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# Importance sampling for randomly excited dynamical systems

M. Macke*, C. Bucher

*Institute of Structural Mechanics, Bauhaus-University Weimar, Marienstr. 15, D-99421 Weimar, Germany*

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**Abstract**

An importance sampling technique for linear and non-linear dynamical systems subjected to random excitations is presented. Applying a transformation of probability measures, controls are introduced in the system of Itô stochastic differential equations such that the sample trajectories can be influenced in a predetermined way. As is shown, there exist controls resulting in unbiased zero-variance estimators. However, these optimal controls are in general not accessible and have to be replaced by sub-optimal ones derived from an optimization procedure analogous to the first order reliability method known from time-invariant problems. The efficiency of the proposed Monte Carlo simulation technique is demonstrated by estimating first-passage probabilities of typical oscillators under external white-noise excitation.


## 1. Introduction

Determining the response of non-linear dynamical systems subjected to random excitations has been a major research topic during the past decades. Nevertheless, the number of mathematically exact solutions available is still restricted to a narrow class of systems and to the knowledge of stationary probability densities or moments only [1–3]. However, in most prominent fields of reliability assessment of dynamical systems, like e.g., earthquake or bridge engineering, the excitation is either of intrinsically non-stationary transient nature or at least of "short duration" such that the structural response (or a significant part of it) is non-stationary as well. Moreover, for assessing the reliability of a dynamical system sufficiently realistically, in addition to the probabilistic characterization of the response at a certain point in time, a reliability measure evolving over a time period is most instrumental. The latter can be achieved by utilizing the first-passage probability, i.e., the probability that the structural response exceeds a prescribed—usually failure-related—boundary in a certain time interval for the first time. Unfortunately, up to now

*Corresponding author. Tel.: +49-3643-58-4509; fax: +49-3643-58-4514.
*E-mail address:* michael.macke@bauing.uni-weimar.de (M. Macke).

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not even in the case of simplest oscillators have exact solutions been obtained for this kind of problem.

In order to enable assessment of the reliability of randomly excited dynamical systems, different approximate solution techniques have been proposed, as e.g., stochastic averaging, cumulant-neglect closure or equivalent linearization, which are applicable for both stationary and non-stationary problems [3–6]. Despite their indisputable merits, especially in problems involving the determination of moments of lower order, they lack sufficient accuracy in quantifying structural responses occurring with low probability—as in the case of extreme load events—which are of paramount importance in the reliability assessment of engineering structures. The only solution technique not showing any of the above-mentioned deficits is presumably Monte Carlo simulation. However, it is most often applied as a last resort only—due to numerical answers of supposedly limited accuracy or prohibitive computational costs.

Whereas these drawbacks of the Monte Carlo simulation technique are an inherent characteristic of its crude or direct form, this does not hold for variance reduction techniques [7,8]. Although this fact is widely recognized for time-invariant problems (cf. Ref. [9]), for problems in random vibration the application of variance reduction techniques is still eschewed—despite different, technically sound methods proposed during the last years [10–14]. The most versatile variance reduction technique appears to be importance sampling utilizing the measure transformation method [15–18] based on the Girsanov theorem [19], which is known from stochastic control [20] or system identification problems [21]. Modelling the time evolution of the structural response by a system of Itô stochastic differential equations, the drift terms of these equations are changed according to a minimization criterion for the variance of the estimator. In other words, controls are utilized which allow one to influence the response paths, in addition to which the unlikeliness of such extreme paths is taken into account by a correction process. Although there exist controls which result in unbiased zero-variance estimators, these optimal controls are in general not accessible and have to be replaced by sub-optimal ones—as will be shown below.

A different and at first sight much simpler approach are so-called (multi-level) splitting methods which can be traced back at least to Ref. [22]. These methods have been brought to a wider attention by Refs. [23,24]. Typical applications to civil engineering systems can be found e.g., in Refs. [25,26]. In multi-level splitting promising sample paths are split into sub-paths at intermediate levels to increase the number of observations of rare events. As has been reported as early as in Ref. [24], however, this procedure can possibly result in an increase of the variance of the estimator. More precarious event is an ''apparent bias'' [27] of the estimator, when the levels for splitting are not chosen consistently. Indeed, multi-level splitting requires for being effective that the splitting is performed along the most likely path leading to an out-crossing [27]. However, in this case a more effective sampling method like importance sampling can be used right from start.

## 2. Dynamical response and Monte Carlo estimators

Assume that the dynamical response of a system at time $t$ ($s \leqslant t \leqslant T$) is described by a $p$-dimensional Itô process $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_p(t))$ in terms of the stochastic differential

equations

$$\mathrm{d}X_i(t) = \alpha_i(t, \mathbf{X})\mathrm{d}t + \sum_{j=1}^{q} \gamma_{ij}(t, \mathbf{X})\,\mathrm{d}W_j(t) \tag{1}$$

with $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_q(t))$ as a $q$-dimensional unit Wiener process, subject to the deterministic initial conditions $\mathbf{x} = (x_1, x_2, \ldots, x_p)$, i.e.,

$$X_i(s) = x_i. \tag{2}$$

Modelling the system response by Eq. (1) does not represent a serious restriction, since a broad class of practical load cases can be covered by augmenting Eq. (1) with filter equations of the same type [28].

In reliability assessment of dynamical systems one is generally not interested in an exact pathwise representation of $\mathbf{X}(t)$, but rather in expectations of functionals of the Itô process—the so-called weak solutions—of the form

$$v(s, \mathbf{x}) = \mathrm{E}[f(\mathbf{X}^{s,\mathbf{x}}(T))], \tag{3}$$

in which $f(\cdot)$ denotes some real-valued function, $\mathbf{X}^{s,\mathbf{x}}(T)$ is the system response at time $T$ subject to the initial conditions $\mathbf{x}$ at time $s$, and $\mathrm{E}[\cdot]$ is the expected value. Typical examples of $v(s, \mathbf{x})$ are the statistical moments of $\mathbf{X}^{s,\mathbf{x}}(T)$, i.e.,

$$v(s, \mathbf{x}) = \mathrm{E}[X_1^{m_1}(T)X_2^{m_2}(T)\cdots X_p^{m_p}(T)] \tag{4}$$

or the probability that the $j$th component of $\mathbf{X}^{s,\mathbf{x}}(t)$ exceeds a given barrier $x_c$ in the time interval $(s \leqslant t \leqslant T)$, i.e.,

$$v(s, \mathbf{x}) = \mathrm{E}\left[\mathbf{I}\left(\max_{s \leqslant t \leqslant T} X_j(t) > x_c\right)\right] \tag{5}$$

with $\mathbf{I}(\cdot)$ denoting an indicator function that equals one if its argument is true, and zero otherwise.

When utilizing Monte Carlo simulation to evaluate Eq. (3), $v(s, \mathbf{x})$ is replaced by its sample-mean formula [8]

$$\hat{v}(s, \mathbf{x}) = \hat{\mathrm{E}}[f(\mathbf{X}^{s,\mathbf{x}}(T))] = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{X}_{(i)}^{s,\mathbf{x}}(T)) \tag{6}$$

with $\mathbf{X}_{(i)}^{s,\mathbf{x}}(t)$ denoting the $i$th sample trajectory of $\mathbf{X}^{s,\mathbf{x}}(t)$ and $N$ representing the number of sample trajectories. The quantity $\hat{v}(s, \mathbf{x})$ is an unbiased estimator of $v(s, \mathbf{x})$ with variance

$$\mathrm{Var}[\hat{v}(s, \mathbf{x})] = \frac{1}{N}\mathrm{Var}[f(\mathbf{X}^{s,\mathbf{x}}(T))]. \tag{7}$$

This variance, or the more commonly used statistical error, which—if it exists—is defined as

$$e = \frac{(\mathrm{Var}[f(\mathbf{X}^{s,\mathbf{x}}(T))]/N)^{1/2}}{\mathrm{E}[f(\mathbf{X}^{s,\mathbf{x}}(T))]} \tag{8}$$

can be conceived as an indicator of the accuracy of the calculated Monte Carlo estimator. The main concern in Monte Carlo simulation, therefore, is to obtain an estimator with sufficiently small statistical error. Whereas this can always be achieved by simply increasing the number $N$ of sample trajectories, it is rarely a rewarding procedure, since—as can be seen from Eq. (8)—the

statistical error is inversely proportional to the square root of the sample size $N$. In other words, to reduce the statistical error by a factor $a$, the sample size has to be increased $a^2$ times. This becomes especially crucial when calculating events of low probability, e.g., to obtain a statistical error of 50% in calculating an estimator of $v(s,\mathbf{x})$ equal to $10^{-6}$ a sample size of $N = 4 \times 10^6$ is needed. Reducing the statistical error to be 10%, however, already requires $N = 10^8$ sample trajectories.

For being still able to apply Monte Carlo simulation in estimating $v(s,\mathbf{x})$ in such cases a variance reduction technique like importance sampling has to be utilized. Importance sampling can be described roughly as a method to control the sample paths in such a way that the samples are concentrated in parts of the sample space which are "most important", instead of spreading them out evenly [8]. However, choosing an arbitrary change of probability measure that makes the event of interest only happen more frequently is not sufficient at all. Indeed, an arbitrary change of measure may even result in an estimator with infinite variance. Because of this possible adverse effect, the design of sampling measures becomes crucial [29].

## 3. Transformation of probability measures

As has been mentioned above, the evaluation of $v(s,\mathbf{x})$ requires weak solutions of Eq. (1) only. Therefore, the measure transformation method, based on the Girsanov theorem [19,30], can be applied. Instead of Eq. (1), the stochastic differential equations ($i = 1, 2, \ldots, p$)

$$\begin{aligned} \mathrm{d}\tilde{X}_i(t) &= \alpha_i(t,\tilde{\mathbf{X}})\,\mathrm{d}t + \sum_{j=1}^{q} \gamma_{ij}(t,\tilde{\mathbf{X}})u_j(t,\tilde{\mathbf{X}})\,\mathrm{d}t + \sum_{j=1}^{q} \gamma_{ij}(t,\tilde{\mathbf{X}})\,\mathrm{d}W_j(t),\\ \mathrm{d}Y(t) &= -\sum_{j=1}^{q} u_j(t,\tilde{\mathbf{X}})\,Y\,\mathrm{d}W_j(t) \end{aligned} \tag{9}$$

are evaluated, augmented by the one-dimensional correction process $Y(t)$ with initial conditions

$$\tilde{\mathbf{X}}(s) = \mathbf{x} \tag{10}$$

and

$$Y(s) = y, \tag{11}$$

respectively. The components of the vector $\mathbf{u}(t,\tilde{\mathbf{X}}) = (u_1(t,\tilde{\mathbf{X}}), \ldots, u_q(t,\tilde{\mathbf{X}}))$ in Eq. (9) can be interpreted as controls which allow to influence the structural response. Under certain restrictions for $\mathbf{u}(t,\tilde{\mathbf{X}})$, which are usually met in practical applications (cf. Ref. [30]), the functional $v(s,\mathbf{x})$ of Eq. (3) is evaluated according to the Girsanov theorem as

$$v(s,\mathbf{x}) = \mathrm{E}[f(\mathbf{X}^{s,\mathbf{x}}(T))] = \mathrm{E}[Y^{s,y}(T)f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T))]/y. \tag{12}$$

In other words, the controls $\mathbf{u}(t,\tilde{\mathbf{X}})$ will change the Itô process $\mathbf{X}(t)$ to $\tilde{\mathbf{X}}(t)$, whereas this "change" is taken into account by the Radon–Nikodym derivative

$$\frac{Y^{s,y}(T)}{y} = \exp\left[-\sum_{j=1}^{q}\int_s^T u_j(t,\tilde{\mathbf{X}})\,\mathrm{d}W_j(t) - \frac{1}{2}\sum_{j=1}^{q}\int_s^T (u_j(t,\tilde{\mathbf{X}}))^2\,\mathrm{d}t\right]. \tag{13}$$

The functional $v(s,\mathbf{x})$ of Eq. (12) can be replaced again by its unbiased Monte Carlo estimator

$$\hat{v}(s,\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N} Y_{(i)}^{s,1}(T) f(\tilde{\mathbf{X}}_{(i)}^{s,\mathbf{x}}(T)), \tag{14}$$

whereby the initial condition for the correction process $Y(t)$ is chosen—just for convenience—as $y = 1$. Whereas the mean of the estimator in Eq. (14) is not influenced by the choice of the controls—this is precisely what the Girsanov theorem states—the variance of the estimator is affected. Hence, by an appropriate choice of $\mathbf{u}(t,\tilde{\mathbf{X}})$, these controls can be utilized for efficiently reducing the variance of the estimator $\hat{v}(s,\mathbf{x})$.

## 4. Optimal controls

In the above terms, optimal controls $\mathbf{u}^*(t,\tilde{\mathbf{X}}) = (u_1^*(t,\tilde{\mathbf{X}}),\ldots,u_q^*(t,\tilde{\mathbf{X}}))$ are defined as those controls for which $\mathrm{E}[(Y^{s,1}(T)f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)))^2]$ becomes minimal. This is a well-known problem from stochastic control theory [20,30]. Defining

$$r(s,\mathbf{x}) = \min_{u_1,\ldots,u_q} \mathrm{E}[(Y^{s,1}(T)f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)))^2] \tag{15}$$

and assuming that such (optimal) controls indeed exist, the Hamilton–Jacobi–Bellman equation can be invoked [30]

$$\min_{u_1^*,\ldots,u_q^*}\left(\mathrm{L}r(s,\mathbf{x}) - \sum_{i=1}^{p}\sum_{j=1}^{q}\gamma_{ij}(s,\mathbf{x})u_j^*(s,\mathbf{x})\frac{\partial r}{\partial x_i}(s,\mathbf{x}) + \sum_{j=1}^{q}(u_j^*(s,\mathbf{x}))^2 r(s,\mathbf{x})\right) = 0 \tag{16}$$

in which the operator L is defined as

$$\begin{aligned}\mathrm{L}r(s,\mathbf{x}) = {} & \frac{\partial r}{\partial s}(s,\mathbf{x}) + \sum_{i=1}^{p}\alpha_i(s,\mathbf{x})\frac{\partial r}{\partial x_i}(s,\mathbf{x}) \\ & + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{q}\gamma_{ik}(s,\mathbf{x})\gamma_{jk}(s,\mathbf{x})\frac{\partial^2 r}{\partial x_i \partial x_j}(s,\mathbf{x}).\end{aligned} \tag{17}$$

Unfortunately, Eq. (16) is even more difficult to solve for $\mathbf{u}^*(t,\tilde{\mathbf{X}})$ than evaluating $v(s,\mathbf{x})$ directly from the original system of Itô stochastic differential equations (1). However, it is possible to find a solution for the optimal controls $\mathbf{u}^*(t,\tilde{\mathbf{X}})$ by a simple reasoning proposed in Refs. [16].

Assume for the time being that the optimal controls $\mathbf{u}^*(t,\tilde{\mathbf{X}})$ are known for which the variance $\mathrm{Var}[Y^{s,1}(T)f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T))]$ becomes zero. This implies that the quantity $Y^{s,1}(T)f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T))$ is deterministic, i.e.,

$$v(s,\mathbf{x}) = Y^{s,1}(T)f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) \tag{18}$$

as well as

$$r(s,\mathbf{x}) = v^2(s,\mathbf{x}). \tag{19}$$

Inserting Eq. (19) in Eq. (16) and solving for $\mathbf{u}^*(t,\tilde{\mathbf{X}})$, thereby taking into account that

$$\mathrm{L}v(s,\mathbf{x}) = 0, \tag{20}$$

the optimal controls are derived as [13,16]

$$u_j^*(t,\tilde{\mathbf{X}}) = \sum_{i=1}^{p} \frac{\gamma_{ij}(t,\tilde{\mathbf{X}})}{v(t,\tilde{\mathbf{X}})} \frac{\partial v}{\partial \tilde{X}_i}(t,\tilde{\mathbf{X}}). \tag{21}$$

When utilizing these controls $u_j^*(t,\tilde{\mathbf{X}})$ in Eq. (9), $\hat{v}(s,\mathbf{x})$ of Eq. (14) will be an unbiased zero-variance estimator of $v(s,\mathbf{x})$, as can be shown by Itô's formula.

There exists only a mild restriction with respect to the function $f(\cdot)$ of Eq. (3) in applying Eq. (21), i.e., there has to exist an $\varepsilon > 0$ such that either the probability $P(f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) > \varepsilon) = 1$ or $P(f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) \leqslant -\varepsilon) = 1$ [13]. In other words, it has to be avoided that $f(\cdot)$ has both negative and positive outcomes. This restriction, however, can be quite easily removed. In case that $f(\cdot)$ is bounded from below or above, i.e., $f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) > -k$ or $f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) < k$, where $k$ denotes an arbitrary real constant, $f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T))$ can be replaced by $f'(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) = f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) + k$ or $f'(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) = -f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) + k$, respectively, e.g., in the first case the control is given by

$$u_j^*(t,\tilde{\mathbf{X}}) = \sum_{i=1}^{p} \frac{\gamma_{ij}(t,\tilde{\mathbf{X}})}{v(t,\tilde{\mathbf{X}}) + k} \frac{\partial v}{\partial \tilde{X}_i}(t,\tilde{\mathbf{X}}) \tag{22}$$

and $v(s,\mathbf{x})$ is evaluated as

$$v(s,\mathbf{x}) = \mathrm{E}[Y^{s,1}(T)(f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) + k)] - k. \tag{23}$$

If $f(\cdot)$ is unbounded, $f(\cdot)$ can be split in two real-valued functions $f_\chi(\cdot) > 0$ and $f_\theta(\cdot) > 0$, such that $f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) = f_\chi(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T)) - f_\theta(\tilde{\mathbf{X}}^{s,\mathbf{x}}(T))$, which have to be evaluated separately. In this case $v(s,\mathbf{x})$ is determined by

$$v(s,\mathbf{x}) = \mathrm{E}[Y_\chi^{s,1}(T) f_\chi(\tilde{\mathbf{X}}_\chi^{s,\mathbf{x}}(T))] - \mathrm{E}[Y_\theta^{s,1}(T) f_\theta(\tilde{\mathbf{X}}_\theta^{s,\mathbf{x}}(T))] \tag{24}$$

It should be noted that in reliability analysis of dynamical systems one is mostly interested in evaluating probabilities which by definition are non-negative.

## 5. Sub-optimal controls

Utilizing the optimal controls $\mathbf{u}^*(t,\tilde{\mathbf{X}})$ a zero-variance estimator of $v(s,\mathbf{x})$ has been obtained. Closer inspection of Eq. (21), however, reveals that in order to construct $\mathbf{u}^*(t,\tilde{\mathbf{X}})$ the sought solution $v(s,\mathbf{x})$ has to be known in advance. Indeed, we not only have to know $v(s,\mathbf{x})$ for a single set of initial conditions $\mathbf{x}$ at time $t = s$, but we have to know $v(t,\tilde{\mathbf{X}}(t))$ and its derivatives $\partial v(t,\tilde{\mathbf{X}}(t))/\partial \tilde{X}_i$ for all values of $t$ and $\tilde{\mathbf{X}}(t)$. This is a drawback typical for variance reduction techniques, which can be essentially characterized as methods to utilize known information about a problem at hand. In other words, knowing the solution allows one to construct a zero-variance estimator, but renders Monte Carlo simulation superfluous, whereas not knowing anything means also that no variance reduction can be achieved [8].

Nevertheless, Eq. (21) can still be utilized advantageously for variance reduction as long as an approximation $\bar{v}(s,\mathbf{x})$ for $v(s,\mathbf{x})$ is readily available. Then, instead of the optimal controls $\mathbf{u}^*(t,\tilde{\mathbf{X}})$, sub-optimal controls $\bar{\mathbf{u}}(t,\tilde{\mathbf{X}})$ can be constructed. The resulting estimator of $v(s,\mathbf{x})$ will furthermore

be unbiased, but now with non-zero variance [18]

$$\mathrm{Var}[\hat{v}(s,\mathbf{x})] = \frac{1}{N}\mathrm{E}\left[\left(\int_s^T \sum_{j=1}^{q} v(t,\tilde{\mathbf{X}})(u_j^*(t,\tilde{\mathbf{X}}) - \bar{u}_j(t,\tilde{\mathbf{X}}))\,\mathrm{d}W_j(t)\right)^2\right]. \tag{25}$$

As can be seen from Eq. (25), the difference between the two controls $\bar{\mathbf{u}}(t,\tilde{\mathbf{X}})$ and $\mathbf{u}^*(t,\tilde{\mathbf{X}})$ is integrated over time. From this follows that the variance can increase with time as long as the solutions $\bar{v}(s,\mathbf{x})$ and $v(s,\mathbf{x})$, respectively, depend on the initial conditions. Moreover, choosing the controls $\bar{\mathbf{u}}(t,\tilde{\mathbf{X}})$ without taking into account specific characteristics of the dynamical system under investigation can even result in an increase of the variance as compared to crude Monte Carlo simulation.

For constructing the sub-optimal controls $\bar{\mathbf{u}}(t,\tilde{\mathbf{X}})$, in Ref. [18] it has been suggested to replace the generally non-linear dynamical systems by equivalent linearized ones. Employing linear random vibration theory, approximate solutions $\bar{v}(s,\mathbf{x})$ of $v(s,\mathbf{x})$ can be utilized in Eq. (21) to construct a possible set of sub-optimal controls. Whereas this strategy gives reasonable results for mildly non-linear dynamical systems, for highly non-linear ones the results have been unsatisfactory, especially in estimating first-passage probabilities [29]. A procedure which prevents such unsatisfactory performance is the construction of sub-optimal controls by solving an optimization problem analogous to the first order reliability method, as has been proposed in [17,31]. These sub-optimal controls will provide the most likely paths leading to an out-crossing of the failure boundary and, therewith, result in an efficient variance reduction of the probability estimates.

Now assume that the boundary of the area that will be out-crossed by the Itô process $\mathbf{X}^{s,\mathbf{x}}(t)$ of Eq. (1) at time $t = \tau$ is described by

$$g(\mathbf{X}^{s,\mathbf{x}}(\tau)) = 0 \tag{26}$$

Defining, furthermore,

$$\mathrm{d}Z_i(t) = \alpha_i(t,\mathbf{Z})\,\mathrm{d}t + \sum_{j=1}^{q} \gamma_{ij}(t,\mathbf{Z})\bar{u}_j^{\tau}(t,\mathbf{Z})\,\mathrm{d}t \tag{27}$$

with initial conditions

$$Z_i(s) = x_i, \tag{28}$$

then the most likely path leading to an out-crossing at time $t = \tau$ is induced by the controls $\bar{u}_j^{\tau}(t)$ which minimize the function

$$\beta(\tau) = \left[\sum_{j=1}^{q} \int_s^{\tau} (\bar{u}_j^{\tau}(t))^2\,\mathrm{d}t\right]^{1/2} \tag{29}$$

subject to

$$g(\mathbf{Z}^{s,\mathbf{x}}(\tau)) = 0. \tag{30}$$

In Ref. [17] the function $\beta(\tau)$ has been coined "minimal distance function". The controls of Eqs. (29) and (30) can now be utilized in Eq. (9), i.e., the Wiener processes $W_j(t)$ are changed to

$\tilde{W}_j(t)$ by

$$\mathrm{d}\tilde{W}_j(t) = \bar{u}_j^{\tau}(t)\,\mathrm{d}t + \mathrm{d}W_j(t) \tag{31}$$

with the Radon–Nikodym derivative

$$R_\tau = \exp\left[-\sum_{j=1}^{q}\int_s^{\tau}\bar{u}_j^{\tau}(t)\,\mathrm{d}W_j(t) - \frac{1}{2}\sum_{j=1}^{q}\int_s^{\tau}(\bar{u}_j^{\tau}(t))^2\,\mathrm{d}t\right]. \tag{32}$$

For estimating the first-passage probability for a certain time interval $s \leqslant t \leqslant T$, it has been suggested in Ref. [17] to employ the control with the smallest value of the minimal distance function only, i.e.,

$$\beta(\tau^*) = \min_{s\leqslant\tau\leqslant T}\beta(\tau). \tag{33}$$

This method has been successfully applied to crack growth problems [17,32], which are dominated by a single exit time. For oscillatory systems, however, a considerable underestimation of the exact results has been reported in Ref. [33]. This is due to the fact that for a time interval $s \leqslant \tau \leqslant T$ there will be a considerable interaction between the different controls $\bar{u}_j^{\tau}(t)$ as well as the corresponding barriers $g(\mathbf{X}^{s,\mathbf{x}}(\tau)) = 0$ for different times $\tau$ of out-crossing (cf. Ref. [34]). In other words, in contrast to fatigue crack growth problems, oscillatory systems show a certain periodicity of the contributions to the magnitude of the first-passage probability which should be taken properly into account by the importance sampling scheme [29,31].

To get a better understanding of the above-mentioned interaction of the different controls, a more detailed look at the change of the excitation processes from $W_j(t)$ to $\tilde{W}_j(t)$ is taken by utilizing the (arbitrary) controls $\bar{u}_j(t)$, i.e.,

$$\mathrm{d}\tilde{W}_j(t) = \bar{u}_j(t)\,\mathrm{d}t + \mathrm{d}W_j(t). \tag{34}$$

The likelihood of these excitation processes $\tilde{W}_j(t)$ with respect to $W_j(t)$ is given by the Radon–Nikodym derivative

$$R = \exp\left[-\sum_{j=1}^{q}\int_s^{T}\bar{u}_j(t)\,\mathrm{d}W_j(t) - \frac{1}{2}\sum_{j=1}^{q}\int_s^{T}(\bar{u}_j(t))^2\,\mathrm{d}t\right]. \tag{35}$$

Nevertheless, the same sample paths of $\tilde{W}_j(t)$ can also be obtained by utilizing different controls $\bar{u}_j^{\tau}(t)$, i.e.,

$$\mathrm{d}\tilde{W}_j(t) = \bar{u}_j^{\tau}(t)\,\mathrm{d}t + \mathrm{d}\tilde{\tilde{W}}_j(t). \tag{36}$$

However, this requires excitation processes $\tilde{\tilde{W}}_j(t)$ as defined by

$$\mathrm{d}\tilde{\tilde{W}}_j(t) = (\bar{u}_j(t) - \bar{u}_j^{\tau}(t))\,\mathrm{d}t + \mathrm{d}W_j(t) \tag{37}$$

to start with. The likelihood of those excitation processes $\tilde{\tilde{W}}_j(t)$ is given by the ratio

$$\tilde{\tilde{R}}_\tau = \exp\left[-\sum_{j=1}^{q}\int_s^{T}(\bar{u}_j(t) - \bar{u}_j^{\tau}(t))\,\mathrm{d}W_j(t) - \frac{1}{2}\sum_{j=1}^{q}\int_s^{T}(\bar{u}_j(t) - \bar{u}_j^{\tau}(t))^2\,\mathrm{d}t\right]. \tag{38}$$

Eq. (38) determines the likelihood that the sample paths generated by utilizing the controls $\bar{u}_j(t)$ could have also be obtained by utilizing the controls $\bar{u}_j^{\tau}(t)$. As can be seen, if the controls $\bar{u}_j^{\tau}(t)$ are

similar to the controls $\bar{u}_j(t)$ then there will be a significant contribution to the value of the probability of out-crossing.

The likelihood that the processes $\tilde{W}_j(t)$ are generated by the controls $\bar{u}_j^\tau(t)$ can be determined by dividing the ratio of Eq. (38) by the ratio of Eq. (35), i.e.,

$$R_\tau = \exp\left[-\sum_{j=1}^{q} \int_s^T \bar{u}_j^\tau(t)\,\mathrm{d}W_j(t) - \frac{1}{2}\sum_{j=1}^{q} \int_s^T (2\bar{u}_j(t) - \bar{u}_j^\tau(t))\bar{u}_j^\tau(t)\,\mathrm{d}t\right]. \tag{39}$$

However, the likelihood that the controls $\bar{u}_j^\tau(t)$ are acting instead of the controls $\bar{u}_j(t)$ is not restricted to a single exit time, but has to be extended to cover the entire time interval $s \leqslant t \leqslant T$. Moreover, not all exit times—and consequently not all controls $\bar{u}_j^\tau(t)$—are equally likely, but they follow a probability density function $w(t)$. A first order approximation of the density function of the exit times has been derived in Ref. [31] as

$$w(t) = \Phi(-\beta(t))\left[\int_s^T \Phi(-\beta(\tau))\,\mathrm{d}\tau\right]^{-1} \quad \text{with } s \leqslant t \leqslant T, \tag{40}$$

whereby $\Phi(\cdot)$ denotes the cumulative standard Gaussian distribution and $\beta(t)$ is the minimal distance function of Eq. (29). From this it follows, on the one hand, that the controls $\bar{u}_j^\tau(t)$ for the time interval $s \leqslant t \leqslant T$ have to be chosen with respect to density function of Eq. (40). And on the other hand, the Radon–Nikodym derivative $R_\tau$ of Eq. (39) has to be weighted with respect to all likely controls $\bar{u}_j^\tau(t)$, i.e.,

$$\frac{1}{Y^{s,1}(T)} = \int_s^T w(\tau)\frac{1}{R_\tau}\,\mathrm{d}\tau. \tag{41}$$

It should be mentioned that Eq. (41) takes implicitly into account the interaction between the different controls.

## 6. Time discretization

When determining the solution of the stochastic differential equations (1) by numerical methods, a discrete time approximation becomes mandatory. Using—without any restriction—an equidistant time discretization of the interval ($s \leqslant t \leqslant T$) with step size $\Delta t = (T - s)/M$, i.e.,

$$s = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant \cdots \leqslant t_M = T \quad \text{with } t_m = s + m\Delta t. \tag{42}$$

Therewith the stochastic differential equations (1) can be written—e.g., in form of the stochastic Euler scheme—as ($i = 1, 2, \ldots, p$)

$$X_i(t_{m+1}) = X_i(t_m) + \alpha_i(t_m, \mathbf{X})\Delta t + \sum_{j=1}^{q} \gamma_{ij}(t_m, \mathbf{X})\Delta W_j(t_m) \tag{43}$$

with initial conditions

$$X_i(t_0) = x_i. \tag{44}$$

It should be noted, that for the following excursus it is not necessary to write Eq. (43) in form of the Euler scheme. Indeed, any other stochastic integration scheme [15,16] can be utilized.

Replacing the $k$th increment of the $j$th Wiener process by

$$\Delta W_j(t_k) = \sqrt{\Delta t}\zeta_{jk} \quad (j = 1, 2, \ldots, q;\ k = 0, 1, \ldots, M-1) \tag{45}$$

with $\zeta_{jk}$ being mutually independent standard Gaussian random variates, the functional of Eq. (3) is approximated by

$$v(s, \mathbf{x}) = \int_{-\infty}^{+\infty} \underset{(qM)}{\cdots} \int_{-\infty}^{+\infty} f(\mathbf{X}^{s,\mathbf{x}}(t_m))\varphi(\boldsymbol{\zeta})\, \mathrm{d}\boldsymbol{\zeta}, \tag{46}$$

whereby $\varphi(\cdot)$ is the $qM$-dimensional joint density function of the standard normal random variates $\boldsymbol{\zeta}$.

Describing the boundary of the area which will be out-crossed by the Itô process $\mathbf{X}^{s,\mathbf{x}}(t)$ at time $t_m$ as

$$g(\mathbf{X}^{s,\mathbf{x}}(t_m)) = \bar{g}(\zeta_{10}, \ldots, \zeta_{q(M-1)}) = 0, \tag{47}$$

then the most likely excitation leading to an out-crossing of this boundary is defined as $\bar{\boldsymbol{\zeta}}^m = (\bar{\zeta}^m_{10}, \ldots, \bar{\zeta}^m_{q(M-1)})$ which minimizes

$$\beta(t_m) = \left[ \sum_{j=1}^{q} \sum_{k=0}^{M-1} (\bar{\zeta}^m_{jk})^2 \right]^{1/2} \tag{48}$$

subject to

$$\bar{g}(\bar{\zeta}^m_{10}, \ldots, \bar{\zeta}^m_{q(M-1)}) = 0. \tag{49}$$

To solve Eqs. (48) and (49) standard techniques from first order reliability method can be applied [35–37]. It should also be emphasized that an analogous formulation is given in Refs. [38,39] for determining the mean out-crossing rate of randomly excited systems.

With the above given most likely excitation $\bar{\boldsymbol{\zeta}}^m$, Eq. (43) is modified such that

$$\begin{aligned} \tilde{X}_i(t_{k+1}) = \tilde{X}_i(t_k) + \alpha_i(t_k, \tilde{\mathbf{X}})\Delta t + \sum_{j=1}^{q} \gamma_{ij}(t_k, \tilde{\mathbf{X}})\bar{u}_j^{t_m}(t_k)\Delta t \\ + \sum_{j=1}^{q} \gamma_{ij}(t_k, \tilde{\mathbf{X}})\Delta W_j(t_k), \end{aligned} \tag{50}$$

whereby the above utilized controls $\bar{u}_j^{t_m}(t_k)$ are defined as

$$\bar{u}_j^{t_m}(t_k) = \frac{1}{\sqrt{\Delta t}} \bar{\zeta}^m_{jk} \quad (j = 1, 2, \ldots, q;\ k = 0, 1, \ldots, M-1). \tag{51}$$

Taking into account the unlikeliness of such a modification of Eq. (50), the functional of Eq. (46) is evaluated by the importance sampling integral [7,8]

$$v(s, \mathbf{x}) = \int_{-\infty}^{+\infty} \underset{(qM)}{\cdots} \int_{-\infty}^{+\infty} f(\tilde{\mathbf{X}}^{s,\mathbf{x}}(t_m)) \frac{\varphi(\tilde{\boldsymbol{\zeta}})}{h(\tilde{\boldsymbol{\zeta}}, t_m)} h(\tilde{\boldsymbol{\zeta}}, t_m)\, \mathrm{d}\tilde{\boldsymbol{\zeta}}, \tag{52}$$

whereby the importance sampling density $h(\tilde{\boldsymbol{\zeta}}, t_m)$ in Eq. (52) is defined as

$$h(\tilde{\boldsymbol{\zeta}}, t_m) = \frac{1}{(2\pi)^{qM/2}} \exp\left[-\frac{1}{2}\sum_{j=1}^{q}\sum_{k=0}^{M-1}(\tilde{\zeta}_{jk} - \bar{\zeta}_{jk}^{m})^2\right] \tag{53}$$

and the likelihood ratio is given by

$$\frac{\varphi(\tilde{\boldsymbol{\zeta}})}{h(\tilde{\boldsymbol{\zeta}}, t_m)} = \exp\left[-\frac{1}{2}\sum_{j=1}^{q}\sum_{k=0}^{M-1}(2\tilde{\zeta}_{jk} - \bar{\zeta}_{jk}^{m})\bar{\zeta}_{jk}^{m}\right]. \tag{54}$$

Inserting in Eq. (54) now the sample $(\bar{u}_j^{t_m}(t_k)\sqrt{\Delta t} + \Delta W_j(t_k)/\sqrt{\Delta t})$ for $\tilde{\zeta}_{jk}^{j}$ results in

$$R_{t_m} = \exp\left[-\sum_{j=1}^{q}\sum_{k=0}^{M-1}\bar{u}_j^{t_m}(t_k)\Delta W_j(t_k) - \frac{1}{2}\sum_{j=1}^{q}\sum_{k=0}^{M-1}(\bar{u}_j^{t_m}(t_k))^2\Delta t\right]. \tag{55}$$

It should be noted that Eq. (55) is nothing else but a discrete approximation of Eq. (32).

When determining the first-passage probability in the time interval $s \leqslant t \leqslant T$, there are—according to the time discretization—$M$ most likely excitations leading to an out-crossing. Weighting these excitations by $\Phi(-\beta(t_m))$, i.e., their probability of occurrence, results in the $M$-modal importance sampling density for $\tilde{\boldsymbol{\zeta}}$ [31]

$$h(\tilde{\boldsymbol{\zeta}}) = \frac{1}{c}\sum_{m=1}^{M}\Phi(-\beta(t_m))h(\tilde{\boldsymbol{\zeta}}, t_m) = \sum_{m=1}^{M} w_m h(\tilde{\boldsymbol{\zeta}}, t_m) \tag{56}$$

with the normalizing constant

$$c = \sum_{m=1}^{M}\Phi(-\beta(t_m)). \tag{57}$$

The (inverse) likelihood ratio is given as

$$\frac{h(\tilde{\boldsymbol{\zeta}})}{\varphi(\tilde{\boldsymbol{\zeta}})} = \sum_{m=1}^{M} w_m \frac{h(\tilde{\boldsymbol{\zeta}}, m)}{\varphi(\tilde{\boldsymbol{\zeta}})}. \tag{58}$$

It should be mentioned that Eq. (58) is simply a discrete version of Eq. (41) and that Eqs. (56) to (58) take implicitly into account the interaction between the different controls.

## 7. Numerical examples

### 7.1. Linear oscillator

Before applying the above outlined importance sampling procedure to oscillators with typical non-linear restoring forces, its very principle is demonstrated for a linear oscillator, described by the non-dimensional equation of motion

$$\ddot{X}(t) + 2\eta\dot{X}(t) + X(t) = \sqrt{4\eta}\xi(t), \quad X(0) = \dot{X}(0) = 0. \tag{59}$$

Here $\xi(t)$ is a zero-mean Gaussian white noise with $\mathrm{E}[\xi(t)\xi(t+\tau)] = \delta(\tau)$ and $\eta = 0.05$ is a viscous damping coefficient. In the following, it is required to estimate the probability that $X(t)$ crosses up

the level $x_c = 4$ for the first time in the interval $(0 \leqslant t \leqslant T = 50)$, i.e.,

$$v(0, \mathbf{x}) = \mathrm{E}\left[\mathbf{I}\left(\max_{0 \leqslant t \leqslant T} X(t) > x_c\right)\right]. \tag{60}$$

For estimating $v(0, \mathbf{x})$, Eq. (59) is written in form of the Itô stochastic differential equations

$$\begin{aligned} \mathrm{d}X_1(t) &= X_2 \,\mathrm{d}t, \\ \mathrm{d}X_2(t) &= (-2\eta X_2 - X_1)\,\mathrm{d}t + \sqrt{4\eta}\,\mathrm{d}W(t). \end{aligned} \tag{61}$$

Utilizing an equidistant time discretization with step size $\Delta t = T/M$, the $k$th increment $\Delta W(t_k = k\Delta t)$ of the Wiener process is replaced by

$$\Delta W(t_k) = \sqrt{\Delta t}\zeta_k \quad (k = 0, 1, \ldots, M-1), \tag{62}$$

whereby $\zeta_k$ are standard normal random variates. Therewith, the response of the oscillator is given as

$$X(t) = \sum_{k=0}^{M-1} \sqrt{4\eta\Delta t}\zeta_k h(t - k\Delta t), \quad 0 \leqslant t < M\Delta t \tag{63}$$

with $h(\cdot)$ denoting the unit-impulse response function

$$h(t) = \frac{1}{\omega}\exp(-\eta t)\sin(\omega t), \quad \omega = \sqrt{1-\eta^2}. \tag{64}$$

For determining the control $\bar{u}^{t_m}(t)$ for up-crossing the level $x_c$ at time $t_m = m\Delta t$, the quantities $\bar{\zeta}_k^m$ in

$$\bar{u}^{t_m}(t) = \sum_{k=0}^{m-1} \frac{1}{\sqrt{\Delta t}} \bar{\zeta}_k^m \delta(t - t_k) \tag{65}$$

have to be chosen now in such a way that, on the one hand, $X(t)$ reaches the level $x_c$ at time $m\Delta t$, i.e.,

$$x_c - \sum_{k=0}^{m-1} \sqrt{4\eta\Delta t}\bar{\zeta}_k^m h(t_m - t_k) = 0 \tag{66}$$

and, on the other hand, the $\beta$-index

$$\beta(t_m) = \left[\sum_{k=0}^{m-1} (\bar{\zeta}_k^m)^2\right]^{-1/2} \tag{67}$$

becomes minimal. As is well known from first order reliability method the solutions to Eqs. (66) and (67), respectively, are

$$\bar{\zeta}_k^m = \frac{(\beta(t_m))^2}{x_c} \sqrt{4\eta\Delta t} h(t_m - t_k) \tag{68}$$

and

$$\beta(t_m) = x_c \left[ \sum_{k=0}^{m-1} 4\eta \Delta t h^2(t_m - t_k) \right]^{-1/2} = \frac{x_c}{\sigma(t_m)} \tag{69}$$

with $\sigma(t_m)$ denoting the standard deviation of the response $X(t)$ at time $t_m$. Therewith, the controls are given (in their continuous form) as

$$\bar{u}^{t_m}(t) = \frac{\sqrt{4\eta} x_c}{\sigma^2(t_m)} h(t_m - t), \quad 0 \leqslant t < t_m. \tag{70}$$

In Fig. 1 the control $\bar{u}^T(t)$ according to Eq. (70) for an up-crossing of the level $x_c$ at time $T = 50$ is displayed. In Fig. 2 a sample path of $X(t)$ is compared with a sample path of $\tilde{X}(t)$ utilizing this control. Only small differences in the trajectories can be observed until time $t = 30$. From there on, however, the control $\bar{u}^T(t)$ becomes dominant and excites the oscillator clearly in its resonant frequency such that $\tilde{x}(t)$ crosses up the threshold $x_c$ when approaching the end of the time interval. Moreover, by comparing the values of the control $\bar{u}^T(t)$ for two times $t_1 = 1.2$ and $t_2 = 1.7$ in Fig. 3, it becomes evident from the elliptically inward spiraling form that, on the one hand, there exists a considerable dependency between the controls and, on the other hand, there is a manifest periodicity of the contributions of the controls in time, which diminishes with increasing distance from the chosen up-crossing time $T$.

In Fig. 4 the importance sampling estimators ($N = 10^3$) of the up-crossing probability of the threshold $x_c = 4$ are compared, respectively, with the results from crude Monte Carlo simulation (solid line, $N = 10^6$) and an approximate solution from linear random vibration theory (dashed line, [40]). As can be seen, there is an excellent agreement between the estimators from importance sampling and crude Monte Carlo simulation for up-crossing probabilities greater than $1.0 \times 10^{-5}$. Beyond this value, however, crude Monte Carlo simulation breaks down—despite the enormous sample size. The importance sampling procedure, on the other hand, is capable to provide estimates for any absolute value of the first-passage probability. Moreover, as indicated in Fig. 4 by the 99% confidence intervals (cf. Ref. [7]), in case of the importance sampling estimators there

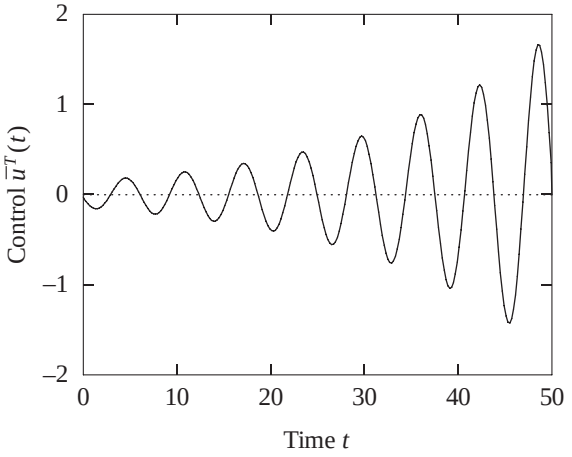


Fig. 1. Control $\bar{u}^T(t)$ of linear oscillator for an up-crossing of level $x_c = 4$ at time $T = 50$.

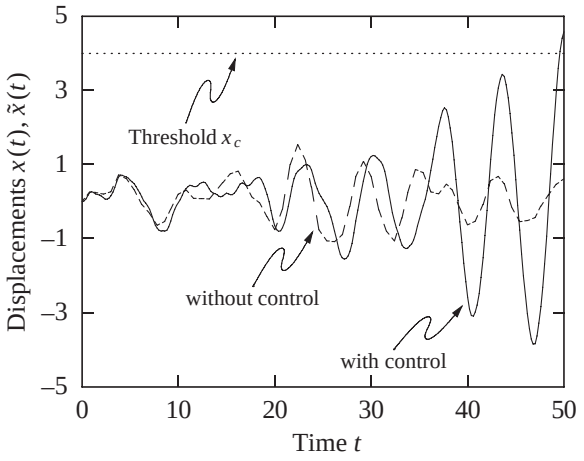


Fig. 2. Sample paths of linear oscillator without and with control $\bar{u}^T(t)$ from Fig. 1 ($T = 50$, $x_c = 4$).

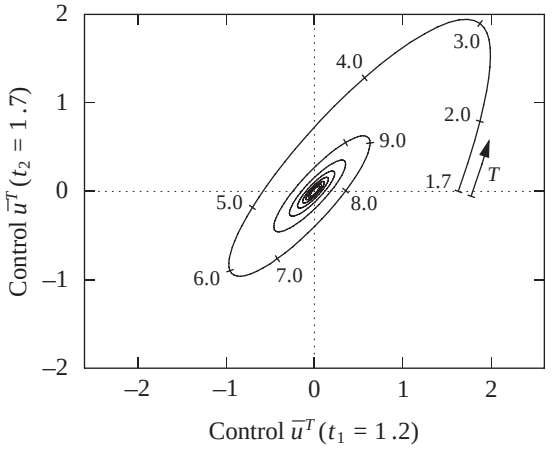


Fig. 3. Controls $\bar{u}^T(t_1)$ and $\bar{u}^T(t_2)$ with $t_1 = 1.2$ and $t_2 = 1.7$, respectively, for different up-crossing times $T$ of the linear oscillator ($x_c = 4.0$).

is only a small statistical error present which is indeed independent of the estimated absolute value—as should be expected from any consistently working importance sampling procedure.

### 7.2. Duffing oscillator

Given is a Duffing oscillator under external white noise excitation $\xi(t)$ with $\mathrm{E}[\xi(t)\xi(t+\tau)] = \delta(\tau)$, described by the dimensionless equation of motion

$$\ddot{X} + 2\eta\dot{X} + X + \varepsilon X^3 = \sqrt{4\eta}\xi(t), \quad X(0) = \dot{X}(0) = 0, \tag{71}$$

whereby $\eta = 0.05$ denotes a viscous damping coefficient and $\varepsilon$ is the degree of non-linearity. In Figs. 5 and 6, respectively, the control for an up-crossing of the level $x_c = 2$ at time $T = 50$ and the influence of this control on the deflection $X(t)$ of the Duffing oscillator are depicted for a

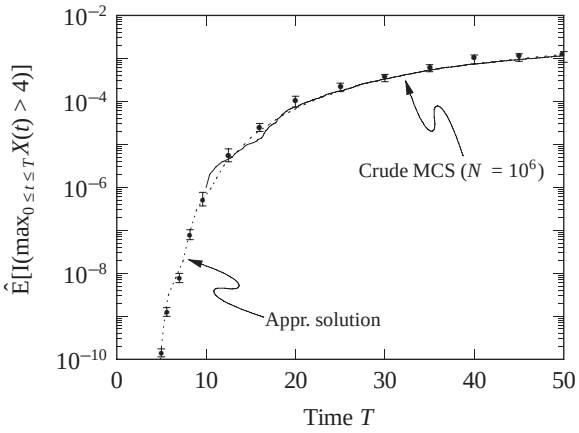


Fig. 4. Importance sampling estimators (•) and 99% confidence intervals (I) of the first-passage probability of the linear oscillator ($N = 10^3$, $x_c = 4$).

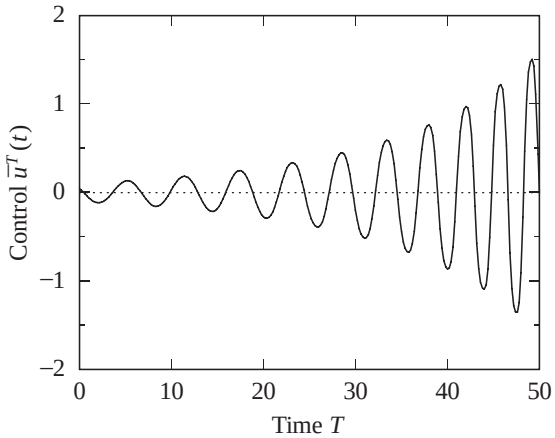


Fig. 5. Control $\bar{u}^T(t)$ of Duffing oscillator for an up-crossing of level $x_c = 2$ at time $T = 50$ ($\varepsilon = 1$).

degree of non-linearity $\varepsilon = 2$. As can be seen, the utilized control $\bar{u}^T(t)$ results in a strong amplification of the response of the Duffing oscillator such that it crosses up the threshold $x_c$ near the specified up-crossing time $T$.

To study the influence of the non-linearity on the estimators of the first-passage probability, $\varepsilon$ is varied between $0 \leqslant \varepsilon \leqslant 5$. In Figs. 7 and 8 the importance sampling estimators ($N = 10^3$) of the first up-crossing probability are compared with the results from crude Monte Carlo simulation (solid line, $N = 10^6$). As can be seen, there is a very good agreement between the two different estimators. Moreover and maybe even more worthwhile to be pointed out, almost independent of the degree of non-linearity—at least for values $\varepsilon > 1$—there exists a sufficient confidence in the importance sampling estimators as indicated by the 99% confidence intervals.

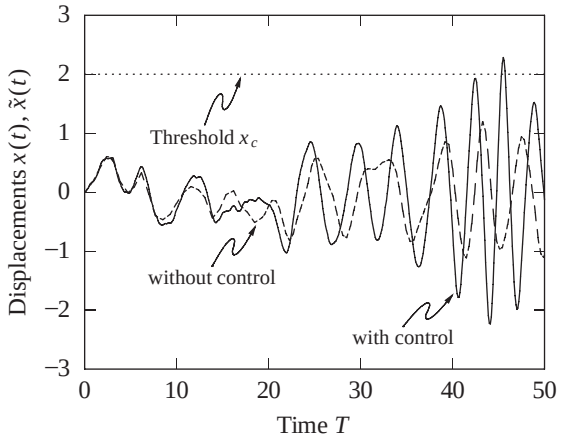


Fig. 6. Sample paths of Duffing oscillator without and with control $\bar{u}^T(t)$ from Fig. 5 ($T = 50$, $x_c = 2$, $\varepsilon = 1$).

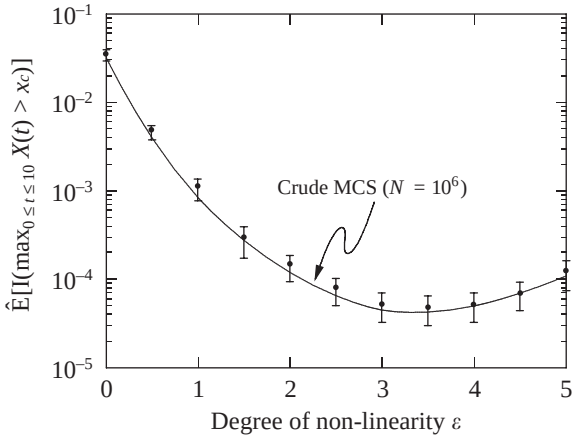


Fig. 7. Importance sampling estimators (•) and 99%-confidence intervals (I) of first-passage probability of the Duffing oscillator for different degrees of non-linearity $\varepsilon$ ($N = 10^3$, $T = 10$, $x_c = 1.9 - 0.1\varepsilon$).

### 7.3. Oscillator with non-linear damping

Given is an oscillator with non-linear damping described by the non-dimensional equation of motion [41]

$$\ddot{X} + 2\eta\dot{X}(1 + \varepsilon|\dot{X}|^{\nu}) + X = \sqrt{4\eta}\xi(t), \quad X(0) = \dot{X}(0) = 0. \tag{72}$$

In Eq. (72) $\eta$ is a viscous damping coefficient, $\varepsilon$ denotes a non-linearity parameter and $\xi(t)$ is a Gaussian white noise with $E[\xi(t)\xi(t+\tau)] = \delta(\tau)$. In the following are choosen $\eta = 0.05$, $\varepsilon = 2$ and $\nu = 2$.

In Fig. 9 the control for an up-crossing of the level $x_c = 2.1$ at time $T = 40$ is depicted. As can be seen, the non-linearity in the damping term results in an control which deviates strongly from the control of a linear oscillator as given by Eq. (70). In particular, there is comparatively more

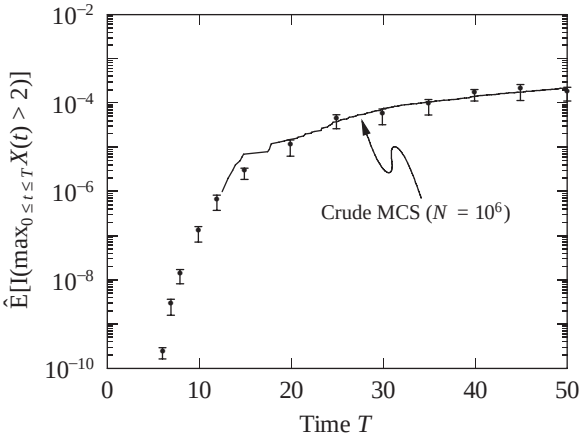


Fig. 8. Importance sampling estimators (•) and 99% confidence intervals (I) of first-passage probability of the Duffing oscillator ($N = 10^3$, $\varepsilon = 2$, $x_c = 2$).

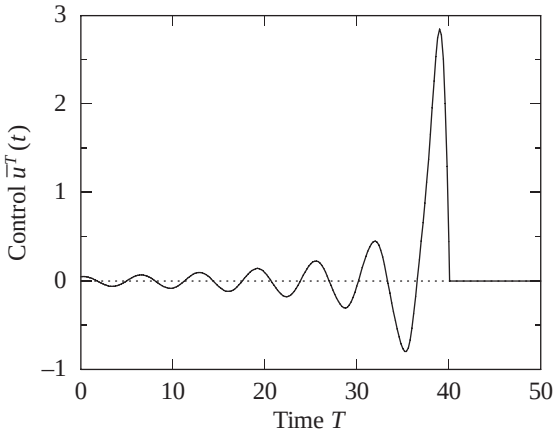


Fig. 9. Control $\bar{u}^T(t)$ of oscillator with non-linear damping for an up-crossing of the level $x_c = 2$ at time $T = 40$ ($\eta = 0.05$).

emphasis on values of the excitation closer to the time of excursion. This is a consequence of the higher energy dissipation present in the system at large amplitudes of vibration. In Figs. 10 and 11 the first passage probabilities as obtained from the suggested importance sampling approach ($N = 10^3$) together with the 99% confidence intervals are shown. These results are compared to those from crude Monte Carlo simulation (solid line, $N = 10^6$) in the range where they are available. In Fig. 10 the threshold level $x_c$ is varied between 2.0 and 3.0, and in Fig. 11 the length of the time interval $T$ is varied between 0 and 50. In both cases, the importance sampling results capture the extremely small probabilities at high threshold levels and at small values of $T$ with a very high level of confidence.

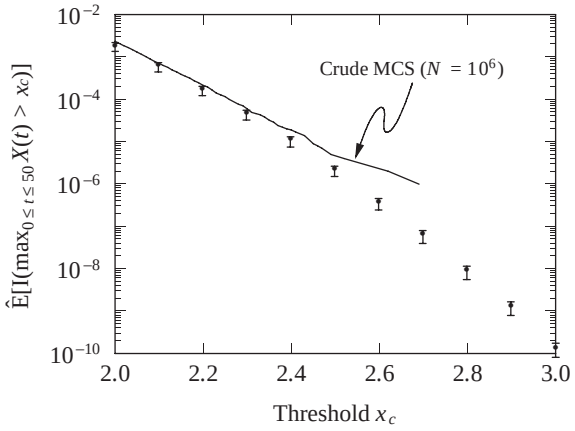


Fig. 10. Importance sampling estimators (•) and 99% confidence intervals (I) of first-passage probability of the oscillator with non-linear damping for different levels $x_c$ ($N = 10^3$, $\varepsilon = 2$, $v = 2$).

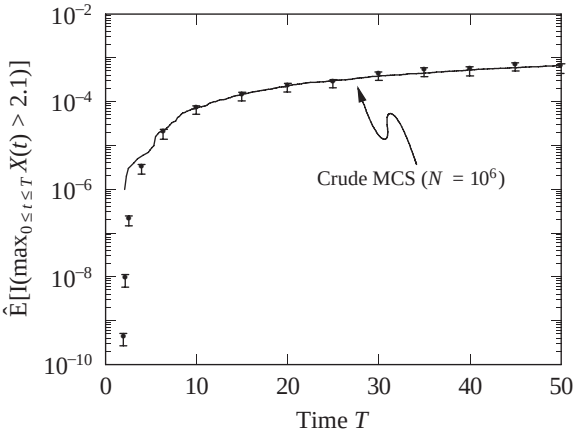


Fig. 11. Importance sampling estimators (•) and 99% confidence intervals (I) of first-passage probability of the oscillator with non-linear damping ($N = 10^3$, $\alpha = 0.05$, $x_c = 2.1$).

### 7.4. Oscillator with hysteretic restoring force

As a last example a hysteretic oscillator is investigated described by the non-dimensional equation of motion [42]

$$\begin{aligned} &\ddot{X} + 2\eta\dot{X} + \alpha X + (1-\alpha)Z = \sqrt{2\pi S_0}\,\xi(t), \quad X(0) = \dot{X}(0) = 0, \\ &\dot{Z} = -\gamma|\dot{X}|Z|Z|^{v-1} - \beta\dot{X}|Z|^{v} + A\dot{X}, \quad Z(0) = 0. \end{aligned} \tag{73}$$

Here $\eta$ denotes a viscous damping coefficient, $\alpha$ is the post- to pre-yielding stiffness ratio, $\xi(t)$ is a Gaussian white noise with $E[\xi(t)\xi(t+\tau)] = \delta(\tau)$, and $v$, $\gamma$, $\beta$ and $A$ are adjustable parameters to describe the hysteretic behavior. (In the following $\eta = 0.05$, $S_0 = (16\,\pi)^{-1/2}$, $v = 1$, $\gamma = \beta = 0.5$ and $A = 1$, respectively.) To study the influence of the hysteretic non-linearity on the estimator of

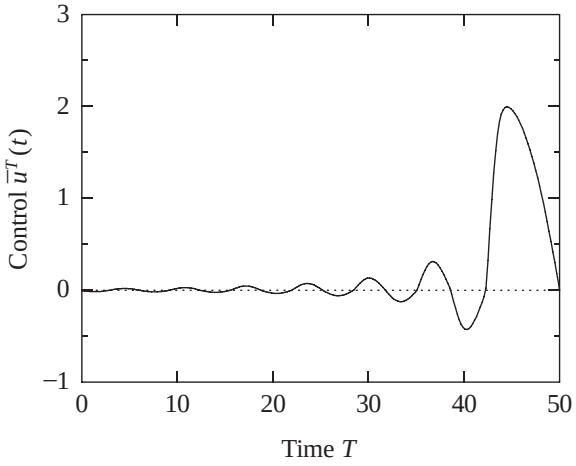


Fig. 12. Control $\bar{u}^T(t)$ of hysteretic oscillator for an up-crossing of the level $x_c = 15$ at time $T = 50$ ($\alpha = 0.05$).

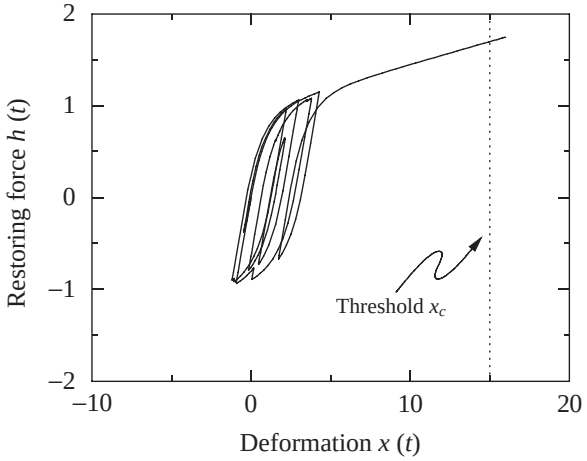


Fig. 13. Sample path of the restoring force–deformation relation for the hysteretic oscillator when utilizing the control $\bar{u}^T(t)$ from Fig. 12 for an up-crossing of the level $x_c = 15$ at time $T = 50$ ($\alpha = 0.05$).

the first passage probability, the stiffness ratio $\alpha$ is varied between $0 \leqslant \alpha \leqslant 1$, whereby $\alpha = 1$ describes a linear system.

In Figs. 12 and 13 the control $\bar{u}^T(t)$ for an up-crossing of the level $x_c = 15$ at time $T = 50$ and its influence on the restoring force–deformation relation is shown, respectively. As can be seen, the control results mainly in permanent deformations in the direction of the threshold $x_c$, with the major contribution from the last half-period of vibration. It should also be noted that the control for the hysteretic oscillator shows thereby a substantially different behavior as e.g., the control for the Duffing oscillator, reflecting the likewise different nature of the respective dynamic systems.

In Figs. 14 and 15 the importance sampling estimators ($N = 10^3$) are compared with the crude Monte Carlo simulation results (solid line, $N = 10^6$). Again, there is an excellent agreement between the estimators for the strongly non-linear behavior when utilizing the stiffness ratio

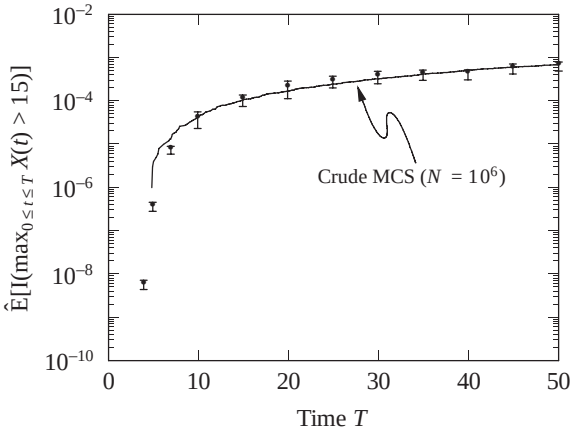


Fig. 14. Importance sampling estimators (•) and 99% confidence intervals (I) of first-passage probability of the hysteretic oscillator ($N = 10^3$, $\alpha = 0.05$, $x_c = 15$).

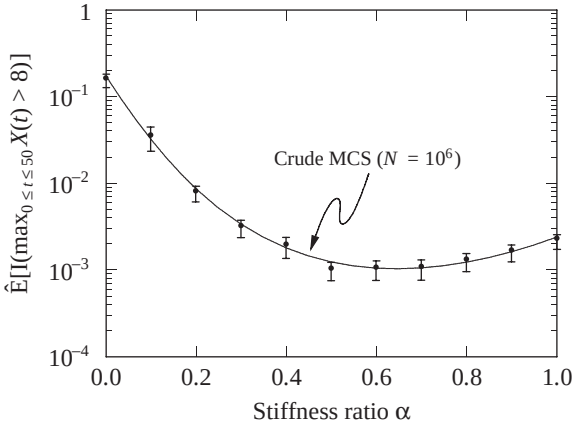


Fig. 15. Importance sampling estimators (•) and 99%-confidence intervals (I) of first-passage probability of the hysteretic oscillator for different stiffness ratios $\alpha$ ($N = 10^3$, $T = 50$, $x_c = 8$).

$\alpha = 0.05$, which is also implicitly depicted by the small 99% confidence intervals (see Fig. 14). Moreover, as can be seen from Fig. 15, when utilizing the importance sampling procedure equally reliable estimates can be obtained for the entire range of stiffness ratios $0 \leqslant \alpha \leqslant 1$.

## 8. Conclusions

A versatile importance sampling procedure for linear and non-linear dynamical systems under random excitations has been presented. The procedure allows one—at least theoretically—to construct optimal, i.e., unbiased zero-variance estimators of the system response. Nevertheless, by utilizing sub-optimal controls constructed via the solution of an optimization problem analogous to the one known from the first order reliability method, the variance of the estimators can be

decreased drastically as compared to crude Monte Carlo simulation. This is achieved by spending additional computational effort in the determination of a set of sub-optimal controls. While this effort is not negligible, it is primarily dominated by the number of random variables used for the time discretization of the random excitation process. This is due to the numerical gradient calculations required for the solution of the optimization problem. In this form, it is acceptable for a sufficiently small number of random variables, i.e., in significantly non-stationary or transient situations. On the other hand, in a strictly stationary situation the determination of the most likely excitations turns out to be trivial since the different controls are constructed by simply shifting one control along the time axis. Moreover, in some special stationary cases only one non-linear dynamic analysis needs to be carried out as shown in Ref. [43]. This means that for a wide range of problems the suggested approach can be considered to be extremely efficient. The advantages become especially clear when considering the system response in the low-probability regions (distribution tails) which are not accessible to crude Monte Carlo simulation.

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