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Active sound scatterers based on the JMC method

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Abstract

The principle of formulating the JMC method to produce secondary sources that function as active scatterers on a hypothetical scattering surface is established, to be applied, e.g., in concert halls. The examination is based on the modified JMC method, to ensure that the logic does not lead to the need of changing the primary sources. The actively reflecting plane serves as an example of the JMC formulation for the active scatterer. The solution is extended to a general planar JMC element with well-defined reflection and transmission properties. The solution works on the local control principle: each reflecting subarea needs information of the primary field only at that subarea. The solution can also apply approximately to piecemeal planar surfaces and to smooth convex surfaces. Further, general active boundary condition elements are defined. Based on the element definitions, simple reflecting source, the pressure- and velocity-reflecting boundaries, and the impedance boundary are introduced. Boundary condition elements do not work on the local control principle: in a general case secondary sources on each subarea need information of the primary field at each subarea. True boundary condition elements are also defined based on their net sound power radiation.

1. Introduction

The JMC method is suitable for formulating the problem of active noise control with the general system theory. More generally, the method applies to the reshaping of acoustic or any other fields [1–7], to wave reconstruction (holochory, holophony) [1,7–10], and to wave propagation problems [11–13]. Its name originates from the first three pioneers of the method: **J**essel, **M**angiante and **C**anévet [4] (the JMC group). In principle, whatever the primary sound field is, it can be changed (reshaped) into any other field by using the JMC method. Thus, it forms a general theoretical approach for the active control of sound.

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Jessel presented the idea of active absorption for acoustic fields based on the complementary configuration of his formulation for the Huygens' principle in 1968 [14]. In 1972, Jessel and Mangiante formulated the operator presentation of the principle of the active absorption by Huygens' sources with the help of Jessel's general perturbation lemma [15]. The aim was also the capability of handling more complex field-reshaping problems. Jessel gave the operator presentation in more detail in 1979 [16], with a general theorem for dividing a field into a couple of complementary fields, by first dividing the space into two complementary fuzzy parts. One special case of that theorem was the complementary couple of the Huygens' principle and the principle of active absorption.

In 1973, Jessel presented the generalized Huygens' principle in which only some of the primary sources were replaced with the secondary ones [1]. Furthermore, Jessel presented the very generalized Huygens' theorem in 1991 [17]. By using the very generalized Huygens' theorem for active absorption, it is possible to introduce separate secondary source zones for separate primary source regions or point sources, or for separate zones to be silenced.

Jessel extended the wave decomposing method by applying it to holophony (reproduction or reconstruction of an acoustic field, acoustical counterpart of holography) in 1973 [1]. Illényi and Jessel discussed its generalization holochory, which can be applied to any fields, in 1983 [8], and in more detail in 1988–1989 [7,9]. In 1991, Mangiante stated that the JMC method offers a definite approach to holochory [10]: with it an arbitrary physical field may be reconstructed exactly (the Huygens' principle, holophony) and a given field may be remodelled arbitrarily (active absorption).

Canévet proposed using Jessel's decomposing method to solve acoustic propagation problems in inhomogeneous transition layers and in waveguides with a changing cross-section in 1980 [11]. The space is divided into Urysohnian subspaces so that the propagation problem can be solved more easily and separately in the subspaces.

In 1983, Jessel attached the JMC method to the general system theory [4]. Resconi and Jessel introduced a general system logical theory in 1986 [5]. It was a combination of Resconi's logical theory of systems and Jessel's theory of secondary sources. By the help of the theory, the JMC method was assigned to a more general framework. In that framework many field-theoretical problems, besides the Huygens' principle and the principle of field-reshaping, can be approached. The theory can also apply to other than field-theoretical problems, geometries and chemical controls given by Resconi and Jessel as two application areas. The general system logical theory can apply to complex problems, due to its ability to deal with the networks of elementary logical systems.

Mangiante introduced the generalized JMC method in 1989–1990 [18,19]. In the generalized JMC method, it is possible to define various boundary conditions at the boundaries of the secondary source zone by using different kinds of source types in the secondary source zone. This enables defining many existing source configurations used in active control of sound, so they may fall into the category of special cases of the generalized JMC method. Uosukainen presented the modified JMC method in 1989–1990 [20–22]. The modified JMC method differs from the original one such that in the former the primary sources are not changed in any case.

The JMC method has not been applied to active scatterers previously. The purpose of this paper is to establish the principle of formulating the JMC method to produce secondary sources that function as active scatterers on a hypothetical scattering surface. As examples, an actively reflecting plane and active boundary condition elements are introduced. The examination is based on the modified JMC method, to ensure that the logic does not lead to the need of changing the primary sources. Because the main idea of this paper has not been published previously in literature, this paper concentrates on the theoretical basis, and the testing and measuring of real applications are left for future research.

Firstly, the modified JMC method is introduced. Secondly, the JMC formulation for the active scatterer is presented in a general operator formulation and applied to acoustic fields especially. The actively reflecting plane is given as an example by the help of the reflection transformation, and the solution is extended to a general planar JMC element with well-defined reflection and transmission properties. Thirdly, general active boundary condition elements are defined in a general operator formulation and applied to acoustic fields especially. The Huygens' principle is utilized to connect the field variables at the hypothetical surface of the scatterer so that the scattered field obeys the field equation automatically. With the acoustic fields, the Huygens' principle is presented by the help of the scalar Green's function for scalar fields and the dyadic Green's function for irrotational vector fields. Based on the element definitions, simple reflecting source, pressure- and velocity-reflecting boundaries, and impedance boundary are introduced. Except for the simple reflecting source, the derivations of the formulae of the secondary source strengths utilize the matrix formulation of the Huygens' principle. The derivation of the formulae of the secondary source strengths of the impedance boundary utililizes a duality transformation to change the boundary condition to a form that can be handled similarly as with an ideal reflector. True boundary condition elements are also defined based on their net sound power radiation.

2. Modified JMC method [20–22]

In the original situation there is a deterministic field (of any type) in which linear operator $\mathbf{L}$ (typically a differential operator) connects sources $\boldsymbol{S}$ and field $\boldsymbol{F}$ via

$$\mathbf{L}\boldsymbol{F} = \boldsymbol{S}. \tag{1}$$

Instead of field $\boldsymbol{F}$, field $\boldsymbol{F}'$ is desired, which can be obtained from the original field using operator $\mathbf{M}$ as

$$\mathbf{M}\boldsymbol{F} = \boldsymbol{F}'. \tag{2}$$

In the original JMC method, operator $\mathbf{M}$ also weights the original sources to sources $\boldsymbol{S}'$. In the modified JMC method, the original sources always remain unchanged; that is, they are not weighted in any case, i.e.,

$$\boldsymbol{S}' = \boldsymbol{S}. \tag{3}$$

Both in the original and modified JMC method, there is a need for additional sources $\boldsymbol{S}''$ such that field equation (1) for the modified field is valid. The field equation of desired field $\boldsymbol{F}'$ with the original sources unchanged is

$$\mathbf{L}\boldsymbol{F}' = \boldsymbol{S} + \boldsymbol{S}''. \tag{4}$$

The expression above, together with Eqs. (1) and (2), yield for the secondary sources in the modified JMC method

$$\boldsymbol{S}'' = \mathbf{L}\boldsymbol{F}' - \boldsymbol{S} = \mathbf{L}\mathbf{M}\boldsymbol{F} - \mathbf{L}\boldsymbol{F} = \mathbf{M}'\boldsymbol{F}, \tag{5}$$

where

$$\mathbf{M}' = \mathbf{L}(\mathbf{M} - \mathbf{I}), \tag{6}$$

where $\mathbf{I}$ is the identity operator.

3. JMC formulation of the active scatterer

In this section the JMC formulation for the active scatterer is presented in a general operator formulation and applied to acoustic fields especially, and the actively reflecting plane serves as an example. The JMC formulation has not been applied to active scatterers previously. The examination is based on the modified JMC method, to ensure that the logic does not lead to the need of changing the primary sources.

3.1. General formulation

A hypothetical scattering obstacle with its boundary surface A is defined according to Fig. 1.

The modified field is assumed to be the sum of an original field $\boldsymbol{F}$ and some extra field $\boldsymbol{F}_s$ (scattered field)

$$\boldsymbol{F}' = \boldsymbol{F} + \boldsymbol{F}_s, \quad \boldsymbol{F}_s(\mathbf{r}) = \mathbf{M}_s\boldsymbol{F}(\mathbf{M}_r\mathbf{r}), \tag{7}$$

where $\mathbf{r}$ is a spatial co-ordinate vector, and $\mathbf{M}_s$ and $\mathbf{M}_r$ are operators. It is supposed that operator $\mathbf{M}_r$ maps vector $\mathbf{r}$ on the other side of surface A, i.e.,

$$\left\{\begin{array}{cc} \mathbf{M}_r\mathbf{r} & \text{inside } A \text{ if } \mathbf{r} \text{ is outside } A \\ \mathbf{M}_r\mathbf{r} & \text{outside } A \text{ if } \mathbf{r} \text{ is inside } A \\ \mathbf{M}_r\mathbf{r} = \mathbf{r} & \text{if } \mathbf{r} \text{ is at } A \end{array}\right\}, \tag{8}$$

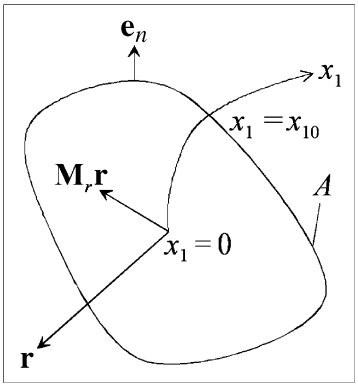

Fig. 1. A hypothetical scattering obstacle.

see Fig. 1. It is further supposed that extra field $\boldsymbol{F}_s$ vanishes inside A and obeys the homogeneous field equation outside A,

$$\boldsymbol{F}_s = 0, \text{inside } A; \quad \mathbf{L}\boldsymbol{F}_s = 0, \text{outside } A. \tag{9}$$

The latter formula, together with Eqs. (1) and (4), implicates that the only possible place for the secondary sources are on surface A.

The fact that the modified field generally obeys Eq. (2) yields

$$\mathbf{M} = \mathbf{I} + \mathbf{M}_s', \tag{10}$$

where operator $\mathbf{M}_s'$ operates so that

$$\mathbf{M}_s'\boldsymbol{F}(\mathbf{r}) = \mathbf{M}_s\boldsymbol{F}(\mathbf{M}_r\mathbf{r}). \tag{11}$$

The secondary sources, according to the modified JMC method, are now as stated in Eq. (5), where, according to Eqs. (6) and (10),

$$\mathbf{M}' = \mathbf{L}(\mathbf{M} - \mathbf{I}) = \mathbf{L}\mathbf{M}_s', \tag{12}$$

so

$$\boldsymbol{S}''(\mathbf{r}) = \mathbf{L}\mathbf{M}_s'\boldsymbol{F}(\mathbf{r}) = \mathbf{L}\mathbf{M}_s\boldsymbol{F}(\mathbf{M}_r\mathbf{r}). \tag{13}$$

According to Eqs. (7) and (9), operator $\mathbf{M}_s$ has to be of the form

$$\mathbf{M}_s = \mathbf{M}_{s0}\varepsilon(x_1 - x_{10}), \tag{14}$$

where $\mathbf{M}_{s0}$ is a continuous function of spatial co-ordinates, $\varepsilon(x_1 - x_{10})$ is a step function, and where it is supposed that boundary A is formed of a constant x_1 surface $x_1 = x_{10}$, see Fig. 1. The secondary sources on A are due to the discontinuity of $\mathbf{M}_s$ at $x_1 = x_{10}$. Eq. (9) can be written outside A, utilizing Eqs. (7) and (9), as

$$\mathbf{L}\boldsymbol{F}_s(\mathbf{r}) = \mathbf{L}\mathbf{M}_s\boldsymbol{F}(\mathbf{M}_r\mathbf{r}) = \mathbf{L}\mathbf{M}_{s0}\boldsymbol{F}(\mathbf{M}_r\mathbf{r}) = 0 \text{ outside } A. \tag{15}$$

Due to the continuity of $\mathbf{M}_{s0}$, this must hold also at A. Because $\boldsymbol{F}_s$ vanishes inside A, according to Eq. (9), the equation above is valid everywhere, i.e.,

$$\mathbf{L}\mathbf{M}_{s0}\boldsymbol{F}(\mathbf{M}_r\mathbf{r}) = 0. \tag{16}$$

Now the secondary sources are, according to Eqs. (13), (14) and (16),

$$\begin{aligned}\boldsymbol{S}''(\mathbf{r}) &= \mathbf{L}(\mathbf{M}_{s0}\varepsilon(x_1 - x_{10})\boldsymbol{F}(\mathbf{M}_r\mathbf{r})) \\ &= \mathbf{L}(\mathbf{M}_{s0}\boldsymbol{F}(\mathbf{M}_r\mathbf{r}))\varepsilon(x_1 - x_{10}) + \mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{M}_{s0}\boldsymbol{F}(\mathbf{M}_r\mathbf{r}) \\ &= \mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{M}_{s0}\boldsymbol{F}(\mathbf{r}) \text{ at } A.\end{aligned} \tag{17}$$

The final general solution above depends on the original field at A, operator $\mathbf{M}_{s0}$, and the field operator operating on the step function at A.

3.2. Application to acoustic fields

In acoustic fields the field and the sources can be represented by vectors $\mathbf{F}$ and $\mathbf{S}$ correspondingly, and in flowless and homogenous ideal fluids the operator connecting them can be presented with matrix operator $\mathbf{L}$ as

$$\mathbf{L} = \begin{bmatrix} Q_0 \dfrac{\partial}{\partial t} & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & \rho_0 \dfrac{\partial}{\partial t} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} p \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} q \\ \mathbf{f} \end{bmatrix}, \tag{18}$$

where t is time, Q_0 and ρ_0 are the compressibility and the density of the unperturbed fluid, p and $\mathbf{u}$ are the sound pressure and the particle velocity of the acoustic field, and q and $\mathbf{f}$ are the monopole and dipole distributions per unit volume.

Operator $\mathbf{L}$ operating on the step function yields now

$$\mathbf{L}(\varepsilon(x_1 - x_{10})) = \begin{bmatrix} 0 & \boldsymbol{\nabla}\varepsilon(x_1 - x_{10})\cdot \\ \boldsymbol{\nabla}\varepsilon(x_1 - x_{10}) & 0 \end{bmatrix} = \delta(x_1 - x_{10}) \begin{bmatrix} 0 & \mathbf{e}_n\cdot \\ \mathbf{e}_n & 0 \end{bmatrix}, \tag{19}$$

where $\delta(x_1 - x_{10})$ is the Dirac delta function and $\mathbf{e}_n$ is a unit outward normal vector on surface A, see Fig. 1.

The secondary sources can now be presented with vector $\mathbf{S}''$, being according to Eqs. (17)–(19)

$$\begin{aligned} \mathbf{S}'' &= \begin{bmatrix} q'' \\ \mathbf{f}'' \end{bmatrix} = \delta(x_1 - x_{10}) \begin{bmatrix} 0 & \mathbf{e}_n\cdot \\ \mathbf{e}_n & 0 \end{bmatrix} \mathbf{M}_{s0} \begin{bmatrix} p \\ \mathbf{u} \end{bmatrix} = \delta(x_1 - x_{10}) \begin{bmatrix} \mathbf{M}_{su}\mathbf{u}\cdot\mathbf{e}_n \\ \mathbf{M}_{sp} p \mathbf{e}_n \end{bmatrix} \\ &= \delta(x_1 - x_{10}) \begin{bmatrix} \mathbf{M}_{su}\mathbf{u} \\ \mathbf{M}_{sp} p \mathbf{I} \end{bmatrix} \cdot \mathbf{e}_n, \end{aligned} \tag{20}$$

where operator $\mathbf{M}_{s0}$ has been divided into two operators, $\mathbf{M}_{sp}$ operating on the sound pressure, and $\mathbf{M}_{su}$ operating on the particle velocity

$$\mathbf{M}_{s0} \begin{bmatrix} p \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{sp} p \\ \mathbf{M}_{su}\mathbf{u} \end{bmatrix}, \tag{21}$$

and where $\mathbf{I}$ is an identic dyadic ($\mathbf{I}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{I} = \mathbf{a}$). Integrating expression (20) with respect to x_1 yields surface secondary source distribution vector $\mathbf{S}''_s$ on A as

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{su}\mathbf{u}\cdot\mathbf{e}_n \\ \mathbf{M}_{sp} p \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{su}\mathbf{u} \\ \mathbf{M}_{sp} p \mathbf{I} \end{bmatrix} \cdot \mathbf{e}_n \text{ at } A. \tag{22}$$

The solution above for the acoustic fields depends on the original sound pressure and the normal component of the original particle velocity at A, and operator $\mathbf{M}_{s0}$.

3.3. Reflecting plane

The actively reflecting plane is treated here based on the examination above. The reflection transformation is utilized. Furthermore, the solution is extended to a general planar JMC element with well-defined reflection and transmission properties.

3.3.1. Reflection dyadic

Dyadic K producing the reflection transformation of the original field with respect to the plane $x = 0$ can be presented as [23]

$$\mathsf{K} = \mathsf{I} - 2\mathbf{e}_x\mathbf{e}_x = -\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z, \tag{23}$$

where $\mathbf{e}_x$ is a unit vector in the x direction (normal to the reflecting plane), see Fig. 2. The dyadic of the reflection transformation inverts the normal component (with respect to the reflecting plane) of the vector as opposite without changing the other components in any way. The reflection transformation operates on both the actual field vectors and co-ordinate vector $\mathbf{r}$, see Fig. 2. The transformed field may be interpreted to be caused by a mirror image of the original source with respect to the surface. The strength of the mirror image and its distance from the reflecting surface are equal to those of the original source.

If the reflecting surface is not ideal, the amplitude of the reflected field is smaller than that of the original field on the reflecting surface. The reflection may also change the phase of the field. This can be taken into account with complex reflection coefficient R. The reflection coefficient must be properly chosen to ensure that the reflected field satisfies the homogeneous field equation in the half-space $x > 0$. One possibility is to use a reflection coefficient independent of the angle of incidence. In that case the reflection coefficient only weights the field of the image source(s) similarly in the half-space $x > 0$. With a properly chosen reflection coefficient, the reflected acoustic fields (subscript r) obey

$$p_r(\mathbf{r}) = Rp(\mathsf{K} \cdot \mathbf{r}), \; \mathbf{u}_r(\mathbf{r}) = R\mathsf{K} \cdot \mathbf{u}(\mathsf{K} \cdot \mathbf{r}). \tag{24}$$

According to the presentation of the spatial variable in the reflection transformation, the propagation direction with respect to the normal of the plate is changed into the opposite, remaining original in lateral directions. Multiplying the original amplitudes by R produces the amplitudes of the reflected sound pressure and the reflected lateral components of the particle velocity at the reflecting surface. Contradictorily, multiplying the original amplitude by $-R$

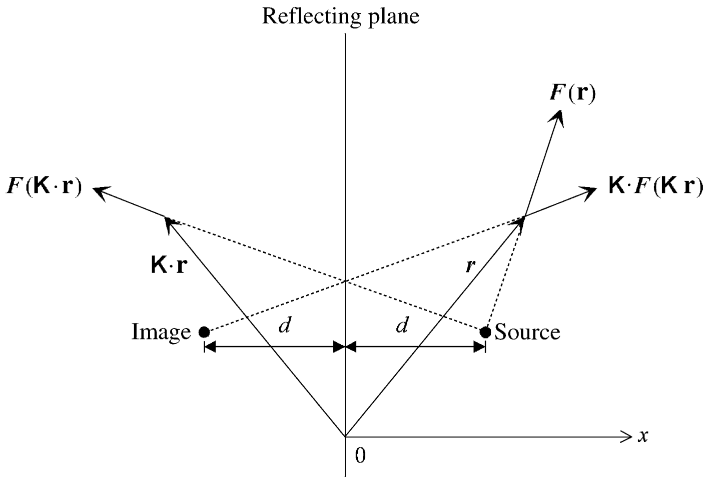

Fig. 2. The effect of the reflection transformation to field vector $\boldsymbol{F}$ and co-ordinate vector $\mathbf{r}$.

produces the amplitude of the reflected normal component of the particle velocity at the reflecting surface.

3.3.2. Secondary sources and sound field

In the next the reflected acoustic field is generated by the help of the modified JMC method in the half-space $x > 0$, see Fig. 2.

Operators $\mathbf{M}_s$ and $\mathbf{M}_r$, defined in Eqs. (7), (8), (14) and (21) are now

$$\mathbf{M}_s = \mathbf{M}_{s0}\varepsilon(x), \quad \mathbf{M}_{s0} = \begin{bmatrix} \mathbf{M}_{sp} \\ \mathbf{M}_{su} \end{bmatrix} = R\begin{bmatrix} 1 \\ \mathsf{K}\cdot \end{bmatrix}, \quad \mathbf{M}_r = \mathsf{K}\cdot. \tag{25}$$

The secondary source vector, according to Eqs. (22), (23) and (25), is now

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{su}\mathbf{u} \\ \mathbf{M}_{sp}p\mathsf{I} \end{bmatrix} \cdot \mathbf{e}_x = R\begin{bmatrix} \mathsf{K}\mathbf{u} \\ p\mathsf{I} \end{bmatrix} \cdot \mathbf{e}_x = R\begin{bmatrix} -\mathbf{u}\cdot\mathbf{e}_x \\ p\mathbf{e}_x \end{bmatrix} \text{ at } x = 0. \tag{26}$$

The solution above works on the local control principle: the secondary source strengths at any point on A depend on the original fields only at the same point.

The total field vector at $x = 0$ is, according to Eqs. (7), (25) and (23),

$$\begin{aligned} \mathbf{F}' &= \mathbf{F} + \mathbf{M}_s\mathbf{F} = \begin{bmatrix} (\mathbf{I} + \mathbf{M}_{sp})p \\ (\mathbf{I} + \mathbf{M}_{su})\mathbf{u} \end{bmatrix} = \begin{bmatrix} (1+R)p \\ (\mathsf{I} + R\mathsf{K})\cdot\mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} (1+R)p \\ (\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z + R(-\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z))\cdot\mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} (1+R)p \\ (1+R)\mathbf{u}\cdot(\mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z) + (1-R)\mathbf{u}\cdot\mathbf{e}_x\mathbf{e}_x \end{bmatrix} \text{ at } x = 0. \end{aligned} \tag{27}$$

In the case of an active rigid surface, $R = +1$, and Eq. (27) yields

$$\mathbf{F}' = \begin{bmatrix} (1+R)p \\ (\mathbf{I} + R\mathsf{K})\cdot\mathbf{u} \end{bmatrix} = 2\begin{bmatrix} p \\ \mathbf{u}\cdot(\mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z) \end{bmatrix} \text{ at } x = 0. \tag{28}$$

The sound pressure and the tangential components of the particle velocity double on the surface at $x = 0$, and the normal component of the particle velocity vanishes on the surface, as they should.

In the case of an active pressure-release surface, $R = -1$, in which case Eq. (27) yields

$$\mathbf{F}' = \begin{bmatrix} (1+R)p \\ (\mathbf{I} + R\mathsf{K})\mathbf{u}\cdot \end{bmatrix} = 2\begin{bmatrix} 0 \\ \mathbf{u}\cdot\mathbf{e}_x\mathbf{e}_x \end{bmatrix} \text{ at } x = 0. \tag{29}$$

In that case the sound pressure and the tangential components of the particle velocity vanish on the surface at $x = 0$, and the normal component of the particle velocity doubles on the surface, as they should.

3.3.3. General planar JMC element

Comparing the secondary surface source densities in Eq. (26) with those of the plane absorptive JMC elements (see [24], absorption in the region $x<0$)

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{e}_x \\ p\mathbf{e}_x \end{bmatrix} \text{ at } x = 0 \text{ (absorbing element, absorption in } x<0) \tag{30}$$

shows that they are of the same form as in Eq. (26), only the sign of the monopole distribution is the opposite and reflection coefficient R is multiplying the source strengths with the reflecting element. The reflecting action in $x>0$ and the absorbing action in $x<0$ can be realized simultaneously by superimposing the source strengths to yield the absorbing-reflecting element

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} (1-R)\mathbf{u}\cdot\mathbf{e}_x \\ (1+R)p\mathbf{e}_x \end{bmatrix} \text{ at } x = 0 \text{ (reflection in } x>0,\text{ absorption in } x<0). \tag{31}$$

Eq. (31) yields for the actively rigid ($R=+1$) absorbing–reflecting element

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} 0 \\ 2p\mathbf{e}_x \end{bmatrix} \text{ at } x = 0, \tag{32}$$

in which case there is no need for monopoles.

Similarly, Eq. (31) yields for the actively pressure-release ($R=-1$) absorbing–reflecting element

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} 2\mathbf{u}\cdot\mathbf{e}_x \\ 0 \end{bmatrix} \text{ at } x = 0, \tag{33}$$

in which case there is no need for dipoles.

The transmitting Huygens' source is similar to the absorbing source, only their signs are opposite. So, the transmitting source, radiating sound in the region $x<0$, has source strengths according to

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = -\begin{bmatrix} \mathbf{u}\cdot\mathbf{e}_x \\ p\mathbf{e}_x \end{bmatrix} \text{ at } x = 0 \text{ (transmitting element, transmission in } x<0). \tag{34}$$

Thus, the transmitting functioning with transmission coefficient T can also be superimposed on the element, having source strengths according to Eq. (31), yielding a general transmitting–reflecting JMC element with desired transmission and reflection properties

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} (1-R-T)\mathbf{u}\cdot\mathbf{e}_x \\ (1+R-T)p\mathbf{e}_x \end{bmatrix} \text{ at } x = 0 \text{ (reflection in } x>0,\text{ transmission in } x<0). \tag{35}$$

The solution for the general planar JMC element above works on the local control principle: the secondary source strengths at any point on A depend on the original fields at the same point only.

The planar secondary source expressions in the equation above approximately applies to piecemeal planar surfaces and even to smooth convex surfaces. In those cases the unit vector in x direction has to be replaced with a unit vector normal to the surface.

4. Boundary condition elements

In some cases it may be difficult to define the scattered field using operators $\mathbf{M}_s$ and $\mathbf{M}_r$ as in Eq. (7), and to be sure that the field satisfies the field equation as well. There is another way to approach the problem in that case. In this section general active boundary condition elements are defined in a general operator formulation and applied to acoustic fields especially. The Huygens' principle is utilized, in order to make the scattered field to obey the field equation at the hypothetical scattering surface automatically.

4.1. General definitions

Eq. (17) for the secondary sources can be written by the help of Eqs. (7), (8) and (14) so that scattered field $\boldsymbol{F}_s$ appears explicitly in the expression

$$\boldsymbol{S}'' = \mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{M}_{s0}\boldsymbol{F} = \mathbf{L}(\varepsilon(x_1 - x_{10}))\boldsymbol{F}_s \text{ at } A. \tag{36}$$

Because the total field $\boldsymbol{F}' = \boldsymbol{F} + \boldsymbol{F}_s$ obeys the field equation, as stated in Eq. (4), with the original and secondary sources included, and the original field obeys it with the original sources included as stated in Eq. (1), the scattered field alone satisfies the field equation as well, with only the secondary sources included,

$$\mathbf{L}\boldsymbol{F}_s = \boldsymbol{S}''. \tag{37}$$

So the scattered field can be obtained from

$$\boldsymbol{F}_s = \mathbf{L}^{-1}\boldsymbol{S}'' = \mathbf{L}^{-1}[\mathbf{L}(\varepsilon(x_1 - x_{10}))\boldsymbol{F}_s], \tag{38}$$

where $\mathbf{L}^{-1}$ is the inverse of operator $\mathbf{L}$. The expression above is in fact the Huygens' principle for the scattered field, i.e., the scattered field calculated by the help of the scattered field at surface A. It is well known that the Huygens' principle as presented above gives zero field inside A and, if the surface is smooth (assumed in the following), half of the field at surface A (see, e.g., [25], p. 182). So if it is applied at surface A itself, the right side of the expression has to be doubled to yield

$$\boldsymbol{F}_s = 2\mathbf{L}^{-1}\boldsymbol{S}'' = 2\mathbf{L}^{-1}[\mathbf{L}(\varepsilon(x_1 - x_{10}))\boldsymbol{F}_s] \text{ at } A. \tag{39}$$

The sum of scattered field $\boldsymbol{F}_s$ with original field $\boldsymbol{F}$ at surface A can now be defined to obey a homogeneous boundary condition attached to the hypothetical scatterer

$$\varphi(\boldsymbol{F} + \boldsymbol{F}_s) = 0 \text{ at } A, \tag{40}$$

where φ is a boundary condition operator. Sometimes it is convenient to express the boundary condition so that the original and scattered fields are separated

$$\varphi'\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{F}_s \end{pmatrix} = 0 \text{ at } A, \tag{41}$$

where φ' is the corresponding boundary condition operator for that case.

Scattered field $\boldsymbol{F}_s$ at A can thus be obtained from original field $\boldsymbol{F}$ using Eq. (39) and boundary condition φ, Eq. (40), or φ', Eq. (41). Once the scattered field at A has been calculated, the secondary sources on A can be obtained from expression (36), and after this the scattered field everywhere outside A can be computed from expression (38) if needed. Depending on the boundary condition, the system does not work on the local control principle, generally.

4.2. Application to acoustic fields

The examination above is here specified to acoustic fields. Based on that, simple reflecting source, and pressure- and velocity-reflecting boundaries are defined. In the Huygens' principle, the scalar Green's function is utilized with scalar fields, and the dyadic Green's function for irrotational fields is utilized with vector fields. With the pressure- and velocity-reflecting boundaries, the derivations of the formulae of the secondary source strengths utilize the matrix formulation of the Huygens' principle.

4.2.1. Basic equations

With acoustic fields the scattered fields can be presented with vector $\mathbf{F}_s$, as with the original ones in Eq. (18),

$$\mathbf{F}_s = \begin{bmatrix} p_s \\ \mathbf{u}_s \end{bmatrix}, \tag{42}$$

where p_s and $\mathbf{u}_s$ are the scattered sound pressure and particle velocity. Inserting this and expression (19) into equation (36) yields

$$\begin{aligned} \mathbf{S}'' = \begin{bmatrix} q'' \\ \mathbf{f}'' \end{bmatrix} &= \delta(x_1 - x_{10}) \begin{bmatrix} 0 & \mathbf{e}_n \cdot \\ \mathbf{e}_n & 0 \end{bmatrix} \begin{bmatrix} p_s \\ \mathbf{u}_s \end{bmatrix} \\ &= \delta(x_1 - x_{10}) \begin{bmatrix} \mathbf{u}_s \cdot \mathbf{e}_n \\ p_s \mathbf{e}_n \end{bmatrix} = \delta(x_1 - x_{10}) \begin{bmatrix} \mathbf{u}_s \\ p_s \mathbf{I} \end{bmatrix} \cdot \mathbf{e}_n. \end{aligned} \tag{43}$$

The corresponding surface source density vector on A is

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{u}_s \cdot \mathbf{e}_n \\ p_s \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_s \\ p_s \mathbf{I} \end{bmatrix} \cdot \mathbf{e}_n \text{ at } A. \tag{44}$$

In acoustic fields operator $\mathbf{L}^{-1}$ is a volume integration over source distributions; in time harmonic fields it is, according to Appendix A,

$$\mathbf{L}^{-1} = \int_V \mathrm{d}V(\mathbf{r}_0) \begin{bmatrix} \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \boldsymbol{\nabla}_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) \cdot \\ \boldsymbol{\nabla}_0 \cdot \mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega Q_0 \mathbf{G}_\mathrm{T}(\mathbf{r}|\mathbf{r}_0) \cdot \end{bmatrix}, \tag{45}$$

where the integration is performed with respect to co-ordinate $\mathbf{r}_0$, $\boldsymbol{\nabla}_0$ operates to co-ordinate $\mathbf{r}_0$, g is the scalar Green's function, $\mathbf{G}$ is the corresponding dyadic Green's function for irrotational fields, $\mathbf{G}_\mathrm{T}$ is its transpose, j is the imaginary unit, and ω is angular frequency ($=2\pi f$, f is frequency). So the scattered field vector $\mathbf{F}_s$ at A is, according to Eqs. (39), (43) and (45),

$$\begin{aligned}\mathbf{F}_s(\mathbf{r}) &= \begin{bmatrix} p_s(\mathbf{r}) \\ \mathbf{u}_s(\mathbf{r}) \end{bmatrix} \\ &= 2\int_V \mathrm{d}V(\mathbf{r}_0) \begin{bmatrix} \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \boldsymbol{\nabla}_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)\cdot \\ \boldsymbol{\nabla}_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega Q_0 \mathbf{G}_\mathrm{T}(\mathbf{r}|\mathbf{r}_0)\cdot \end{bmatrix} \delta(x_1 - x_{10}) \begin{bmatrix} \mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0} \\ p_s(\mathbf{r}_0)\mathbf{e}_{n0} \end{bmatrix} \\ &= 2\oint_A \mathrm{d}A(\mathbf{r}_0) \begin{bmatrix} \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \mathbf{e}_{n0}\cdot\boldsymbol{\nabla}_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) \\ \boldsymbol{\nabla}_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega Q_0 \mathbf{e}_{n0}\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) \end{bmatrix} \begin{bmatrix} \mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0} \\ p_s(\mathbf{r}_0) \end{bmatrix} \text{ at } A, \end{aligned} \tag{46}$$

where $\mathbf{e}_{n0}$ is the unit normal vector at the integration point on surface A. The scattered field vector where only the normal component of the particle velocity, instead of the total particle velocity, is present, is denoted by $\mathbf{F}_{sn}$. According to Eq. (46), this is at surface A (in the last vector in the expression, the order of the field quantities has been changed)

$$\begin{aligned}\mathbf{F}_{sn}(\mathbf{r}) &= \begin{bmatrix} p_s(\mathbf{r}) \\ \mathbf{u}_s(\mathbf{r})\cdot\mathbf{e}_n \end{bmatrix} \\ &= 2\oint_A \mathrm{d}A(\mathbf{r}_0) \begin{bmatrix} \mathbf{e}_{n0}\cdot\boldsymbol{\nabla}_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) \\ \mathrm{j}\omega Q_0 \mathbf{e}_{n0}\mathbf{e}_n{:}\mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathbf{e}_n\boldsymbol{\nabla}_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) \end{bmatrix} \begin{bmatrix} p_s(\mathbf{r}_0) \\ \mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0} \end{bmatrix} \text{ at } A, \end{aligned} \tag{47}$$

where $\mathbf{e}_n$ is the unit normal vector at the field point on surface A and the double dot product of two dyadics $\mathbf{ab}$ and $\mathbf{cd}$ is defined as $\mathbf{ab}:\mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})$. The necessary parts of Green's dyadic $\mathbf{G}$ are obtained from scalar Green's function g, according to Appendix A, as

$$\begin{aligned}\mathbf{e}_{n0}\mathbf{e}_n{:}\mathbf{G} &= \mathbf{e}_{n0}\cdot\mathbf{e}_n \mathrm{g} - \left(\frac{1}{k^2}\right)\mathbf{e}_{n0}\mathbf{e}_n{:}[\boldsymbol{\nabla}_0\times(\boldsymbol{\nabla}_0 \mathrm{g}\times\mathbf{I})] \\ \mathbf{e}_n\cdot\boldsymbol{\nabla}_0\mathbf{G} &= \mathbf{e}_n\cdot\boldsymbol{\nabla}_0 \mathrm{g}. \end{aligned} \tag{48}$$

In fact, the operator in Eq. (47) is such that if one field quantity at the surface is known, the other can be calculated from either of the two equations included in the expressions.

Similar expressions at surface A can be presented for the original field if the reference directions of the unit normal vector and the particle velocity are reversed. This leads to the expression of original field vector $\mathbf{F}_n$ (subscript n meaning: with only the normal component of the particle velocity present)

$$\begin{aligned}\mathbf{F}_n(\mathbf{r}) &= \begin{bmatrix} p(\mathbf{r}) \\ \mathbf{u}(\mathbf{r})\cdot\mathbf{e}_n \end{bmatrix} \\ &= -2\oint_A \mathrm{d}A(\mathbf{r}_0) \begin{bmatrix} \mathbf{e}_{n0}\cdot\boldsymbol{\nabla}_0 \mathrm{g}_i(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega\rho_0 \mathrm{g}_i(\mathbf{r}|\mathbf{r}_0) \\ \mathrm{j}\omega Q_0 \mathbf{e}_{n0}\mathbf{e}_n{:}\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0) & \mathbf{e}_n\cdot\boldsymbol{\nabla}_0\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0) \end{bmatrix} \begin{bmatrix} p(\mathbf{r}_0) \\ \mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0} \end{bmatrix} \text{ at } A. \end{aligned} \tag{49}$$

Generally, Green's functions g and g_i, and the corresponding Green's dyadics, in expressions (47) and (49), respectively are not the same unless the free space Green's function is selected, e.g., if the Green's function is selected so that it fulfills either the Dirichlet's or Neumann's boundary condition at A and A is finite, Green's function g in Eq. (47) represents a field in an infinite space (outside A), and in Eq. (49) Green's function g_i represents a field in a resonator (inside A). Only in the case of surface A being infinite, it divides the space in two semi-infinite volumes and there are possibilities to have similar Green's functions fulfilling the same boundary conditions at A in both subspaces.

Boundary conditions (40) and (41) are now formally

$$\varphi\left(\begin{bmatrix} p \\ \mathbf{u}\cdot\mathbf{e}_n \end{bmatrix} + \begin{bmatrix} p_s \\ \mathbf{u}_s\cdot\mathbf{e}_n \end{bmatrix}\right) = 0 \text{ at } A \Leftrightarrow \varphi\left(\begin{bmatrix} p+p_s \\ (\mathbf{u}+\mathbf{u}_s)\cdot\mathbf{e}_{\boldsymbol{n}} \end{bmatrix}\right) = 0 \text{ at } A$$

or

$$\varphi'\left(\begin{matrix} \begin{bmatrix} p \\ \mathbf{u}\cdot\mathbf{e}_{\boldsymbol{n}} \end{bmatrix} \\ \begin{bmatrix} p_s \\ \mathbf{u}_s\cdot\mathbf{e}_{\boldsymbol{n}} \end{bmatrix} \end{matrix}\right) = 0 \text{ at } A \Leftrightarrow \varphi'\left(\begin{bmatrix} p \\ \mathbf{u}\cdot\mathbf{e}_{\boldsymbol{n}} \\ p_s \\ \mathbf{u}_s\cdot\mathbf{e}_{\boldsymbol{n}} \end{bmatrix}\right) = 0 \text{ at } A. \tag{50}$$

Once the scattered field at A has been calculated from expressions (47) and (50), the secondary sources on A can be obtained from expression (44), and after that, if needed, the scattered field vector everywhere outside A can be computed from expression (38), which is now, according to equations (45) and (43),

$$\begin{aligned} \mathbf{F}_s(\mathbf{r}) = \begin{bmatrix} p_s(\mathbf{r}) \\ \mathbf{u}_s(\mathbf{r}) \end{bmatrix} &= \int_V \mathrm{d}V(\mathbf{r}_0) \begin{bmatrix} \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)\cdot \\ \nabla_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega Q_0 \mathbf{G}_\mathrm{T}(\mathbf{r}|\mathbf{r}_0)\cdot \end{bmatrix} \begin{bmatrix} q''(\mathbf{r}_0) \\ \mathbf{f}''(\mathbf{r}_0) \end{bmatrix} \\ &= \oint_A \mathrm{d}A(\mathbf{r}_0) \begin{bmatrix} \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)\cdot \\ \nabla_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega Q_0 \mathbf{G}_\mathrm{T}(\mathbf{r}|\mathbf{r}_0)\cdot \end{bmatrix} \begin{bmatrix} q_s''(\mathbf{r}_0) \\ \mathbf{f}''_s(\mathbf{r}_0) \end{bmatrix} \text{ outside } A. \end{aligned} \tag{51}$$

4.2.2. Simple reflecting source

A reflecting source is defined here to be such that either the sound pressure or the normal component of the particle velocity of the scattered field is directly proportional to the corresponding original field quantity at surface A. For the pressure-reflecting source boundary condition operator φ' in Eq. (50) (lower alternative) can be presented as a vector $\boldsymbol{\varphi}'$

$$\boldsymbol{\varphi}' = [-R \quad 0 \quad 1 \quad 0], \tag{52}$$

where R is the reflection coefficient. The boundary condition is now simply

$$p_s = Rp \text{ at } A. \tag{53}$$

For the velocity-reflecting source the vector $\boldsymbol{\varphi}'$ representing the boundary condition operator is

$$\boldsymbol{\varphi}' = [0 \quad R \quad 0 \quad 1], \tag{54}$$

and the boundary condition is thus

$$\mathbf{u}_s\cdot\mathbf{e}_n = -R\mathbf{u}\cdot\mathbf{e}_n \text{ at } A. \tag{55}$$

The minus sign has been selected above in order to have the expression for the sound pressure consistent with that of the pressure-reflecting source in the case of an infinite reflective plane, as is evident later.

In the following, simple reflecting sources are defined, "simple" meaning that the sources own some additional properties. Firstly, the reflection coefficient is supposed to be constant on surface A, so it can be extracted from the surface integration in Eq. (47). Secondly, it is supposed that the Green's functions fulfill the same boundary conditions on both sides of A. For a *simple pressure-reflecting source*, obeying Eq. (53), this can be defined most conveniently as

$$\mathrm{g}_i(\mathbf{r}|\mathbf{r}_0) = \mathbf{L}_g \mathrm{g}(\mathbf{r}|\mathbf{r}_0) \text{ or } \mathbf{L}_g \mathbf{G}_i(\mathbf{r}|\mathbf{r}_0) = \mathbf{G}(\mathbf{r}|\mathbf{r}_0), \tag{56}$$

where operator $\mathbf{L}_g$ is such that it can be extracted from the surface integration in expressions (47) and (49). In that case the scattered field is, according to Eq. (47),

$$p_s(\mathbf{r}) = 2\oint_A [\mathbf{e}_{n0}\cdot\nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0) + \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0) \text{ at } A$$

or

$$\mathbf{u}_s(\mathbf{r})\cdot\mathbf{e}_n = 2\mathbf{L}_g\oint_A [\mathrm{j}\omega Q_0 \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:}\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0) + \mathbf{e}_n\cdot\nabla_0\cdot\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0) \text{ at } A, \tag{57}$$

and the original field is correspondingly, according to Eq. (49),

$$p(\mathbf{r}) = -2\mathbf{L}_g\oint_A \left[\mathbf{e}_{n0}\cdot\nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0) + \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\right]\mathrm{d}A(\mathbf{r}_0) \text{ at } A$$

or

$$\mathbf{u}(\mathbf{r})\cdot\mathbf{e}_n = -2\oint_A [\mathrm{j}\omega Q_0 \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:}\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0) + \mathbf{e}_n\cdot\nabla_0\cdot\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0) \text{ at } A. \tag{58}$$

Now the Green's functions and the Green's dyadics in the integrations in Eqs. (57) and (58) are the same, and they can be selected so that either of the following boundary conditions is fulfilled

$$\mathbf{e}_{n0}\cdot\nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) = 0 \text{ at } A \text{ or } \mathbf{e}_n\cdot\nabla_0\cdot\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0) = 0 \text{ at } A. \tag{59}$$

In those cases the following alternate expressions for the scattered field, according to Eq. (57), are obtained correspondingly

$$p_s(\mathbf{r}) = 2\mathrm{j}\omega\rho_0\oint_A \mathrm{g}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\mathrm{d}A(\mathbf{r}_0) \text{ at } A$$

or

$$\mathbf{u}_s(\mathbf{r})\cdot\mathbf{e}_n = 2\mathrm{j}\omega Q_0\mathbf{L}_g\oint_A \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:}\mathbf{G}_i(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0)\mathrm{d}A(\mathbf{r}_0) \text{ at } A. \tag{60}$$

The corresponding equations for the original fields are in that case, according to Eq. (58),

$$p(\mathbf{r}) = -2\mathrm{j}\omega\rho_0\mathbf{L}_g \oint_A \mathrm{g}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\mathrm{d}A(\mathbf{r}_0)\ \mathrm{at}A$$

or

$$\mathbf{u}(\mathbf{r})\cdot\mathbf{e}_n = -2\mathrm{j}\omega Q_0 \oint_A \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}_i(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0)\mathrm{d}A(\mathbf{r}_0)\ \text{at}\ A. \tag{61}$$

Inserting Eq. (53) in Eqs. (60) and (61) yields

$$\oint_A \mathrm{g}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\mathrm{d}A(\mathbf{r}_0) = -R\mathbf{L}_g \oint_A \mathrm{g}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\mathrm{d}A(\mathbf{r}_0)\ \text{at}\ A$$

or

$$\mathbf{u}_s(\mathbf{r})\cdot\mathbf{e}_n = -R\mathbf{L}_g\mathbf{u}(\mathbf{r})\cdot\mathbf{e}_n\ \text{at}\ A, \tag{62}$$

which alternatives in fact give the same result, best presented by the latter. So in this case the expressions for the scattered fields are such that the surface integrations and Green's functions are no longer needed. The expression for the secondary source strength vector of the simple pressure-reflecting source is now, according to Eqs. (44), (53) and (62),

$$\mathbf{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{u}_s\cdot\mathbf{e}_n \\ p_s\mathbf{e}_n \end{bmatrix} = R\begin{bmatrix} -\mathbf{L}_g\mathbf{u}\cdot\mathbf{e}_n \\ p\mathbf{e}_n \end{bmatrix}\ \text{at}\ A. \tag{63}$$

According to the result above, the secondary dipole source distribution works on the local control principle in the case of a pressure-reflecting source. Generally, that is not the case with the secondary monopole distribution.

Similarly, if the Green's functions fulfill the same boundary conditions on both sides of A, the *simple velocity-reflecting source*, obeying Eq. (55), has

$$\mathbf{L}_{gu}\mathrm{g}_i(\mathbf{r}|\mathbf{r}_0) = \mathrm{g}(\mathbf{r}|\mathbf{r}_0)\ \text{or}\ \mathbf{G}_i(\mathbf{r}|\mathbf{r}_0) = \mathbf{L}_{gu}\mathbf{G}(\mathbf{r}|\mathbf{r}_0), \tag{64}$$

where operator $\mathbf{L}_{gu}$ is such that it can be extracted from the surface integration in expressions (47) and (49). In that case the scattered field is, according to Eq. (47),

$$p_s(\mathbf{r}) = 2\mathbf{L}_{gu}\oint_A [\mathbf{e}_{n0}\cdot\nabla_0\mathrm{g}_i(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0) + \mathrm{j}\omega\rho_0\mathrm{g}_i(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0)\ \text{at}\ A$$

or

$$\mathbf{u}_s(\mathbf{r})\cdot\mathbf{e}_n = 2\oint_A [\mathrm{j}\omega Q_0\mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0) + \mathbf{e}_n\cdot\nabla_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0)\ \text{at}\ A, \tag{65}$$

and the original field is correspondingly, according to Eq. (49),

$$p(\mathbf{r}) = -2\oint_A [\mathbf{e}_{n0}\cdot\nabla_0\mathrm{g}_i(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0) + \mathrm{j}\omega\rho_0\mathrm{g}_i(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0)\ \text{at}\ A$$

or

$$\mathbf{u}(r)\cdot\mathbf{e}_n = -2\mathbf{L}_{gu}\oint_A [\mathrm{j}\omega Q_0\mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0) + \mathbf{e}_n\cdot\boldsymbol{\nabla}_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})]\mathrm{d}A(\mathbf{r}_0) \text{ at } A. \tag{66}$$

Also in this situation, Green's functions and the Green's dyadics in the integrations in Eqs. (65) and (66) are the same, and they can be selected so that either of the following boundary conditions is fulfilled:

$$\mathbf{e}_{n0}\cdot\boldsymbol{\nabla}_0 \mathrm{g}_i(\mathbf{r}|\mathbf{r}_0) = 0 \text{ at } A \text{ or } \mathbf{e}_n\cdot\boldsymbol{\nabla}_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) = 0 \text{ at } A. \tag{67}$$

In those cases, correspondingly, the following alternate expressions for the scattered fields, according to Eq. (65), are obtained

$$p_s(\mathbf{r}) = 2\mathrm{j}\omega\rho_0\mathbf{L}_{gu}\oint_A \mathrm{g}_i(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}_s(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\mathrm{d}A(\mathbf{r}_0) \text{ at } A$$

or

$$\mathbf{u}_s(\mathbf{r})\cdot\mathbf{e}_n = 2\mathrm{j}\omega Q_0\oint_A \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0)\mathrm{d}A(\mathbf{r}_0) \text{ at } A. \tag{68}$$

The corresponding equations for the original fields are in that case, according to Eq. (66),

$$p(\mathbf{r}) = -2\mathrm{j}\omega\rho_0\oint_A \mathrm{g}_i(\mathbf{r}|\mathbf{r}_0)(\mathbf{u}(\mathbf{r}_0)\cdot\mathbf{e}_{n0})\mathrm{d}A(\mathbf{r}_0) \text{ at } A$$

or

$$\mathbf{u}(\mathbf{r})\cdot\mathbf{e}_n = -2\mathrm{j}\omega Q_0\mathbf{L}_{gu}\oint_A \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0)\mathrm{d}A(\mathbf{r}_0) \text{ at } A. \tag{69}$$

Inserting Eq. (55) in Eqs. (68) and (69) yields

$$p_s(\mathbf{r}) = R\mathbf{L}_{gu}p(\mathbf{r}) \text{ at } A$$

or

$$\oint_A \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}(\mathbf{r}|\mathbf{r}_0)p_s(\mathbf{r}_0)\mathrm{d}A(\mathbf{r}_0) = R\mathbf{L}_{gu}\oint_A \mathbf{e}_{n0}\mathbf{e}_n\mathbf{:G}(\mathbf{r}|\mathbf{r}_0)p(\mathbf{r}_0)\mathrm{d}A(\mathbf{r}_0) \text{ at } A, \tag{70}$$

which alternatives in fact give the same result, best presented by the former alternative. So in this case the expressions for the scattered fields are also such that the surface integrations and Green's functions are no longer needed. The expression for the secondary source strength vector of the simple velocity-reflecting source is now, according to Eqs. (44), (55) and (70),

$$\boldsymbol{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{u}_s\cdot\mathbf{e}_n \\ p_s\mathbf{e}_n \end{bmatrix} = R\begin{bmatrix} -\mathbf{u}\cdot\mathbf{e}_n \\ \mathbf{L}_{gu}p\mathbf{e}_n \end{bmatrix} \text{ at } A. \tag{71}$$

According to the result above, the secondary monopole distribution works on the local control principle in the case of a simple velocity-reflecting source. Generally, that is not the case with the secondary dipole distribution.

If surface A is an infinite plane, $\mathbf{L}_g = \mathbf{L}_{gu} = 1$. In that case the simple pressure-reflecting source and the simple velocity-reflecting source are identical and, according to Eqs. (62) and (70),

$$p_s = Rp \text{ at } A \text{ and } \mathbf{u}_s \cdot \mathbf{e}_n = -R\mathbf{u} \cdot \mathbf{e}_n \text{ at } A. \tag{72}$$

The secondary source vector in this case is, according to Eq. (63) or Eq. (71),

$$\boldsymbol{S}''_s = \begin{bmatrix} q''_s \\ \mathbf{f}''_s \end{bmatrix} = \begin{bmatrix} \mathbf{u}_s \cdot \mathbf{e}_n \\ p_s \mathbf{e}_n \end{bmatrix} = R \begin{bmatrix} -\mathbf{u} \cdot \mathbf{e}_n \\ p\mathbf{e}_n \end{bmatrix} \text{ at } A. \tag{73}$$

This result is compatible with the one obtained directly with the reflection transformation, see Eq. (26).

According to Eq. (73), the simple planar reflecting source is a special case of the boundary condition element such that it works purely on the local control principle.

4.2.3. Matrix formulation of the equations

For numerical calculations in more complicated situations, surface A can be divided into subareas so that the field variables can be regarded as constant on each subarea. Subarea pressure vector $\mathbf{P}_s$ and subarea velocity vector $\mathbf{U}_s$ of the scatterer on surface A are defined as

$$\begin{aligned} \mathbf{P}_s &= [p_{s1} \quad p_{s2} \quad p_{s3} \quad \ldots]_\mathrm{T} \\ \mathbf{U}_s &= [\mathbf{u}_{s1} \cdot \mathbf{e}_n \quad \mathbf{u}_{s2} \cdot \mathbf{e}_n \quad \mathbf{u}_{s3} \cdot \mathbf{e}_n \quad \ldots]_\mathrm{T}. \end{aligned} \tag{74}$$

Similar definitions can be made for the original field, to yield vectors $\mathbf{P}$ and $\mathbf{U}$ with components $p_1, p_2, p_3, \cdots$, and $u_1 = \mathbf{u}_1 \cdot \mathbf{e}_n$, $u_2 = \mathbf{u}_2 \cdot \mathbf{e}_n$, $u_3 = \mathbf{u}_3 \cdot \mathbf{e}_n \cdots$. Expression (47) now has the form

$$\begin{bmatrix} \mathbf{P}_s \\ \mathbf{U}_s \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{P}_s \\ \mathbf{U}_s \end{bmatrix} \text{ at } A, \tag{75}$$

where components a_{ij}, b_{ij}, c_{ij}, and d_{ij} of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, respectively, are

$$\begin{aligned} & a_{ij} = 2\Delta A_j \mathbf{e}_{nj} \cdot \boldsymbol{\nabla}_j g(r_i|r_j); b_{ij} = 2\Delta A_j \mathrm{j}\omega\rho_0 g(r_i|r_j); \\ & c_{ij} = 2\Delta A_j \mathrm{j}\omega Q_0 \mathbf{e}_{nj}\mathbf{e}_{ni}\mathbf{:}\mathbf{G}(r_i|r_j); d_{ij} = 2\Delta A_j \mathbf{e}_{ni} \cdot \boldsymbol{\nabla}_j \mathbf{G}(r_i|r_j), \end{aligned} \tag{76}$$

where ΔA_j is the area of the subarea j, the subscript j in $\boldsymbol{\nabla}$ denotes that the spatial derivative is attached to co-ordinate r_j, the last subscript in $\mathbf{e}_n$ denotes the point where the unit normal vector is defined, and, according to Eq. (48),

$$\begin{aligned} & \mathbf{e}_{nj}\mathbf{e}_{ni}\mathbf{:}\mathbf{G}(r_i|r_j) = \mathbf{e}_{nj} \cdot \mathbf{e}_{ni} g(r_i|r_j) - \frac{1}{k^2}\mathbf{e}_{nj}\mathbf{e}_{ni}\mathbf{:}[\boldsymbol{\nabla}_j \times (\boldsymbol{\nabla}_j g(r_i|r_j) \times \mathbf{I})] \\ & \mathbf{e}_{ni} \cdot \boldsymbol{\nabla}_j \cdot \mathbf{G}(r_i|r_j) = \mathbf{e}_{ni} \cdot \boldsymbol{\nabla}_j g(r_i|r_j). \end{aligned} \tag{77}$$

The matrix terms with $i = j$ should be integrated to avoid singularity in the expressions above.

The equation for the field quantities at A, Eq. (75), can be expressed also as

$$\begin{bmatrix} (\mathbf{I} - \mathbf{A})\mathbf{P}_s \\ (\mathbf{I} - \mathbf{D})\mathbf{U}_s \end{bmatrix} - \begin{bmatrix} \mathbf{B}\mathbf{U}_s \\ \mathbf{C}\mathbf{P}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at } A, \tag{78}$$

where $\mathbf{I}$ is a unit matrix.

Matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are not totally independent. Three of them yield always the remaining one. Eq. (78) yields, e.g., matrix $\mathbf{C}$ by the help of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ as

$$\mathbf{C} = (\mathbf{I} - \mathbf{D})\mathbf{B}^{-1}(\mathbf{I} - \mathbf{A}). \tag{79}$$

Boundary condition (50) is now

$$\begin{aligned} &\varphi\left(\begin{bmatrix} \mathbf{P} + \mathbf{P}_s \\ \mathbf{U} + \mathbf{U}_s \end{bmatrix}\right) = 0 \text{ at } A \text{ or} \\ &\varphi'\left(\begin{bmatrix} \mathbf{P} \\ \mathbf{U} \\ \mathbf{P}_s \\ \mathbf{U}_s \end{bmatrix}\right) = 0 \text{ at } A. \end{aligned} \tag{80}$$

With the matrix formulation, monopole strength vector $\mathbf{Q}_s''$ with components $q_{s1}'', q_{s2}'', q_{s3}'', \dots,$ and dipole strength vector $\mathbf{F}_s''$ with components $\mathbf{f}_{s1}'', \mathbf{f}_{s2}'', \mathbf{f}_{s3}'', \dots$ in Eq. (44) are

$$\mathbf{Q}_s'' = \mathbf{U}_s, \quad \mathbf{F}_s'' = \mathbf{P}_{ns}, \tag{81}$$

where the ith element in the pressure vector $\mathbf{P}_{ns}$ vector is $p_{si}\mathbf{e}_{ni}$. Once the scattered field vectors have been calculated from boundary condition (80) by the help of the original field vectors and Eq. (78), the secondary sources needed are obtained from the scattered fields at A according to Eq. (81).

4.2.4. Pressure-reflecting boundary

With the pressure-reflecting boundary, the boundary condition is as in Eq. (53). In the general case the reflection coefficient R has to be substituted by a diagonal reflection coefficient matrix $\mathbf{R}$. If the reflection coefficient is not a function of spatial co-ordinates on A, the matrix is a multiple of the unit matrix of the form $\mathbf{R} = R\mathbf{I}$. The boundary condition and presentation (78) yield now

$$\begin{bmatrix} (\mathbf{I} - \mathbf{A})\mathbf{R}\mathbf{P} \\ (\mathbf{I} - \mathbf{D})\mathbf{U}_s \end{bmatrix} = \begin{bmatrix} \mathbf{B}\mathbf{U}_s \\ \mathbf{C}\mathbf{R}\mathbf{P} \end{bmatrix} \text{ at } A, \tag{82}$$

from which the normal component of the scattered particle velocity at A can be calculated. The upper and lower equations give two possible presentations for the normal component of the scattered particle velocity at A:

$$\mathbf{U}_s = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A})\mathbf{R}\mathbf{P} \text{ at } A \text{ or } \mathbf{U}_s = (\mathbf{I} - \mathbf{D})^{-1}\mathbf{C}\mathbf{R}\mathbf{P} \text{ at } A. \tag{83}$$

The secondary source strength vector is now, according to Eqs. (44), (53) and (83),

$$\begin{bmatrix} \mathbf{Q}_s'' \\ \mathbf{F}_s'' \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A})\mathbf{R}\mathbf{P} \\ \mathbf{R}\mathbf{P}_n \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{D})^{-1}\mathbf{C}\mathbf{R}\mathbf{P} \\ \mathbf{R}\mathbf{P}_n \end{bmatrix}, \tag{84}$$

where the ith element in the pressure vector $\mathbf{P}_n$ is $p_i\mathbf{e}_{ni}$.

In the case of a pressure-release boundary, the sum of the sound pressures vanishes on the boundary. The reflection coefficient matrix is now $\mathbf{R} = -\mathbf{I}$ and

$$\mathbf{U}_s = -\mathbf{B}^{-1}[\mathbf{I} - \mathbf{A}]\mathbf{P} \text{ at } A \text{ or } \mathbf{U}_s = -[\mathbf{I} - \mathbf{D}]^{-1}\mathbf{CP} \text{ at } A. \tag{85}$$

The secondary source strength vector is now

$$\begin{bmatrix} \mathbf{Q}_s'' \\ \mathbf{F}_s'' \end{bmatrix} = -\begin{bmatrix} \mathbf{B}^{-1}[\mathbf{I} - \mathbf{A}]\mathbf{P} \\ \mathbf{P}_n \end{bmatrix} = -\begin{bmatrix} [\mathbf{I} - \mathbf{D}]^{-1}\mathbf{CP} \\ \mathbf{P}_n \end{bmatrix}. \tag{86}$$

In the matrix presentation for the pressure-reflecting boundary, only the sound pressure is needed from the original fields. The secondary dipole sources work on the local control principle with the pressure-reflecting boundary, as stated before with the operator presentation.

4.2.5. Velocity-reflecting boundary

With the velocity-reflecting boundary, the boundary condition is as in Eq. (55). In the general case the reflection coefficient R has to be substituted by a diagonal reflection coefficient matrix $\mathbf{R}$. The boundary condition and presentation (78) yield now

$$\begin{bmatrix} (\mathbf{I} - \mathbf{A})\mathbf{P}_s \\ (\mathbf{I} - \mathbf{D})\mathbf{RU} \end{bmatrix} = -\begin{bmatrix} \mathbf{BRU} \\ \mathbf{CP}_s \end{bmatrix} \text{ at } A, \tag{87}$$

from which the scattered sound pressure can be calculated. The upper and lower equations give two possible presentations for the scattered sound pressure at A

$$\mathbf{P}_s = -(\mathbf{I} - \mathbf{A})^{-1}\mathbf{BRU} \text{ at } A \text{ or } \mathbf{P}_s = -\mathbf{C}^{-1}(\mathbf{I} - \mathbf{D})\mathbf{RU} \text{ at } A. \tag{88}$$

The secondary source strength vector is now, according to Eqs. (44), (55) and (88),

$$\begin{bmatrix} \mathbf{Q}_s'' \\ \mathbf{F}_s'' \end{bmatrix} = -\begin{bmatrix} \mathbf{RU} \\ (\mathbf{I} - \mathbf{A})^{-1}\mathbf{BRU}_n \end{bmatrix} = -\begin{bmatrix} \mathbf{RU} \\ \mathbf{C}^{-1}(\mathbf{I} - \mathbf{D})\mathbf{RU}_n \end{bmatrix}, \tag{89}$$

where the ith element in the velocity vector $\mathbf{U}_n$ is $\mathbf{u}_i\mathbf{e}_{ni}$.

In the case of a rigid boundary, the sum of the normal components of the particle velocity vanishes on the boundary. In that case $\mathbf{R} = \mathbf{I}$ and

$$\mathbf{P}_s = -(\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU} \text{ at } A \text{ or } \mathbf{P}_s = -\mathbf{C}^{-1}(\mathbf{I} - \mathbf{D})\mathbf{U} \text{ at } A. \tag{90}$$

The secondary source strength vector of the rigid boundary is

$$\begin{bmatrix} \mathbf{Q}''_s \\ \mathbf{F}''_s \end{bmatrix} = -\begin{bmatrix} \mathbf{U} \\ (\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}_n \end{bmatrix} = -\begin{bmatrix} \mathbf{U} \\ \mathbf{C}^{-1}(\mathbf{I} - \mathbf{D})\mathbf{U}_n \end{bmatrix}. \tag{91}$$

In the matrix presentation for the velocity-reflecting boundary, only the normal component of the particle velocity is needed from the original fields. The secondary monopole sources work on the local control principle with the velocity-reflecting boundary, as stated before with the operator presentation.

4.3. Impedance boundary condition

In a general case the boundary conditions cannot be presented as easily as above, e.g., in the case of an impedance boundary. In that case a proper duality transformation [23] makes the problem much similar to those presented. In the next the impedance boundary is introduced, based on a proper duality transformation.

4.3.1. Duality transformed fields

It is supposed that the field at A is composed of original and scattered components $\mathbf{F}_n$ and $\mathbf{F}_{sn}$ (subscript n with only normal components of vector quantities) having both two components

$$\mathbf{F}_n = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \mathbf{F}_{sn} = \begin{bmatrix} f_{s1} \\ f_{s2} \end{bmatrix}, \tag{92}$$

and that there exists duality transformation $\mathbf{T}_D$ such that with the transformed fields at surface A

$$\mathbf{F}_d = \begin{bmatrix} f_{1d} \\ f_{2d} \end{bmatrix} = \mathbf{T}_D\mathbf{F}_n = \mathbf{T}_D \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \mathbf{F}_{sd} = \begin{bmatrix} f_{s1d} \\ f_{s2d} \end{bmatrix} = \mathbf{T}_D\mathbf{F}_{sn} = \mathbf{T}_D \begin{bmatrix} f_{s1} \\ f_{s2} \end{bmatrix} \tag{93}$$

the boundary condition operator can be represented by a vector $\boldsymbol{\varphi}_d$ as

$$\boldsymbol{\varphi}_d = [0 \quad 1]. \tag{94}$$

Boundary condition (50) (upper version) for the transformed fields is in this case

$$\boldsymbol{\varphi}_d(\mathbf{F}_d + \mathbf{F}_{sd}) = 0 \text{ at } A \Leftrightarrow [0 \quad 1] \begin{bmatrix} f_{1d} + f_{s1d} \\ f_{2d} + f_{s2d} \end{bmatrix} = 0 \text{ at } A, \tag{95}$$

which is simply

$$f_{2d} + f_{s2d} = 0 \text{ at } A \Leftrightarrow f_{s2d} = -f_{2d} \text{ at } A. \tag{96}$$

The other transformed scattered field component is thus readily known on the surface. Substituting the transformed field variables with $\mathbf{F}_{sn} = \mathbf{T}_D^{-1}\mathbf{F}_{sd}$, according to Eq. (93), in Eq. (39) (or Eq. (47)), or its discretized version Eq. (75), and utilizing Eq. (96) yield

$$\mathbf{T}_D^{-1}\mathbf{F}_{sd} = \mathbf{T}_D^{-1} \begin{bmatrix} f_{s1d} \\ -f_{2d} \end{bmatrix} = 2\mathbf{L}^{-1}[\mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{T}_D^{-1}\mathbf{F}_{sd}] = 2\mathbf{L}^{-1} \left[\mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{T}_D^{-1} \begin{bmatrix} f_{s1d} \\ -f_{2d} \end{bmatrix} \right] \text{ at } A \text{ or}$$

$$\mathbf{T}_D^{-1}\mathbf{F}_{sd} = \mathbf{T}_D^{-1} \begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \mathbf{T}_D^{-1}\mathbf{F}_{sd} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \mathbf{T}_D^{-1} \begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} \text{ at } A, \tag{97}$$

where $\mathbf{F}_{1d}$, $\mathbf{F}_{2d}$, $\mathbf{F}_{s1d}$ and $\mathbf{F}_{s2d}$ are subarea vector presentations of f_{1d}, f_{2d}, f_{s1d}, and f_{s2d} correspondingly. Multiplying the equation above by $\mathbf{T}_D$ yields

$$\begin{bmatrix} f_{s1d} \\ -f_{2d} \end{bmatrix} = 2\mathbf{T}_D\mathbf{L}^{-1} \left[\mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{T}_D^{-1} \begin{bmatrix} f_{s1d} \\ -f_{2d} \end{bmatrix} \right] \text{ at } A \text{ or}$$

$$\begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} = \mathbf{T}_D \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \mathbf{T}_D^{-1} \begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} \text{ at } A. \tag{98}$$

Now the other transformed scattered field component at the surface can also be calculated from either of the expressions above. After the transformed scattered field at the surface has been calculated, the actual scattered field at the surface is obtained from, according to Eq. (93),

$$\mathbf{F}_{sn} = \begin{bmatrix} f_{s1} \\ f_{s2} \end{bmatrix} = \mathbf{T}_D^{-1}\mathbf{F}_{sd} = \mathbf{T}_D^{-1}\begin{bmatrix} f_{s1d} \\ f_{s2d} \end{bmatrix} \text{ at } A, \tag{99}$$

after which the secondary sources are known, according to Eq. (36),

$$\mathbf{S}'' = \mathbf{L}(\varepsilon(x_1 - x_{10}))\mathbf{F}_s. \tag{100}$$

Operator $\mathbf{L}(\varepsilon(x_1 - x_{10}))$ picks up only component $\mathbf{F}_{sn}$ from $\mathbf{F}_s$ at A in the equation above.

4.3.2. Impedance boundary

In the case of an impedance boundary, the boundary condition is

$$\mathbf{P} + \mathbf{P}_s = \mathbf{Z}(\mathbf{U} + \mathbf{U}_s) \text{ at } A, \tag{101}$$

where the diagonal impedance matrix $\mathbf{Z}$ is the desired ratio of the total sound pressure and normal component of the particle velocity at the surface as a function of spatial co-ordinates. If the impedance is not dependent on spatial co-ordinates on A, it is a multiple of the unit matrix of the form $\mathbf{Z} = Z\mathbf{I}$. The fields can be transformed in order to change the boundary condition into the form of Eq. (95) or (96). The proper field transformation at the surface is [23]

$$\mathbf{F}_{sd} = \begin{bmatrix} \mathbf{F}_{s1d} \\ \mathbf{F}_{s2d} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{Z} \\ \mathbf{I} & -\mathbf{Z} \end{bmatrix}\mathbf{F}_{sn} = \begin{bmatrix} \mathbf{I} & \mathbf{Z} \\ \mathbf{I} & -\mathbf{Z} \end{bmatrix}\begin{bmatrix} \mathbf{P}_s \\ \mathbf{U}_s \end{bmatrix},$$
$$\mathbf{F}_d = \begin{bmatrix} \mathbf{F}_{1d} \\ \mathbf{F}_{2d} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{Z} \\ \mathbf{I} & -\mathbf{Z} \end{bmatrix}\mathbf{F}_n = \begin{bmatrix} \mathbf{I} & \mathbf{Z} \\ \mathbf{I} & -\mathbf{Z} \end{bmatrix}\begin{bmatrix} \mathbf{P} \\ \mathbf{U} \end{bmatrix}. \tag{102}$$

So, the duality transformation operator and its inverse are

$$\mathbf{T}_D = \begin{bmatrix} \mathbf{I} & \mathbf{Z} \\ \mathbf{I} & -\mathbf{Z} \end{bmatrix}, \quad \mathbf{T}_D^{-1} = \tfrac{1}{2}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{Y} & -\mathbf{Y} \end{bmatrix}, \tag{103}$$

where

$$\mathbf{Y} = \mathbf{Z}^{-1} \tag{104}$$

The transformed fields could be obtained from expressions (98). However, using expressions (97) instead yields simpler results. Now

$$\mathbf{T}_D^{-1}\mathbf{F}_{sd} = \tfrac{1}{2}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{Y} & -\mathbf{Y} \end{bmatrix}\begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \mathbf{F}_{s1d} - \mathbf{F}_{2d} \\ \mathbf{Y}(\mathbf{F}_{s1d} + \mathbf{F}_{2d}) \end{bmatrix},$$
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\mathbf{T}_D^{-1} = \tfrac{1}{2}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{Y} & -\mathbf{Y} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \mathbf{A} + \mathbf{BY} & \mathbf{A} - \mathbf{BY} \\ \mathbf{C} + \mathbf{DY} & \mathbf{C} - \mathbf{DY} \end{bmatrix}. \tag{105}$$

Now expressions (97) yield

$$\frac{1}{2}\begin{bmatrix} \mathbf{F}_{s1d} - \mathbf{F}_{2d} \\ \mathbf{Y}(\mathbf{F}_{s1d} + \mathbf{F}_{2d}) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{A} + \mathbf{BY} & \mathbf{A} - \mathbf{BY} \\ \mathbf{C} + \mathbf{DY} & \mathbf{C} - \mathbf{DY} \end{bmatrix}\begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} \text{ at } A, \tag{106}$$

which can be rearranged to

$$\begin{bmatrix} \mathbf{F}_{s1d} - \mathbf{F}_{2d} \\ \mathbf{F}_{s1d} + \mathbf{F}_{2d} \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{BY} & \mathbf{A} - \mathbf{BY} \\ \mathbf{ZC} + \mathbf{ZDY} & \mathbf{ZC} - \mathbf{ZDY} \end{bmatrix}\begin{bmatrix} \mathbf{F}_{s1d} \\ -\mathbf{F}_{2d} \end{bmatrix} \text{ at } A. \tag{107}$$

This leads to two alternative solutions

$$\begin{aligned} &\mathbf{F}_{s1d} = [\mathbf{I} - \mathbf{A} - \mathbf{BY}]^{-1}[\mathbf{I} - \mathbf{A} + \mathbf{BY}]\mathrm{F}_{2d} \text{ or} \\ &\mathbf{F}_{s1d} = -[\mathbf{I} - \mathbf{ZC} - \mathbf{ZDY}]^{-1}[\mathbf{I} + \mathbf{ZC} - \mathbf{ZDY}]\mathrm{F}_{2d} \text{ at } A. \end{aligned} \tag{108}$$

Now, according to Eqs. (42), (93), (102), (105) and (108), the two alternative solutions below yield the scattered field vector at the surface

$$\begin{aligned} \mathbf{F}_{sn} = \begin{bmatrix} \mathbf{P}_s \\ \mathbf{U}_s \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} \mathbf{F}_{s1d} - \mathbf{F}_{2d} \\ \mathbf{Y}(\mathbf{F}_{s1d} + \mathbf{F}_{2d}) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} [\mathbf{I} - \mathbf{A} - \mathbf{BY}]^{-1}[\mathbf{I} - \mathbf{A} + \mathbf{BY}] - \mathbf{I} \\ \mathbf{Y}([\mathbf{I} - \mathbf{A} - \mathbf{BY}]^{-1}[\mathbf{I} - \mathbf{A} + \mathbf{BY}] + \mathbf{I}) \end{bmatrix}(\mathbf{P} - \mathbf{ZU}) \text{ at } A, \text{ or} \\ &= -\frac{1}{2}\begin{bmatrix} [\mathbf{I} - \mathbf{ZC} - \mathbf{ZDY}]^{-1}[\mathbf{I} + \mathbf{ZC} - \mathbf{ZDY}] + \mathbf{I} \\ \mathbf{Y}([\mathbf{I} - \mathbf{ZC} - \mathbf{ZDY}]^{-1}[\mathbf{I} + \mathbf{ZC} - \mathbf{ZDY}] - \mathbf{I}) \end{bmatrix}(\mathbf{P} - \mathbf{ZU}) \text{ at } A. \end{aligned} \tag{109}$$

The secondary source strength vector is now, according to Eq. (81),

$$\begin{aligned} \begin{bmatrix} \mathbf{Q}''_s \\ \mathbf{F}''_s \end{bmatrix} &= \begin{bmatrix} \mathbf{U}_s \\ \mathbf{P}_{ns} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \mathbf{Y}([\mathbf{I} - \mathbf{A} - \mathbf{BY}]^{-1}[\mathbf{I} - \mathbf{A} + \mathbf{BY}] + \mathbf{I}) \\ ([\mathbf{I} - \mathbf{A} - \mathbf{BY}]^{-1}[\mathbf{I} - \mathbf{A} + \mathbf{BY}] - \mathbf{I})e_n \end{bmatrix}(\mathbf{P} - \mathbf{ZU}) \text{ at } A, \text{ or} \\ &= -\frac{1}{2}\begin{bmatrix} \mathbf{Y}([\mathbf{I} - \mathbf{ZC} - \mathbf{ZDY}]^{-1}[\mathbf{I} + \mathbf{ZC} - \mathbf{ZDY}] - \mathbf{I}) \\ ([\mathbf{I} - \mathbf{ZC} - \mathbf{ZDY}]^{-1}[\mathbf{I} + \mathbf{ZC} - \mathbf{ZDY}] + \mathbf{I})\mathbf{e}_n \end{bmatrix}(\mathbf{P} - \mathbf{ZU}) \text{ at } A. \end{aligned} \tag{110}$$

Either of the two alternative expressions above yield the secondary sources needed to obtain the active impedance boundary with the desired impedance at surface A. In a general case, the solution does not work on the local control principle, but the secondary source strengths on each subarea need information of the primary fields at each subarea.

4.4. True boundary condition elements

The active scatterer may be categorized as a true boundary condition element if its net sound power radiation is less than or equal to zero

$$\oint_A \mathrm{Re}\{(p+p_s)(\mathbf{u}+\mathbf{u}_s)^* \cdot \mathbf{e}_n\}\mathrm{d}A \leqslant 0, \tag{111}$$

where "*" denotes complex conjugate and Re{·} denotes real part. The zero power means that the scatterer behaves like a reactive element, and negative power means that the scatterer has absorptive properties.

As an example, the simple reflecting source on an infinite plane has, according to Eq. (72),

$$\begin{aligned}\oint_A &\mathrm{Re}\{(p+p_s)(\mathbf{u}+\mathbf{u}_s)^* \cdot \mathbf{e}_n\}\mathrm{d}A \\ &= \oint_A \mathrm{Re}\{p(1+R)\mathbf{u}^*(1-R^*) \cdot \mathbf{e}_n\}\mathrm{d}A = \oint_A \mathrm{Re}\{p\mathbf{u}^* \cdot \mathbf{e}_n(1+R-R^*-|R|^2)\}\mathrm{d}A \\ &= \oint_A [\mathrm{Re}\{p\mathbf{u}^* \cdot \mathbf{e}_n\}(1-|R|^2) - 2\mathrm{Im}\{p\mathbf{u}^* \cdot \mathbf{e}_n\}\mathrm{Im}\{R\}]\mathrm{d}A \leqslant 0,\end{aligned} \tag{112}$$

where Im{·} denotes imaginary part. Because the original field propagates towards the surface, planar scatterers have

$$\mathrm{Re}\{p\mathbf{u}^* \cdot \mathbf{e}_n\} \leqslant 0. \tag{113}$$

Thus, if the original field has no reactive intensity at A (as with an incident plane wave), the condition of the simple reflecting source on an infinite plane to be a true boundary condition element is simply

$$|R| \leqslant 1. \tag{114}$$

As further examples, the pressure-release surface ($R=-1$) and the rigid surface ($R=+1$) elements are always true boundary condition elements. With the first, the total sound pressure vanishes, and with the second, the normal component of the total particle velocity vanishes at A. This leads to the situation that the normal component of the intensity vanishes at A and according to Eq. (112) there is no power flow normal to the surface.

As the last example, the impedance surface is considered. With the impedance surface

$$\begin{aligned}\oint_A \mathrm{Re}\{(p+p_s)(\mathbf{u}+\mathbf{u}_s)^* \cdot \mathbf{e}_n\}\mathrm{d}A &= \oint_A \mathrm{Re}\{Z(\mathbf{u}+\mathbf{u}_s) \cdot \mathbf{e}_n(\mathbf{u}+\mathbf{u}_s)^* \cdot \mathbf{e}_n\}\mathrm{d}A \\ &= \oint_A \mathrm{Re}\{Z\}|(\mathbf{u}+\mathbf{u}_s) \cdot \mathbf{e}_n|^2\mathrm{d}A \leqslant 0.\end{aligned} \tag{115}$$

Now, if the impedance is constant on A, the condition of the impedance boundary to be a true boundary condition element is simply

$$\mathrm{Re}\{Z\} \leqslant 0. \tag{116}$$

This ensures that the net power flow is not directed outwards from the impedance boundary.

5. Conclusions

The principle of formulating the JMC method to produce secondary sources that function as active scatterers on a hypothetical scattering surface has been established. As examples, an actively reflecting plane and active boundary condition elements were introduced.

The JMC formulation for the active scatterer was presented in a general operator formulation and applied to acoustic fields. The actively reflecting plane served as an example, and the solution was extended to a general planar JMC element with well-defined reflection and transmission properties. The solution works on the local control principle: each reflecting subarea needs information of the primary field only at that subarea. The solution can also apply to piecemeal planar surfaces and to smooth convex surfaces, approximately.

General active boundary condition elements were defined in a general operator formulation and applied to acoustic fields. Based on the element definitions, simple reflecting source, pressure- and velocity-reflecting boundaries, and impedance boundary were introduced. The utilization of the Huygens' principle in the derivations connects the field variables at the hypothetical surface of the scatterer so that the scattered field obeys the field equation automatically. Boundary condition elements do not work on the local control principle: in a general case secondary sources on each subarea need information of the primary field at each subarea. Secondary dipole sources work on the local control principle in the case of a pressure-reflecting boundary, and secondary monopole sources in the case of a velocity-reflecting boundary correspondingly. True boundary condition elements were also defined based on their net sound power radiation.

The active scatterers need, as a reference signal, the original field values at the hypothetical scattering surface. This may be one of the main limitations of practical systems. However, there are solutions to avoid the limitation, e.g., unidirectional reference detectors can be used, or the reference signals can be synthesized from the original source signals detected near the sources, utilizing the transfer functions from the sources to the hypothetical scattering surface. One possibility is to implement an electronic model of the feedback path (with the sign reversed) within the controller in order to subtract the outputs of the actuators from the detector signals, as is done in the Internal Model Control [26] in feedback systems.

There is a need for further work in practical problems concerning implementation of the active scatterers, and in testing and measuring their applicability in real systems. Some fields of application could be the active control of the acoustical properties of concert halls, cinemas, open-air concert places, and generally all interior spaces where changeable acoustical properties are needed.

Appendix A. Inverse of operator L

Consider a flowless, homogenous ideal fluid. With time-harmonic fields, Eqs. (1) and (18) yield the wave equations for the field variables in the forms

$$\nabla^2 p + k^2 p = -\mathrm{j}\omega\rho_0 q + \nabla \cdot \mathbf{f}, \quad \nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u} = -\mathrm{j}\omega Q_0 \mathbf{f} + \nabla q, \tag{A.1}$$

where ω is the angular frequency, j is the imaginary unit, and k is the wave number (ω/c_0, c_0 is the speed of sound of the fluid at rest).

Scalar Green's function g is a function obeying (see, e.g., [27])

$$\nabla^2 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) + k^2 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0), \tag{A.2}$$

where δ is the Dirac delta function, $\mathbf{r}$ is the field point, and $\mathbf{r}_0$ is the source point. The scalar Green's function can be regarded as the field of a point source.

Dyadic Green's function G for irrotational fields is a function obeying [23]

$$\nabla\nabla \cdot \mathsf{G}(\mathbf{r}|\mathbf{r}_0) + k^2 \mathsf{G}(\mathbf{r}|\mathbf{r}_0) = -\mathsf{I}\delta(\mathbf{r} - \mathbf{r}_0), \tag{A.3}$$

where I is the identic dyadic. If the scalar Green's function is known, the corresponding dyadic Green's function for irrotational fields can be obtained from the scalar one by using the relation [28]

$$\mathsf{G}(\mathbf{r}|\mathbf{r}_0) = \left(\mathsf{I} + \frac{1}{k^2}\nabla\nabla{}^{\times}_{\times}\mathsf{I}\right)\mathrm{g}(\mathbf{r}|\mathbf{r}_0), \tag{A.4}$$

where

$$\nabla\nabla{}^{\times}_{\times}\mathsf{I}\mathrm{g}(\mathbf{r}|\mathbf{r}_0) = -\nabla \times \nabla \times (\mathsf{I}\mathrm{g}(\mathbf{r}|\mathbf{r}_0)). \tag{A.5}$$

The double cross product of two dyadics $\mathbf{ab}$ and $\mathbf{cd}$ is generally defined as $\mathbf{ab}{}^{\times}_{\times}\mathbf{cd} = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \times \mathbf{d})$.

Comparing Eq. (A.1) with Eqs. (A.2) and (A.3) yields expressions for the sound pressure and the particle velocity as convolution integrals with the scalar Green's function and the corresponding dyadic Green's function, and the monopole and dipole distributions as

$$\begin{aligned}
p &= \int_V [\mathrm{j}\omega\rho_0 q(\mathbf{r}_0) - \nabla_0 \cdot \mathbf{f}(\mathbf{r}_0)]\mathrm{g}(\mathbf{r}|\mathbf{r}_0)\mathrm{d}V(\mathbf{r}_0) \\
&= \int_V [\mathrm{j}\omega\rho_0 q(\mathbf{r}_0)\mathrm{g}(\mathbf{r}|\mathbf{r}_0) + \mathbf{f}(\mathbf{r}_0) \cdot \nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)]\mathrm{d}V(\mathbf{r}_0) \\
\mathbf{u} &= \int_V [\mathrm{j}\omega Q_0 \mathbf{f}(\mathbf{r}_0) - \nabla_0 q(\mathbf{r}_0)] \cdot \mathsf{G}(\mathbf{r}|\mathbf{r}_0)\mathrm{d}V(\mathbf{r}_0) \\
&= \int_V [\mathrm{j}\omega Q_0 \mathbf{f}(\mathbf{r}_0) \cdot \mathsf{G}(\mathbf{r}|\mathbf{r}_0) + q(\mathbf{r}_0)\nabla_0 \cdot \mathsf{G}(\mathbf{r}|\mathbf{r}_0)]\mathrm{d}V(\mathbf{r}_0),
\end{aligned} \tag{A.6}$$

where the integration is performed with respect to co-ordinate $\mathbf{r}_0$, ∇_0 operates to co-ordinate $\mathbf{r}_0$, and where V is the volume under examination. The boundary surface of V such that the source distributions are truly inside it is denoted by S. This property of surface S has been utilized in obtaining the latter versions of the right sides of Eq. (A.6) by the help of partial integration and the Gauss' theorem.

If the field equations are now presented in the operator formulae as in Eq. (1), the inverse of operator L can be identified from the equations above as

$$\mathbf{L}^{-1} = \int_V \mathrm{d}V(\mathbf{r}_0)\begin{bmatrix} \mathrm{j}\omega\rho_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0) & \nabla_0 \mathrm{g}(\mathbf{r}|\mathbf{r}_0)\cdot \\ \nabla_0\cdot\mathbf{G}(\mathbf{r}|\mathbf{r}_0) & \mathrm{j}\omega Q_0\mathbf{G}_\mathrm{T}(\mathbf{r}|\mathbf{r}_0)\cdot \end{bmatrix}, \tag{A.7}$$

where $\mathbf{G}_\mathrm{T}$ is the transpose of $\mathbf{G}$.

Using Eq. (A.4), a couple of useful expressions can be obtained

$$\begin{aligned} \nabla_0\cdot\mathbf{G} &= \nabla_0\cdot\left(\mathbf{I}+\frac{1}{k^2}\nabla_0\nabla_0 \,{}^{\times}_{\times}\mathbf{I}\right)\mathrm{g} = \nabla_0\mathrm{g}\cdot\mathbf{I} = \nabla_0\mathrm{g} \\ \mathbf{a}\cdot\mathbf{G} &= \mathbf{a}\cdot\left(\mathbf{I}+\frac{1}{k^2}\nabla_0\nabla_0 \,{}^{\times}_{\times}\mathbf{I}\right)\mathrm{g} = \mathbf{a}\mathrm{g} - \frac{1}{k^2}\mathbf{a}\cdot\{\nabla_0\times[\nabla_0\times(\mathrm{g}\mathbf{I})]\} \\ &= \mathbf{a}\mathrm{g} - \frac{1}{k^2}\mathbf{a}\cdot[\nabla_0\times(\nabla_0\mathrm{g}\times\mathbf{I})], \end{aligned} \tag{A.8}$$

where $\mathbf{a}$ is an arbitrary vector.

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