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Letter to the Editor

A discrete-time, optimal, active vibration absorber

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1. Introduction

This paper deals with optimal damping of steady state sinusoidal oscillations in vibratory systems. A simple spring–mass system subjected to sinusoidal excitation with known frequency is considered as an example. The discrete-time control algorithm developed by Lindquist and Yakubovich [1] is proposed for the problem.

The control algorithm application and the design of the active absorber presented in this paper is motivated by the problem of suppressing vibrations of a rotor with mass imbalance. The mass unbalance of the rotor gives rise to periodic excitations whose nominal frequencies can be determined. The active absorber designed here will be able to suppress the resulting steady state oscillations for a wide range of excitation frequencies and it will be optimal for all values of amplitudes and phases of the excitation.

The problem of controlling vibration in the presence of sinusoidal excitations was studied by Lee and Sinha [2]. They proposed an optimal control algorithm developed by Johnson [3]. Johnson's algorithm requires the external excitation be modeled by a system of linear differential equations of finite order. Therefore, it can be applied if the excitation is periodic since a periodic excitation can always be resolved into sinusoidal components that can be expressed as solutions of second order differential equations. Lee and Sinha [2] applied this algorithm for the case of a single sinusoidal excitation acting on a spring–mass system and showed that their active vibration absorber gave a better overall performance than optimal passive vibration absorbers [4]. The active absorber was designed assuming that the excitation frequency equals the natural frequency of the system. Such an absorber will completely suppress oscillations at the resonance condition but the system will have non-zero amplitude of vibrations at other excitation frequencies.

Ma and Sinha [5] used a multi-layer neural network (MNN) along with the Johnson algorithm based control system. In this approach, the MNN was used to learn the non-linear mapping between the excitation frequency and the feedback gains for the Johnson algorithm based

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controller. Once the learning phase is completed, the neural network acts as an online optimal controller by taking online measurements of the rotor speeds as inputs to suppress the steady state oscillations. Wang and Sinha [6] designed a neural network based controller to compensate for uncertainties in system parameters and excitation frequencies. They employed two online feedforward MNNs. One MNN was used to compensate for modelling errors in the output model by using an online training algorithm with linkweights chosen such that the errors between the measured system states and states predicted by the system model were minimized. The other MNN was used to produce a correction in the control input for reduction in the vibratory response of the structure by using an online training algorithm with linkweights that minimized an error function.

In this paper we design an optimal and robust regulator in the discrete-time domain using an algorithm developed by Lindquist and Yakubovich [1]. The absorber designed yields a distinctively robust performance for a wide range of design frequencies. It does not completely suppress the oscillations at design frequency but it gives a flat frequency response for a wide range of frequencies. This leads to better overall performance in the presence of varying or uncertain excitation frequencies. Like the Johnson algorithm based absorber, its design depends only on excitation frequencies, and is independent of the amplitudes and phases. The designed regulator is optimal for all values of amplitudes and phases. The design of the controller is done in the discrete-time domain, and hence it can be directly implemented using a digital computer.

The following section presents some important elements of the Lindquist–Yakubovich algorithm required to solve the problem presented. Then the design and simulation results are presented along with a comparative analysis of this and earlier approaches.

2. Theory

Consider a linear, time-invariant, discrete-time dynamical system affected by an additive sinusoidal disturbance with known frequencies but unknown amplitudes:

$$x_{t+1} = Ax_t + Bu_t + C\omega_t, \quad x_0 = a. \tag{1}$$

$\{x_t\}$ is an n-dimensional state sequence, $\{u_t\}$ is a k-dimensional real control sequence and

$$\omega_t = \begin{pmatrix} \alpha_1 \cos(\theta_1 t + \varphi_1) \\ \alpha_2 \cos(\theta_2 t + \varphi_2) \\ \vdots \\ \alpha_v \cos(\theta_v t + \varphi_v) \end{pmatrix} \tag{2}$$

is an v-dimensional real sinusoidal disturbance with known frequencies

$$-\pi < \theta_1 < \theta_2 < \cdots < \theta_v < \pi \tag{3}$$

but unknown amplitudes $\alpha_1, \alpha_2, \ldots, \alpha_v$, and phases $\varphi_1, \varphi_2, \ldots, \varphi_v$, and A, B, C are given real matrices of appropriate dimensions so that (A, B) is a stabilizable pair and C has no trivial (i.e., zero) columns.

The control objective is to minimize the cost functional given by

$$\Phi = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0} \Lambda(x_t, u_t), \tag{4}$$

where $\Lambda(x,u)$ has the real quadratic form

$$\Lambda(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R_0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{5}$$

with Q and R_0 symmetric, i.e., $Q = Q^*$ and $R = R_0^*$.

It is desired that the regulator, which is stabilizing and optimal in the sense that it minimizes the cost functional Φ, satisfies the following conditions:

1. The regulator should be realizable i.e., it should utilize a finite bounded memory,

$$u_t = \hat{f}(x_t, x_{t-1}, \ldots, x_{t-\tau}, u_{t-1}, \ldots, u_{t-\tau}), \tag{6}$$

   for some τ.
2. The function $\hat{f}$, corresponding to the optimal regulator should not depend on the amplitudes and phases of the disturbance signal. However, the cost function, Φ and the process (x_t, u_t) depend on the amplitudes and phases.
3. The optimal regulator should be robust with respect to known frequencies $\theta_1, \theta_2, \ldots, \theta_v$. In practice, the regulator will be computed from estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_v$ of the true frequencies $\theta_1, \theta_2, \ldots, \theta_v$. Therefore, the cost functional must be continuous in the estimation errors $\theta_1 - \hat{\theta}_1, \theta_2 - \hat{\theta}_2, \ldots, \theta_v - \hat{\theta}_v$, and must tend to its true optimal value as the errors tend to zero.

The optimal regulator designed using Lyapunov functions for the standard plant (1) affected by harmonic disturbances (2) is unrealizable. It depends on the amplitudes and phases of the disturbance. However, if it is possible to find a regulator that yields the same closed-loop process (x_t, u_t) and is still realizable, then the problem would be solved.

Lindquist and Yakubovich [1] first solved a complex optimization problem, which encompasses the real optimization problem, and thus derived a regulator that yields an unrealizable but stabilizing closed-loop process (x_t, u_t). Then they considered a class of regulators which was realizable and derived a set of conditions that its parameters need to satisfy to yield the same closed-loop process (x_t, u_t) as that with the optimal but unrealizable regulator. This class encompasses all realizable optimal regulators. it is given by

$$D(\sigma)u_t = N(\sigma)x_t, \tag{7}$$

where σ is the forward shift operator ($\sigma x_t = x_{t+1}$). $D(\lambda)$ and $N(\lambda)$ are $k \times k$ and $k \times n$ matrices respectively. They are given by

$$D(\lambda) = R(\lambda)B + \rho(\lambda)I_k, \tag{8}$$

$$N(\lambda) = R(\lambda)(\lambda I_n - \Gamma), \tag{9}$$

where

1. $\Gamma = A + BK$. K is the control gain for the simple optimal control problem,

$$K = -(B^* PB + R)^{-1}(A^* PB + S)^*, \tag{10}$$

   where P is the stabilizing solution of the algebraic Riccati equation,

$$P = A^* PA - (A^* PB + S)(B^* PB + R)^{-1}(A^* PB + S)^* + Q. \tag{11}$$

   The symmetric solution, P, if it exists renders the feedback matrix, Γ stable.
2. $\rho(\lambda)$ is an arbitrarily chosen, real, scalar, stable polynomial. The order of $\rho(\lambda)$ is equal to the number of distinct complex frequencies in the disturbance signal. The coefficients of $\rho(\lambda)$ should be chosen such that all roots of the polynomial lie within the unit circle. The freedom in choosing the coefficients gives flexibility in the design of the regulator.
3. $R(\lambda)$ is a real $k \times n$ matrix polynomial such that $\deg R(\lambda) \leqslant \deg \rho(\lambda)$, and it is found by solving the interpolation conditions given by

$$R(e^{i\theta_j})Ce_j = \rho(e^{i\theta_j})\Theta(e^{i\theta_j})PCe_j, \quad j = 1, 2, \ldots, m, \tag{12}$$

   where

$$\Theta(\lambda) = (B^* PB + R)^{-1} B^* (\lambda \Gamma^* - I)^{-1}, \tag{13}$$

   e_j is the jth column of the $k \times k$ identity matrix, I_k. Eq. (12) is obtained by equating the closed-loop solution of the plant with the optimal unrealizable regulator with that of the plant with the regulator of Eqs. (8) and (9). If $c_j = Ce_j \neq 0$, the jth interpolation condition of Eq. (12) can be written as

$$R(e^{i\theta_j}) = \rho(e^{i\theta_j})\Theta(e^{i\theta_j})Pc_j(c_j^* c_j)^{-1} c_j^*. \tag{14}$$

3. Numerical results

The continuous time model of a single-degree-of-freedom spring–mass system subjected to external excitation $w(t)$ is given by

$$\dot{\mathbf{x}} = A_c \mathbf{x} + \mathbf{f}_c u(t) + \mathbf{b}_c \omega(t), \quad \mathbf{x}(0) = \mathbf{x_0}, \tag{15}$$

where

$$A_c = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}, \quad \mathbf{f}_c = \mathbf{b}_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{16}$$

The simulations presented are for the spring stiffness $k = 10\ \mathrm{N\ m^{-1}}$ and the mass $m = 1$ kg. The continuous time model (15) is discretized to get

$$x_{t+1} = Ax_t + \mathbf{f}u_t + \mathbf{b}\omega_t, \tag{17}$$

where $A = e^{A_c T_s}$, and $\mathbf{f} = \int_0^{T_s} e^{A_c \tau} \mathbf{f}_c \; d\tau = \mathbf{b}$.

For simulations, the frequency of the excitation was taken to be equal to the resonance frequency of the system. Therefore, the differential equation governing the excitation is

$$\ddot{\omega} + \beta_1 \omega = 0, \quad \beta_1 = k/m, \quad \beta_2 = 0. \tag{18}$$

In a real implementation, the excitation changes continuously whereas the control input changes at discrete time intervals. Simulations were carried such that they reflect the continuous nature of the excitation, and the discrete nature of the controller.

The weighting matrices chosen were

$$Q = \begin{pmatrix} 1.5 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_0 = 0.0001. \tag{19}$$

Q and R_0 were chosen such that the designed absorber requires the same maximum control effort (u_{max}) as the Johnson algorithm based absorber studied in Ref. [2] at the design frequency so that we can make a fair comparison of the performance of the two absorbers. Both absorbers are designed by assuming that the excitation frequency is equal to the natural frequency of the system.

The non-zero real frequency, $\theta = \sqrt{k/m}$ in the disturbance corresponds to two distinct complex frequencies, $\pm i\theta$, in the complex optimization problem. Hence, we choose $\rho(\lambda)$ to be of order 2. Both roots of $\rho(\lambda)$ should be within the unit circle. The choice of the polynomial does not affect the cost functional but it may affect the transient response. For example, it could affect the peak value of control input required in the transient process. The simulation results presented in the paper are for $\rho(\lambda) = \lambda^2 + 1.4\lambda + 0.85$.

Next, a real 1×2 matrix polynomial of degree one,

$$R(\lambda) = (R_1(\lambda) \quad R_2(\lambda)) = (r_{11}\lambda + r_{12} \;\; r_{21}\lambda + r_{22}), \tag{20}$$

was found by solving the two sets of equations resulting from the interpolation conditions (14):

$$\begin{pmatrix} \cos(\theta) & 1 \\ \sin(\theta) & 0 \end{pmatrix} \begin{pmatrix} r_{k1} \\ r_{k2} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(rhs_k) \\ \mathrm{Im}(rhs_k) \end{pmatrix}, \quad k = 1, 2, \tag{21}$$

where ($rhs_1 \;\; rhs_2$) is the right hand side of equation (14), Re(.) denotes the real part and Im(.) the imaginary part.

Upon solving the above equations, $R_1(\lambda) = 0.0640\lambda - 0.0064$, $R_2(\lambda) = 2.5539\lambda - 0.2563$. The eigenvalues of the closed loop system are $-0.7 \pm i0.6$, $-0.7638 \pm i0.5041$, $0.6315 \pm i0.2745$.

Fig. 1 shows the frequency responses of the Lindquist–Yakubovich (LY) algorithm based absorber and the Johnson algorithm based absorber. The Johnson absorber completely suppresses the oscillations at the design frequency (which was taken to be equal to the natural frequency of the system), whereas the LY absorber has a non-zero steady state amplitude at the design frequency. But unlike the Johnson absorber, the LY absorber is robust with respect to deviations of the excitation frequency from the design value. It yields a smaller steady state amplitude than the Johnson absorber at most non-design excitations frequencies.

The performance of the Johnson absorber is highly sensitive to the design frequency. The frequency response of the Johnson absorber closed-loop system with design frequency equal to $\sqrt{10}\,\omega_{nat}$ is shown in Fig. 2. Again, the oscillations are totally suppressed at the design frequency but the performance at other frequencies deteriorates. The response observed at ω_{nat} may be

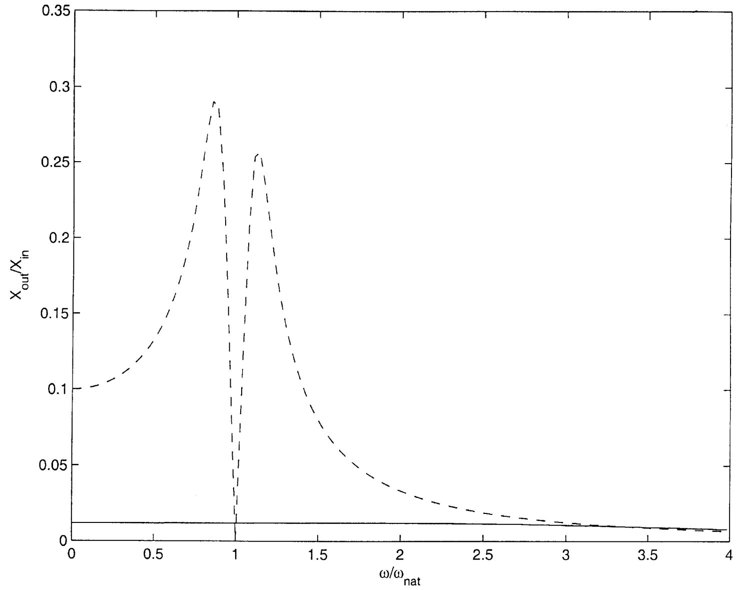

Fig. 1. Frequency response of the closed-loop system: —, LY; - - -, Johnson.

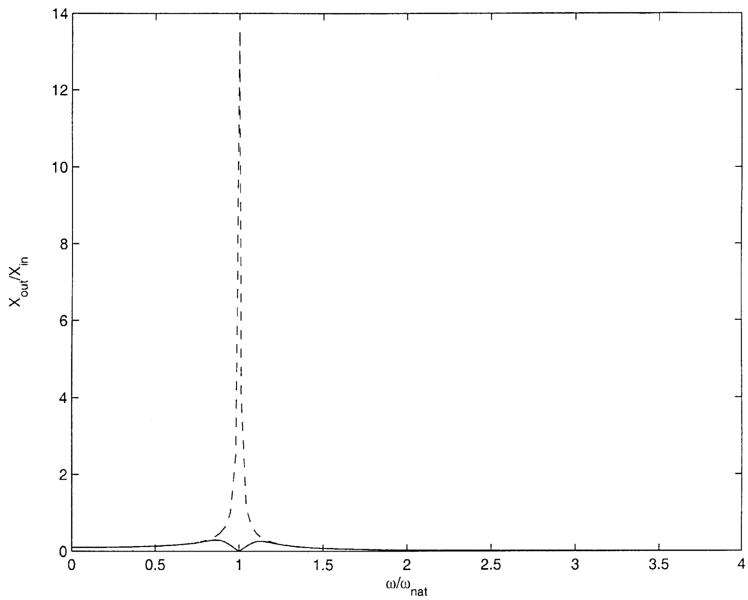

Fig. 2. Frequency response of the closed-loop system with Johnson algorithm based vibration absorber: —, $\omega_{des} = \omega_{nat}$; - - -, $\omega_{des} = 3.16\omega_{nat}$.

unacceptable. We will have this situation for all Johnson absorbers whose design frequency is not equal to the natural frequency of the system. Unlike the Johnson absorber, the LY absorber retains the quality of the closed-loop performance for a wide range of design frequencies. Fig. 3

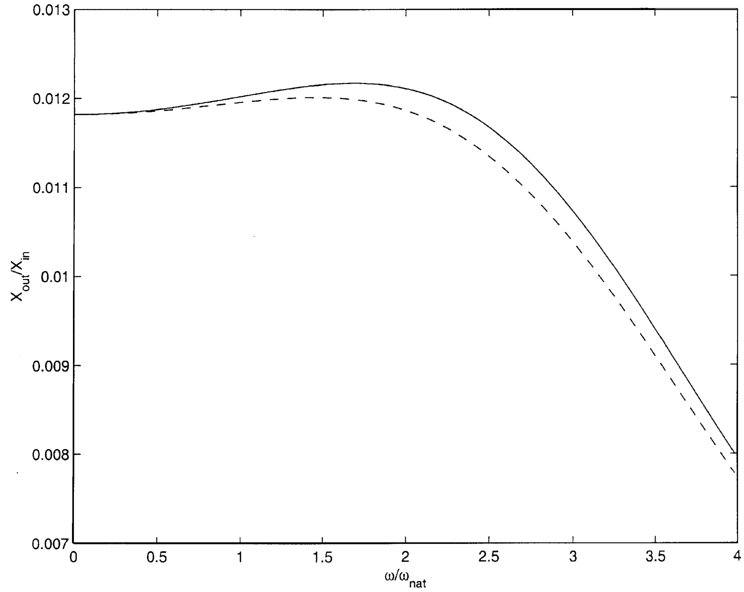

Fig. 3. Frequency response of the closed-loop system with LY algorithm based vibration absorber: —, $\omega_{des} = \omega_{nat}$; - - -, $\omega_{des} = 3.16\omega_{nat}$.

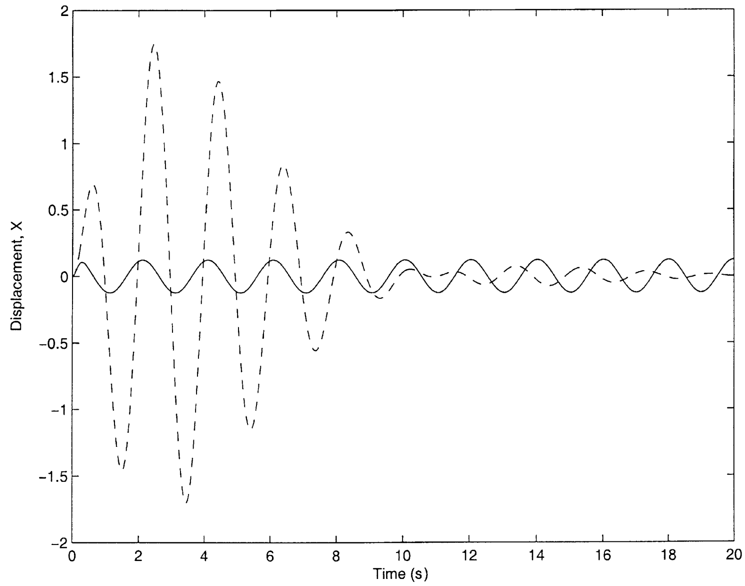

Fig. 4. Transient response of the closed-loop system, $\omega = \omega_{des} = \omega_{nat}$: —, LY; - - -, Johnson.

shows the frequency response of the closed-loop system for $\omega_{dcs} = \sqrt{10}\omega_{nat}$. The LY algorithm yields an optimal regulator which has similar characteristics for frequencies not equal to natural frequency of the system.

The LY absorber has a faster transient response. Fig. 4 shows the responses with $\omega_{des} = \omega_{nat}$. The excitation frequency used is ω_{nat}. Thus, although the steady state amplitude of the closed-loop system with LY absorber is non-zero at the design frequency, its transient response is better than that of the closed-loop system with the Johnson absorber.

4. Conclusions

A linear discrete-time optimal control algorithm developed by Lindquist and Yakubovich [1] has been proposed for vibration control using active absorbers. The algorithm was applied to suppress the oscillations of a spring–mass system subjected to an additive sinusoidal disturbance. Numerical simulations showed that the performance of the resulting closed-loop system is robust with respect to excitation frequencies. The overall performance of the system was found to be better than that of the Johnson algorithm based absorber [2]. Moreover, the design being done in the discrete-time domain, the regulator can be directly implemented using a digital computer.

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