On the influence of lateral vibrations of supports for an axially moving string

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Abstract

In this paper the transverse oscillations in travelling strings due to arbitrary lateral vibrations of the supports will be studied. Using the method of Laplace transforms (exact) solutions will be constructed for the initial-boundary value problems which describe these transverse oscillations.
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1. Introduction

About 50 years ago Sack [1], Mahalingam [2], and Archibald and Emslie [3] started to investigate the transverse oscillations in travelling strings due to the sinusoidal lateral vibrations of the supports over which the string passes. In Refs. [1–3] it is assumed that the transverse displacement of the string can be expressed in a simple trigonometric function which is directly related to the sinusoidal lateral vibrations of the supports. In this way the resonance frequencies were found. Recently Pakdemirli and Boyaci [4] considered the transverse oscillations in travelling strings due to small lateral vibrations of the supports. A perturbation method (based on the truncation method and the method of multiple scales) has been used in Ref. [4] to approximate the transverse oscillations of the string. The applicability of this perturbation method to these moving string problems (in particular the use of the truncation method) will be discussed at the end of this paper.

The following non-dimensional equation of motion for the string problem will be considered in this paper:

$$\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial x \partial t} + (v^2 - 1)\frac{\partial^2 w}{\partial x^2} = 0, \ \ 0 < x < 1, \ \ t > 0, \tag{1}$$

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where $w = w(x,t)$ is the transverse displacement of the string, v is the (constant) axial transport velocity of the string, x is the spatial co-ordinate, and t is the time co-ordinate. These dimensionless quantities are in the following way related to the dimensional ones (see also for instance Refs. [1–7]): $w = w^*/l$, $x = x^*/l$, $t = t^*(T/\varrho Al)^{1/2}$, and $v = v^*/(T/\varrho A)^{1/2}$, in which l is the (constant) distance between the two supports, T is the (constant) tension in the string, and ϱA is the (constant) mass of the string per unit length. Furthermore, it has been assumed that the oscillations are sufficiently small such that the non-linear terms in the equation of motion can be neglected. Also gravity effects, bending stiffness of the string, and damping effects have been neglected. In this paper the following Dirichlet boundary conditions (BC) will be considered:

$$w(0,t) = 0, \quad \text{and} \quad w(1,t) = f(t), \ t \geqslant 0, \tag{2}$$

where $f(t)$ describes arbitrary vibrations of the support at $x = 1$. Only the vibrations of the string due to the arbitrary vibrations of the support(s) will be considered in this paper. For that reason it is assumed that the initial displacement of the string and the initial velocity (IV) of the string are both identically equal to zero, that is,

$$w(x,0) = 0, \ \text{and} \ w_t(x,0) = 0, \ 0 < x < 1 \tag{3}$$

respectively. In this paper the initial-boundary value problem (1)–(3) for $w(x,t)$ will be solved exactly using the Laplace transform method. It should be remarked that the initial-boundary value problem (1)–(3) also has been studied in Refs. [1–4]. The resonance frequencies have been determined in Refs. [1–3]. In Ref. [4] approximations of the solution of the initial-boundary value problem (1)–(3) have been constructed for some specific, prescribed motion of the supports. The results as obtained in Refs. [1–4] can readily be compared with the (exact) results for the solution and for the frequencies as obtained in this paper.

This paper is organized as follows. In Section 2 of this paper the stationary belt problem will be studied (that is, problem (1)–(3) with $v = 0$). Standard techniques from the theory of partial differential equations can be used in this case (see for instance Ref. [8]). In Section 3 of this paper the axially moving string problem (1)–(3) will be solved exactly using the standard method of Laplace transforms. Finally in Section 4 of this paper some conclusions will be drawn and some remarks will be made.

2. The stationary belt

In this section, the initial-boundary value problem (1)–(3) is studied with $v = 0$, that is,

$$
\begin{aligned}
w_{tt} - w_{xx} &= 0, \quad 0 < x < 1, \ t > 0, \\
w(0,t) &= 0, \quad w(1,t) = f(t), \quad t \geqslant 0, \\
w(x,0) &= w_t(x,0) = 0, \quad 0 < x < 1.
\end{aligned}
\tag{4}
$$

This initial-boundary value problem (4) can be solved by using the available standard techniques as for instance described in Section 6 of Chapter 5 in Ref. [8]. Firstly the boundary conditions are made homogeneous by putting

$$w(x,t) = u(x,t) + xf(t), \tag{5}$$

where $u(x, t)$ has to satisfy

$$u_{tt} - u_{xx} = -xf''(t), \quad 0 < x < 1, \ t > 0,$$
$$u(0, t) = u(1, t) = 0, \quad t \geqslant 0,$$
$$u(x, 0) = -xf(0), \quad u_t(x, 0) = -xf'(0), \quad 0 < x < 1. \tag{6}$$

Then, the function $u(x, t)$ is written into two parts

$$u(x, t) = u_1(x, t) + u_2(x, t), \tag{7}$$

where $u_1(x, t)$ satisfies the homogeneous PDE and the inhomogeneous IVs, that is,

$$u_{1_{tt}} - u_{1_{xx}} = 0, \quad 0 < x < 1, \ t > 0,$$
$$u_1(0, t) = u_1(1, t) = 0, \quad t \geqslant 0,$$
$$u_1(x, 0) = -xf(0), \quad u_{1_t}(x, 0) = -xf'(0), \quad 0 < x < 1, \tag{8}$$

and where $u_2(x, t)$ satisfies the inhomogeneous PDE and the homogeneous IVs, that is,

$$u_{2_{tt}} - u_{2_{xx}} = -xf''(t), \quad 0 < x < 1, \ t > 0,$$
$$u_2(0, t) = u_2(1, t) = 0, \quad t \geqslant 0,$$
$$u_2(x, 0) = u_{2_t}(x, 0) = 0, \quad 0 < x < 1. \tag{9}$$

The initial-boundary value problem (8) for $u_1(x, t)$ can readily be solved, yielding

$$u_1(x, t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi t) + B_n \sin(n\pi t)) \sin(n\pi x), \tag{10}$$

where

$$A_n = -2 \int_0^1 xf(0) \sin(n\pi x) \, \mathrm{d}x = \frac{2(-1)^n}{n\pi} f(0),$$

and

$$B_n = \frac{-2}{n\pi} \int_0^1 xf'(0) \sin(n\pi x) \, \mathrm{d}x = \frac{2(-1)^n}{n^2\pi^2} f'(0).$$

The initial-boundary value problem (9) for $u_2(x, t)$ can be solved by putting

$$u_2(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x). \tag{11}$$

Then, Eq. (11) is substituted into the initial-boundary value problem (9) to obtain

$$u_n'' + n^2\pi^2 u_n = -2f''(t) \int_0^1 x \sin(n\pi x) \, \mathrm{d}x = \frac{2(-1)^n}{n\pi} f''(t),$$
$$u_n(0) = u_n'(0) = 0. \tag{12}$$

The initial value problem (12) for $u_n(t)$ can readily be solved, and so from Eq. (11) $u_2(x, t)$ follows, yielding

$$u_2(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2 \pi^2} \left\{ \sin(n\pi t) \int_0^t f''(\tau) \cos(n\pi\tau) \, d\tau - \cos(n\pi t) \int_0^t f''(\tau) \sin(n\pi\tau) \, d\tau \right\} \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2 \pi^2} \int_0^t f''(\tau) \sin(n\pi(t - \tau)) \, d\tau \sin(n\pi x). \tag{13}$$

From Eqs. (5), (7), (10), and (13) the solution $w(x, t)$ of the initial-boundary value problem (4) now follows. By introducing

$$S(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2 \pi^2} \sin(n\pi t) \sin(n\pi x),$$

this solution $w(x, t)$ can also be written in the compact form

$$w(x, t) = f(0)S_t(x, t) + f'(0)S(x, t) + \int_0^t f''(\tau)S(x, t - \tau) \, d\tau + xf(t). \tag{14}$$

3. The axially moving string

In this section the initial-boundary value problem (1)–(3) will be studied with $v^2 \neq 1$. The method of Laplace transforms are used to construct the (exact) solution for this problem. Let $W(x, s)$ be the Laplace transform of $w(x, t)$, that is,

$$W(x, s) = \int_0^{\infty} w(x, t)e^{-st} \, dt. \tag{15}$$

Then, by applying the Laplace transform to the PDE (1) and by using the IVs (3), it follows

$$s^2 W(x, s) - sw(x, 0) - w_t(x, 0) + 2v(sW_x(x, s) - w_x(x, 0)) + (v^2 - 1)W_{xx}(x, s) = 0$$

$$\Rightarrow W_{xx} + (2vs/(v^2 - 1))W_x + (s^2/(v^2 - 1))W = 0. \tag{16}$$

Eq. (16) for $W(x, s)$ can readily be solved, yielding

$$W(x, s) = C_1(s) \exp(-sx/(v + 1)) + C_2(s) \exp(-sx/(v - 1)), \tag{17}$$

where $C_1(s)$ and $C_2(s)$ are still arbitrary functions, which will be determined by the BCs (2). Let $F(s)$ be the Laplace transform of $f(t)$. Then, by applying the Laplace transform to the BCs (2) it follows that

$$W(0, s) = 0 \quad \text{and} \quad W(1, s) = F(s). \tag{18}$$

From (17) and (18) it then follows that

$$C_1(s) = F(s)/(\exp(-s/(v + 1)) - \exp(-s/(v - 1))) \quad \text{and} \quad C_2(s) = -C_1(s),$$

and so,

$$W(x, s) = F(s)(\exp(-sx/(v + 1)) - \exp(-sx/(v - 1)))/(\exp(-s/(v + 1)) - \exp(-s/(v - 1))). \tag{19}$$

The inverse Laplace transform of $W(x, s)$ is defined to be

$$w(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} W(x, s) e^{st} \, ds = \sum_n \text{Res}(s_n) \tag{20}$$

for some $\gamma > 0$, and where $\text{Res}(s_n)$ is the residue of $W(x, s) e^{st}$ at $s = s_n$. Since $f(t)$ is arbitrary it follows that the poles in Eq. (20) due to $F(s)$ are unknown. For that reason Eq. (19) can be rewritten in the form

$$W(x, s) = F(s) H(x, s)$$

with

$$H(x, s) = (\exp(-sx/(v+1)) - \exp(-sx/(v-1)))/(\exp(-s/(v+1)) - \exp(-s/(v-1))). \tag{21}$$

The inverse Laplace transform applied to Eq. (21) will eventually lead to a convolution integral in the (x, t) domain. Before this inverse transform can be determined the singularities of $H(x, s)$ have to be calculated first. These singularities of $H(x, s)$ are given by

$$\exp\left(\frac{-s}{v+1}\right) - \exp\left(\frac{-s}{v-1}\right) = 0$$

$$\Leftrightarrow \quad \exp\left(\frac{2s}{v^2 - 1}\right) = 1$$

$$\Rightarrow \quad s = s_n = (v^2 - 1)n\pi i \tag{22}$$

with $n \in \mathbb{Z}$. It should be observed that $s_0 = 0$ is not a pole of $H(x, s)$, and that all other poles of $H(x, s)$ are simple and are given by Eq. (22) with $n \in \mathbb{Z} \backslash \{0\}$. To obtain a solution form for $w(x, t)$ which is immediately comparable to the case $v = 0$ (see Eq. (14)) expression (21) for $W(x, s)$ is rewritten as

$$W(x, s) = (s^2 F(s) - sf(0) - f'(0)) \frac{H(x, s)}{s^2} + (sf(0) + f'(0)) \frac{H(x, s)}{s^2}. \tag{23}$$

Now it should be observed that $H(x, s)/s^2$ has a pole of order two at $s = 0$, and that $H(x, s)/s^2$ has poles of order one at $s = s_n = (v^2 - 1)n\pi i$ with $n \in \mathbb{Z} \backslash \{0\}$. The inverse Laplace transform of $H(x, s)/s^2$, that is, $L^{inv}(H(x, s)/s^2)$ now easily follows, yielding

$$L^{inv}\left(\frac{H(x, s)}{s^2}\right) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{H(x, s)}{s^2} e^{st} \, ds$$

$$= \frac{1}{1!} \lim_{s \to 0} \frac{\partial}{\partial s} (H(x, s) e^{st}) + \sum_{n \neq 0} \text{Res}(s_n) = \cdots$$

$$= \frac{x}{v^2 - 1} (v - xv + t(v^2 - 1))$$

$$
\begin{aligned}
- \sum_{n=1}^{\infty} & \left\{ \frac{\cos((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)} [\cos((v-1)n\pi)(\cos((v-1)n\pi x) - \cos((v+1)n\pi x)) \right. \\
& + \sin((v-1)n\pi)(\sin((v-1)n\pi x) - \sin((v+1)n\pi x))] \\
& + \frac{\sin((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)} [\sin((v-1)n\pi)(\cos((v+1)n\pi x) - \cos((v-1)n\pi x)) \\
& \left. + \cos((v-1)n\pi)(\sin((v-1)n\pi x) - \sin((v+1)n\pi x))] \right\} = h_2(x,t).
\end{aligned} \tag{24}
$$

Further it should be observed that $H(x,s)/s$ has only poles of order one at $s = (v^2-1)n\pi i$ with $n \in \mathbb{Z}$. The inverse Laplace transform of $H(x,s)/s$ now also easily follows, yielding

$$
\begin{aligned}
L^{inv}\left(\frac{H(x,s)}{s}\right) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{H(x,s)}{s} e^{st} \, ds = \sum_{n \in \mathbb{Z}} \mathrm{Res}(s_n) = \cdots \\
&= x + \sum_{n=1}^{\infty} \left\{ \frac{\sin((v^2-1)n\pi t)}{n\pi} [\cos((v-1)n\pi)(\cos((v-1)n\pi x) - \cos((v+1)n\pi x)) \right. \\
& \qquad + \sin((v-1)n\pi)(\sin((v-1)n\pi x) - \sin((v+1)n\pi x))] \\
& \qquad + \frac{\cos((v^2-1)n\pi t)}{n\pi} [\sin((v-1)n\pi)(\cos((v-1)n\pi x) - \cos((v+1)n\pi x)) \\
& \qquad \left. + \cos((v-1)n\pi)(\sin((v+1)n\pi x) - \sin((v-1)n\pi x))] \right\} \\
&= h_1(x,t) = \frac{\partial h_2}{\partial t}(x,t).
\end{aligned} \tag{25}
$$

Since the inverse Laplace transform of $s^2 F(s) - sf(0) - f'(0)$ is $f(t)$, it then finally follows from Eqs. (20) and (23)–(25) that

$$
w(x,t) = \int_0^t f''(\tau) h_2(x, t-\tau) \, d\tau + f(0)h_1(x,t) + f'(0)h_2(x,t), \tag{26}
$$

where $h_2(x,t)$ and $h_1(x,t)$ are given by Eqs. (24) and (25) respectively. By putting

$$
\begin{aligned}
h(x,t) = - \sum_{n=1}^{\infty} & \left\{ \frac{\cos((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)} [\cos((v-1)n\pi)(\cos((v-1)n\pi x) - \cos((v+1)n\pi x)) \right. \\
& + \sin((v-1)n\pi)(\sin((v-1)n\pi x) - \sin((v+1)n\pi x))] \\
& + \frac{\sin((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)} [\sin((v-1)n\pi)(\cos((v+1)n\pi x) - \cos((v-1)n\pi x)) \\
& \left. + \cos((v-1)n\pi)(\sin((v-1)n\pi x) - \sin((v+1)n\pi x))] \right\},
\end{aligned} \tag{27}
$$

the solution of the initial-boundary value problem (1)–(3) can be rewritten as

$$
w(x, t) = \int_0^t f''(\tau) h(x, t - \tau) \, d\tau + f(0) \frac{\partial h}{\partial t}(x, t)
$$
$$
+ f'(0) h(x, t) + \frac{vx(1 - x)}{v^2 - 1} f'(t) + xf(t). \tag{28}
$$

For $v = 0$ it immediately follows from Eqs. (27) and (28) that resprensation (14) for $w(x, t)$ is again obtained. It can also be observed from Eqs. (27) and (28) that resonances (that is, unbounded behaviour of $w(x, t)$ in time) can occur when for (at least) one $n \in \mathbb{Z}^+$

$$
\int_0^t f''(\tau) \cos((v^2 - 1)n\pi(t - \tau)) \, d\tau \quad \text{and/or}
$$
$$
\int_0^t f''(\tau) \sin((v^2 - 1)n\pi(t - \tau)) \, d\tau \tag{29}
$$

are unbounded in time. For instance for $f(t) = \cos(\omega t)$ with ω equal to (or close to) $(v^2 - 1)n\pi$ it follows from Eq. (29) that resonance occurs (see also Refs. [1,2] or [3] for a similar result). Also for a more general periodic function $f(t)$ with period $2/(m(v^2 - 1))$ (for some fixed $m \in \mathbb{Z}^+$) it easily follows from Eq. (29) that resonances will occur.

4. Conclusions and remarks

In this paper the transverse oscillations in travelling strings due to arbitrary lateral vibrations of the support(s) have been studied. It has been shown how the initial-boundary value problem (1)–(3) which describes these oscillations can be solved using the method of Laplace transforms. The (exact) solution has been constructed explicitly, and resonance conditions have been derived.

In this paper a homogeneous boundary condition at $x = 0$ and an inhomogeneous one at $x = 1$ have been considered. When both boundary conditions are inhomogeneous the method of Laplace transforms can still be applied. In fact the method of Laplace transforms can be applied to the following initial-boundary value problem:

$$
\begin{aligned}
&w_{tt} + 2vw_{xt} + (v^2 - 1)w_{xx} = g(x, t), \quad 0 < x < 1, \ t > 0, \\
&w(0, t) = h(t), \quad w(1, t) = f(t), \quad t \geqslant 0, \\
&w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad 0 < x < 1.
\end{aligned} \tag{30}
$$

The calculations, however, become lengthy but remain completely similar to the calculations as presented in Section 3 of this paper. Also when the Dirichlet boundary conditions in Eq. (30) are replaced by for instance Neumann or Robin boundary conditions similar calculations can be performed to those presented in this paper.

Recently in Ref. [4] the transverse oscillations in travelling strings due to small lateral vibrations of the supports have been considered. A perturbation method which is based on the truncation method has been used in Ref. [4] to approximate the transverse oscillations of the string. In fact in Ref. [4] the approximation is truncated to a single mode of vibration. The authors of Ref. [4] and their co-authors (see the list of references in Ref. [4]) have applied this truncation method to many

problems which are related to axially moving strings or beams. As can be seen from Eqs. (27) and (28), however, the exact solution for the initial-boundary value problem (1)–(3) will always consist of infinitely many modes of vibration. So, truncating the approximation to a single mode of vibration (as has been done in Ref. [4]) will in general not give accurate approximations on long time scales. In fact the accuracy of the approximations as obtained in Ref. [4] on a specified time interval can be readily determined by comparing Eq. (28) with the approximations as obtained in Ref. [4]. A similar conclusion on the applicability of the truncation method to problems for axially moving strings or beams with a time-varying velocity has been drawn in Refs. [9,10].

References

[1] R.A. Sack, Transverse oscillations in travelling strings, British Journal of Applied Physics 5 (1954) 224–226.

[2] S. Mahalingam, Transverse vibrations of power transmission chains, British Journal of Applied Physics 8 (1957) 145–148.

[3] F.R. Archibald, A.G. Emslie, The vibration of a string having a uniform motion along its length, Applied Mechanics 25 (1958) 347–348.

[4] M. Pakdemirli, H. Boyaci, Effect of non-ideal boundary conditions on the vibrations of continuous systems, Journal of Sound and Vibration 249 (2002) 815–823.

[5] A.G. Ulsoy, C.D. Mote Jr., R. Syzmani, Principal developments in band saw vibration and stability research, Holz als Roh und Werkstoff 36 (1978) 273–280.

[6] J.A. Wickert, C.D. Mote Jr., Current research on the vibration and stability of axially moving materials, Shock and Vibration Digest 20 (1988) 3–13.

[7] S. Abrate, Vibrations of belts and belt drives, Mechanisms & Machine Theory 27 (1992) 645–659.

[8] W.A. Strauss, Partial Differential Equations, An Introduction, Wiley, New York, 1992.

[9] G. Suweken, W.T. van Horssen, On the transversal vibrations of a conveyor belt with a low and time-varying velocity. Part I: the string-like case, Journal of Sound and Vibration 264 (2003) 117–133.

[10] G. Suweken, W.T. van Horssen, On the transversal vibrations of a conveyor belt with a low and time-varying velocity. Part II: the beam-like case, Journal of Sound and Vibration 267 (2003) 1007–1027.