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Letter to the Editor

On the periodic solutions of a generalized non-linear Van der Pol oscillator

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1. Introduction

In this paper the following generalized non-linear Van der Pol oscillator equation will be considered:

$$\ddot{X} + X^{(2m+1)/(2n+1)} = \varepsilon(1 - X^2)\dot{X}, \tag{1}$$

where $X = X(t)$, $m, n \in \mathbb{N}$, and where ε is a small parameter satisfying $0 < \varepsilon \ll 1$. The dot represents differentiation with respect to time t. Many researchers have studied the unperturbed non-linear oscillator equation

$$\ddot{X} + f(X) = 0. \tag{2}$$

For instance, Awrejcewicz and Andrianov [1,2] studied Eq. (2) using the so-called small and large δ-method. Using a generalized harmonic balance method Mickens and his co-authors [3–6] also studied Eq. (2). For a particular case of Eq. (2) with $f(X) = X^{1/(2n+1)}$ some results have been presented in Refs. [2,3,5,7]. The periods of the periodic solutions for this particular case have been approximated by Mickens in Refs. [3,5]. Moreover, exact expressions for the periods of the periodic solutions for this particular Eq. (2) have been given by Van Horssen [7]. Eq. (1) with $m = n = 0$ is the well-known Van der Pol equation. Recently, Eq. (1) with $m = 0$ and $n = 1$ has been studied in Ref. [6]. Approximations of the periodic solution are constructed in Ref. [6] by using the method of harmonic balance. In this paper the recently developed perturbation method based on integrating factors (see Refs. [8–12]) is used to approximate first integrals and periodic solutions for the generalized non-linear Van der Pol oscillator (1). In this paper not only asymptotic approximations of first integrals are constructed but also asymptotic approximations of the periodic solutions and their periods are determined. The results presented include existence, uniqueness, and stability properties of the periodic solutions. In this paper it is shown that straightforward expansions in ε can be used to construct asymptotic results on long time-scales.

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This paper is organized as follows. In Section 2 it is shown how approximations of first integrals can be constructed. It will be shown in Section 3 how the existence, stability, and the period of time-periodic solutions can be determined from the constructed approximations of the first integrals. Finally in Section 4, some conclusions will be drawn and some remarks will be made.

2. Approximations of first integrals

In this section it will be shown that the perturbation method based on integrating factors can be applied to approximate first integrals for a generalized non-linear Van der Pol oscillator. Consider the following generalized non-linear Van der Pol oscillator equation:

$$\ddot{X} + X^{(2m+1)/(2n+1)} = \varepsilon(1 - X^2)\dot{X}. \tag{3}$$

The unperturbed solutions of Eq. (3) with $\varepsilon = 0$ form a family of periodic orbits. This family covers the entire "phase plane" $(X, \dot{X})$. Each periodic orbit corresponds to a constant energy level $E = \frac{1}{2}\dot{X}^2 + [(2n+1)/(2m+2n+2)]X^{(2m+2n+2)/(2n+1)}$. To a constant energy level E a phase angle ψ can be defined by

$$\psi = \int_0^X \frac{\mathrm{d}r}{\sqrt{2E - [(2n+1)/(m+n+1)]r^{(2m+2n+2)/(2n+1)}}}.$$

By using the transformation $(X, \dot{X}) \mapsto (E, \psi)$ it follows that

$$\dot{E} = \varepsilon \dot{X} f = g_1(E, \psi),$$

$$\dot{\psi} = 1 - \varepsilon \int_0^X \frac{\mathrm{d}r}{(2E - [(2n+1)/(m+n+1)]r^{(2m+2n+2)/(2n+1)})^{3/2}} \dot{X} f = g_2(E, \psi), \tag{4}$$

where $f = (1 - X^2)\dot{X}$. By multiplying the first and the second equation in Eq. (4) by the integrating factors μ_1 and μ_2, respectively, it follows from the theory of integrating factors as presented in Refs. [8–10] that μ_1 and μ_2 have to satisfy

$$\frac{\partial \mu_1}{\partial \psi} = \frac{\partial \mu_2}{\partial E},$$

$$\frac{\partial \mu_1}{\partial t} = -\frac{\partial}{\partial E}(\mu_1 g_1 + \mu_2 g_2), \quad \frac{\partial \mu_2}{\partial t} = -\frac{\partial}{\partial \psi}(\mu_1 g_1 + \mu_2 g_2). \tag{5}$$

By expanding μ_1 and μ_2 in power series in ε and by substituting g_1, g_2, and the expansions for the integrating factors into Eq. (5), and by taking together terms of equal powers in ε, the usual $\mathcal{O}(\varepsilon^n)$-problems for $n = 0, 1, 2, \ldots$ (see also Refs. [9–12]) are obtained. The $\mathcal{O}(\varepsilon^0)$-problem is

$$\frac{\partial \mu_{1,0}}{\partial \psi} = \frac{\partial \mu_{2,0}}{\partial E},$$

$$\frac{\partial \mu_{1,0}}{\partial t} = -\frac{\partial \mu_{2,0}}{\partial E}, \quad \frac{\partial \mu_{2,0}}{\partial t} = -\frac{\partial \mu_{2,0}}{\partial \psi}, \tag{6}$$

and for $n \geqslant 1$ the $\mathcal{O}(\varepsilon^n)$-problems are

$$\frac{\partial \mu_{1,n}}{\partial \psi} = \frac{\partial \mu_{2,n}}{\partial E},$$

$$\frac{\partial \mu_{1,n}}{\partial t} = -\frac{\partial}{\partial E}(\mu_{1,n-1}g_{1,1} + \mu_{2,n-1}g_{2,1} + \mu_{2,n}),$$

$$\frac{\partial \mu_{2,n}}{\partial t} = -\frac{\partial}{\partial \psi}(\mu_{1,n-1}g_{1,1} + \mu_{2,n-1}g_{2,1} + \mu_{2,n}), \tag{7}$$

where $\varepsilon g_{1,1} = g_1$, $\varepsilon g_{2,1} = g_2 - 1$. The $\mathcal{O}(\varepsilon^0)$-problem (6) can readily be solved, yielding $\mu_{1,0} = h_{1,0}(E, \psi - t)$ and $\mu_{2,0} = h_{2,0}(E, \psi - t)$ with $\partial h_{1,0}/\partial\psi = \partial h_{2,0}/\partial E$. The functions $h_{1,0}$ and h_{20} are still arbitrary and will now be chosen as simple as possible: $h_{1,0} \equiv 1$ and $h_{2,0} \equiv 0$, and so (see also Refs. [8–12])

$$\mu_{1,0} = 1, \quad \mu_{2,0} = 0. \tag{8}$$

Then, from the order ε-problem (7) $\mu_{1,1}$ and $\mu_{2,1}$ can be obtained, yielding

$$\mu_{1,1} = -\frac{\partial}{\partial E}\left(\int^t (\dot{X}^2 - \dot{X}^2 X^2)\,\mathrm{d}\bar{t}\right),$$

$$\mu_{2,1} = -\frac{\partial}{\partial \psi}\left(\int^t (\dot{X}^2 - \dot{X}^2 X^2)\,\mathrm{d}\bar{t}\right). \tag{9}$$

An approximation F_1 of a first integral $F = constant$ of system (4) can now be obtained from Eqs. (8), (9), and the theory of integrating factors as presented in Refs. [8–12], yielding

$$F_1 = E - \varepsilon \int^t (\dot{X}^2 - \dot{X}^2 X^2)\,\mathrm{d}\bar{t}, \tag{10}$$

where

$$\dot{X} = \pm\sqrt{2E - \frac{2n+1}{m+n+1} X^{(2m+2n+2)/(2n+1)}}. \tag{11}$$

The elementary procedure to construct F_1 using the integrating factors is for instance given in Refs. [8–12]. How well F_1 approximates F in a first integral $F = constant$ follows from the theorems as presented in Refs. [9–12]. In this case it can be shown that (using the theory as presented in Refs. [9–12])

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \varepsilon\mu_{1,1}g_1 + \varepsilon\mu_{2,1}(g_2 - 1) = \varepsilon^2 \mathcal{R}_1(E, \psi), \tag{12}$$

where g_1 and g_2, and $\mu_{1,1}$ and $\mu_{2,1}$ are given by Eqs. (4) and (9), respectively. In a similar way, a second (functionally independent) approximation of a first integral can be constructed by taking

$$\mu_{2,0} = 1, \quad \mu_{1,0} = 0, \tag{13}$$

instead of Eq. (8). The $\mathcal{O}(\varepsilon)$-problem (7) can now again be solved, yielding

$$\mu_{1,1} = \frac{\partial}{\partial E}\left(\int^t\left(\int_0^X \frac{\mathrm{d}r}{(2E-[(2n+1)/(m+n+1)]r^{(2m+2n+2)/(2n+1)})^{3/2}}(\dot{X}^2-\dot{X}^2X^2)\right)\mathrm{d}\bar{t}\right),$$

$$\mu_{2,1} = \frac{\partial}{\partial \psi}\left(\int^t\left(\int_0^X \frac{\mathrm{d}r}{(2E-[(2n+1)/(m+n+1)]r^{(2m+2n+2)/(2n+1)})^{3/2}}(\dot{X}^2-\dot{X}^2X^2)\right)\mathrm{d}\bar{t}\right). \quad (14)$$

An approximation F_2 of a first integral $F = constant$ of system (4) can now be obtained from Eqs. (13), (14), and the theory of integrating factors as presented in Refs. [8–12], yielding

$$F_2(E,\psi,t) = (\psi - t) + \varepsilon\left[\int^t\left(\int_0^X \frac{\mathrm{d}r}{(2E-[(2n+1)/(m+n+1)]r^{(2m+2n+2)/(2n+1)})^{3/2}}(\dot{X}^2-\dot{X}^2X^2)\right)\mathrm{d}\bar{t}\right]. \quad (15)$$

How well F_2 approximates a first integral $F = constant$ follows from the theorems as presented in Refs. [9–12]. In this case it can be shown that (using the theory as presented in Refs. [9–12])

$$\frac{\mathrm{d}F_2}{\mathrm{d}t} = \varepsilon\mu_{1,1}g_1 + \varepsilon\mu_{2,1}(g_2-1) = \varepsilon^2\mathcal{R}_2(E,\psi), \quad (16)$$

where g_1 and g_2, and $\mu_{1,1}$ and $\mu_{2,1}$ are given by Eqs. (4) and (14), respectively.

3. Approximations of time-periodic solutions

In Section 2 asymptotic approximations of first integrals have been constructed. In this section, it will be shown how the existence, the stability, and the approximations of non-trivial, time-periodic solutions can be determined from these asymptotic approximations of the first integrals. Let $T < \infty$ be the period of a periodic solution and let c_1 be a constant in the first integral $F(E,\psi,t;\varepsilon) = constant$ for which a periodic solution exists. Consider $F = c_1$ for $t = 0$ and T. Approximating F by F_1 (given by (10)), eliminating c_1 by subtraction (using the fact that $E(0) = E(T)$ for a periodic solution) it follows that

$$\varepsilon\left(\int_0^T(\dot{X}^2-\dot{X}^2X^2)\,\mathrm{d}\bar{t}\right) = \mathcal{O}(\varepsilon^2) \Leftrightarrow \varepsilon\left(\int_{X(0)}^{X(T)}(\dot{X}-\dot{X}X^2)\,\mathrm{d}X\right) = \mathcal{O}(\varepsilon^2). \quad (17)$$

Without loss of generality, it can be assumed that at $t = 0$ $(X(0),\dot{X}(0)) = (A,0)$ with $A > 0$. Because of the symmetry of the unperturbed orbits in the phase plane it follows that $(X(T/2),\dot{X}(T/2)) = (-A,0)$. From Eq. (17) it then follows that

$$\varepsilon I(E) = \mathcal{O}(\varepsilon^2), \quad \text{where } I(E) = 4\int_0^A(\dot{X}-\dot{X}X^2)\,\mathrm{d}X. \quad (18)$$

To have a periodic solution for (3) an energy level E has to be found such that $I(E)$ is equal to zero (see also [11,13,14]). It should be observed that the same problem (that is, finding zeros of $I(E)$) is obtained when the Poincaré return map technique or the Melnikov method is applied (see also

Refs. [13–16]). To find this energy level E the integral $I(E)$ is rewritten in (using Eq. (11))

$$I(E) = 4I_1(E)\left(1 - \frac{I_2(E)}{I_1(E)}\right), \tag{19}$$

where

$$\begin{aligned} I_1(E) &= \int_0^A \left(2E - \frac{(2n+1)}{(m+n+1)} X^{(2m+2n+2)/(2n+1)}\right)^{1/2} \mathrm{d}X, \\ I_2(E) &= \int_0^A X^2 \left(2E - \frac{(2n+1)}{(m+n+1)} X^{(2m+2n+2)/(2n+1)}\right)^{1/2} \mathrm{d}X. \end{aligned} \tag{20}$$

Now it should be observed that $E(t) = \frac{1}{2}\dot{X}(t)^2 + [(2n+1)/(2m+2n+2)]X(t)^{(2m+2n+2)/(2n+1)}$, and $E(0) = [(2n+1)/(2m+2n+2)]A^{(2m+2n+2)/(2n+1)}$. From Eq. (4) it is not difficult to see that E is constant up to $\mathcal{O}(\varepsilon)$ on time-scales of $\mathcal{O}(1)$. By using the transformation $X = Au$ in Eq. (20) and by using the fact that $E = E(0) + \mathcal{O}(\varepsilon)$ for $0 \leqslant t \leqslant T$ it is easy to see from Eqs. (18)–(20) that Eq. (18) can be rewritten in

$$4\varepsilon I_1(E)(1 - Q) = \mathcal{O}(\varepsilon^p) \quad \text{with } p > 1, \tag{21}$$

where

$$Q = \left(2E\frac{m+n+1}{2n+1}\right)^{(2n+1)/(m+n+1)} \frac{J_2(m,n)}{J_1(m,n)}, \tag{22}$$

and where

$$\begin{aligned} J_1(m,n) &= \int_0^1 \sqrt{1 - u^{(2m+2n+2)/(2n+1)}}\, \mathrm{d}u, \\ J_2(m,n) &= \int_0^1 \sqrt{u^4 - u^{(2m+10n+6)/(2n+1)}}\, \mathrm{d}u. \end{aligned} \tag{23}$$

It is easy to see that $J_1(m,n) > 0$ and $J_2(m,n) > 0$ for all values of $m, n \in \mathbb{N}$. It is also easy to see from Eq. (22) that $\mathrm{d}Q/\mathrm{d}E > 0$. This implies that Q is strictly monotonically increasing. Since Q is strictly monotonically increasing in E it can be concluded that there exists a unique, non-trivial E-value such that $I(E) = 0$. From these results it can be concluded (see also for instance Ref. [11], Section 4.2) that there exists a unique, non-trivial, stable time-periodic solution for Eq. (3). Suppose that at $t = 0$ $X(0) = A_0$ and $\dot{X}(0) = 0$ for the periodic solution. Then

$$\tfrac{1}{2}\dot{X}^2 + \frac{2n+1}{2m+2n+2} X^{(2m+2n+2)/(2n+1)} = \frac{(2n+1)}{(2m+2n+2)} A_0^{(2m+2n+2)/(2n+1)} \equiv E_0, \tag{24}$$

where E_0 is the energy such that a periodic solution exists. Obviously, E_0 satisfies (see also Eqs. (19) and (21))

$$\left(2E_0\frac{m+n+1}{2n+1}\right)^{(2n+1)/(m+n+1)} \frac{J_2(m,n)}{J_1(m,n)} = 1, \tag{25}$$

up to $\mathcal{O}(\varepsilon^{p-1})$ with $p>1$. The period of the periodic solution can be calculated up to $\mathcal{O}(\varepsilon^{p})$ with $p>1$ from Eq. (24), yielding

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \pm\sqrt{\frac{2n+1}{m+n+1}}\sqrt{A_0^{(2m+2n+2)/(2n+1)} - X^{(2m+2n+2)/(2n+1)}}, \tag{26}$$

or equivalently

$$\sqrt{\frac{m+n+1}{2n+1}}\frac{\mathrm{d}X}{\mathrm{d}t}\frac{1}{\sqrt{A_0^{(2m+2n+2)/(2n+1)} - X^{(2m+2n+2)/(2n+1)}}} = \pm 1. \tag{27}$$

Then, integrating Eq. (27) with respect to t from $t=0$ to $T/2$ yields

$$T_{m,n} = 4\sqrt{\frac{m+n+1}{2n+1}}A_0^{(n-m)/(2n+1)}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^{(2m+2n+2)/(2n+1)}}}. \tag{28}$$

Using a standard numerical integration routine period (28) of the periodic solution can easily be approximated numerically (up to $\mathcal{O}(\varepsilon^{p-1})$ with $p>1$).

4. Conclusions and remarks

In this paper the perturbation method based on integrating factors has been used to approximate first integrals for a generalized non-linear Van der Pol oscillator equation. From these approximations, the existence, uniqueness, stability, and the periods of the time-periodic solutions have been obtained straightforwardly. Compared to most other perturbation method (see for instance the harmonic balance method in Refs. [3–6] or the small/large δ method in Refs. [1,2]) the presented perturbation method gives explicit approximations for the periods including error estimates. Moreover, the presented perturbation method can be applied to a large class of problems as has been shown in Refs. [7–12].

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