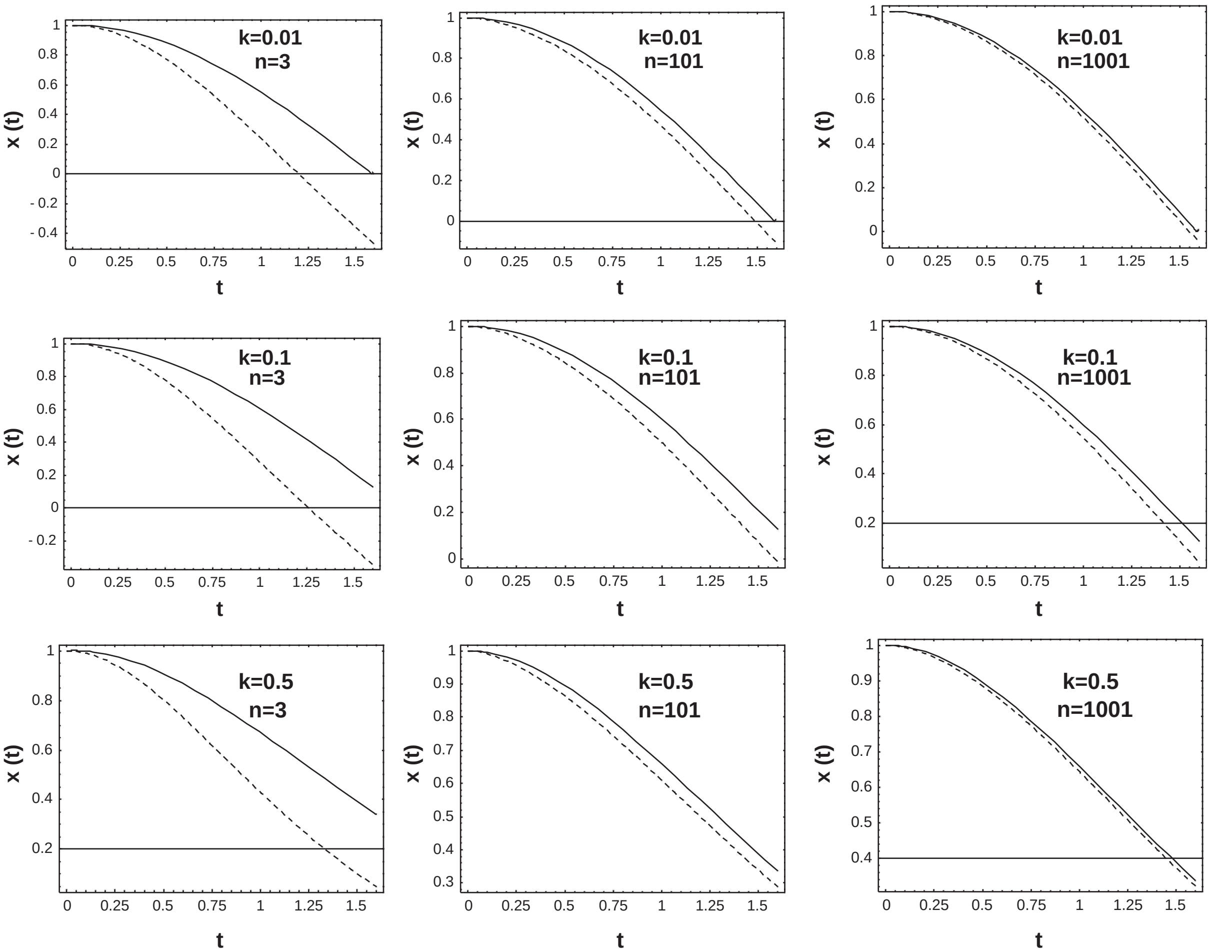

Fig. 1. A comparison between analytical (——) and numerical (- - - - -) solutions for different values of k and n coefficients.

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