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Non-linear vibrations of a simple–simple beam with a non-ideal support in between

Mehmet Pakdemirli*, Hakan Boyacı

Department of Mechanical Engineering, Celal Bayar University, 45140 Muradiye, Manisa, Turkey

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Abstract

A simply supported Euler–Bernoulli beam with an intermediate support is considered. Non-linear terms due to immovable end conditions leading to stretching of the beam are included in the equations of motion. The concept of non-ideal boundary conditions is applied to the beam problem. In accordance, the intermediate support is assumed to allow small deflections. An approximate analytical solution of the problem is found using the method of multiple scales, a perturbation technique. Ideal and non-ideal frequencies as well as frequency-response curves are contrasted.

1. Introduction

Beams are frequently used as design models. The types of support have direct influence on the vibrations of beams. Different support conditions are defined (simple, built-in, guided, free, etc.) and the requirements associated with such supports are stated. In real system applications, usually the support type that resembles best the behaviour is selected. However, real system behaviour may deviate from the idealized support conditions. If the beam is simply supported, the ideal conditions require deflections and moments to be zero at the supports. In reality, however small deviations from the ideal conditions indeed occur. A pinned joint is modelled as a simple support. If the pin-hole assembly is not tightly fixed and if there is some friction, small deflections as well as moments occur which makes deviations from the ideal conditions. This idea can be applied in a similar way to other support conditions such as built-in, guided, free, etc. To represent such behaviour, a non-ideal boundary condition concept has been recently proposed [1,2].

*Corresponding author. Tel.: +90-236-241-2144; fax: +90-236-241-2143.
E-mail address: mpak@bayar.edu.tr (M. Pakdemirli).

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Non-ideal boundary conditions are modelled using perturbations. In the pioneering work [1], linear beam problems of different support conditions and an axially moving string problem has been treated. A non-linear beam problem with stretching has also been considered [2]. Here, in this work, the forced damped case with a non-ideal simple support at an intermediate point is considered further. Ideal and non-ideal frequencies as well as frequency-response curves are contrasted. The effect of a loose intermediate support is analyzed here together with the non-linear effects. As a general rule, the loose support causes odd numbered frequencies to increase, while the effect is reverse for the even numbered frequencies. This general rule does not apply to degenerate frequencies as defined in the text. The non-ideality causes a shift in the frequency-response curves also. By shifting the frequency-response curve, a system under resonance may be brought to a safer operating condition.

2. Problem formulation and solution

The simple–simple Euler–Bernoulli beam considered here has a simple support at an intermediate point at $x = \eta$ $(0 < \eta < 1)$, where x is the spatial co-ordinate (See Fig. 1). The equations of motion and the boundary conditions are (see Ref. [3] for the derivation of equations in a similar case of beam–mass problem)

$$\begin{aligned} \ddot{w}_1 + w_1^{iv} &= \frac{1}{2}\left[\int_0^\eta w_1'2\,\mathrm{d}x + \int_\eta^1 w_2'2\,\mathrm{d}x\right] w_1'' - 2\bar{\mu}\dot{w}_1 + \bar{F}_1 \cos \Omega t, \\ \ddot{w}_2 + w_2^{iv} &= \frac{1}{2}\left[\int_0^\eta w_1'2\,\mathrm{d}x + \int_\eta^1 w_2'2\,\mathrm{d}x\right] w_2'' - 2\bar{\mu}\dot{w}_2 + \bar{F}_2 \cos \Omega t, \end{aligned} \tag{1,2}$$

$$\begin{aligned} &w_1(0,t) = w_1''(0,t) = w_2(1,t) = w_2''(1,t) = 0, \\ &w_1(\eta,t) = w_2(\eta,t) = \varepsilon\sqrt{\varepsilon}\alpha(t), \quad w_1'(\eta,t) = w_2'(\eta,t), \\ &w_1''(\eta,t) = w_2''(\eta,t), \end{aligned} \tag{3}$$

where w_1 is the left side deflection and w_2 is the right side deflection, t is the time variable, $\bar{\mu}$ is the damping coefficient and, $\bar{F}_i$ and Ω are the magnitudes and the frequency of the external excitation respectively. $(\cdot)$ denotes derivation with respect to time variable t and $()'$ denotes derivation with respect to spatial variable x. ε is a small perturbation parameter. All variables are dimensionless. The relation between the dimensional (denoted by *) and dimensionless quantities

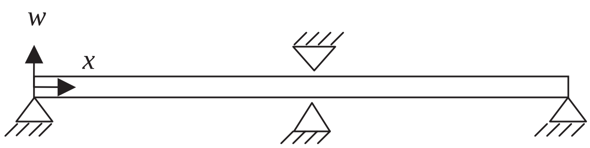


Fig. 1. A simply supported beam of immovable end conditions with a non-ideal simple support in between.

are as follows:

$$x = \frac{x^*}{L}, \ w_i = \frac{w_i^*}{r}, \ \eta = \frac{x_s}{L}, \ t = \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}t^*,$$
$$\Omega = \frac{\Omega * L^2}{\sqrt{EI/\rho A}}, \ \bar{F}_i = \frac{F_i^*}{EIr}, \ 2\bar{\mu} = \frac{\mu * L^2}{\sqrt{\rho AEI}}, \tag{4}$$

where L is the length, ρ is the density, A is the cross-sectional area, E is the Young's modulus, I is the moment of inertia, r is the radius of gyration of the beam cross-section and x_s is the location of the intermediate support. Here, small deflections at the intermediate point η are permitted to indicate deviations from the ideal boundary condition.

Before proceeding further, deflection term w_i and the magnitude of the external excitation $\bar{F}_i$ are scaled as

$$w_i = \sqrt{\varepsilon}u_i, \ \bar{F}_i = \sqrt{\varepsilon}\hat{F}_i, \tag{5}$$

for appropriate ordering of the terms. Eqs. (1)–(3) are rewritten in the new variable u as follows:

$$\ddot{u}_1 + u_1^{iv} = \frac{1}{2}\varepsilon\left[\int_0^\eta u_1'2 + \int_\eta^1 u_2'2\right]u_1'' - 2\bar{\mu}\dot{u}_1 + \hat{F}_1 \cos \Omega t,$$
$$\ddot{u}_2 + u_2^{iv} = \frac{1}{2}\varepsilon\left[\int_0^\eta u_1'2 + \int_\eta^1 u_2'2\right]u_2'' - 2\bar{\mu}\dot{u}_2 + \hat{F}_2 \cos \Omega t, \tag{6, 7}$$

$$u_1(0,t) = u_1''(0,t) = u_2(1,t) = u_2''(1,t) = 0,$$
$$u_1(\eta,t) = u_2(\eta,t) = \varepsilon\alpha(t), \quad u_1'(\eta,t) = u_2'(\eta,t),$$
$$u_1''(\eta,t) = u_2''(\eta,t). \tag{8}$$

Now, approximate solutions of Eqs. (6) and (7) with associated boundary conditions (8) are sought. The method of multiple scales (a perturbation technique) [4] is applied directly to the partial differential system and boundary conditions. Expansions are assumed of the forms

$$u_1(x,t;\varepsilon) = u_{11}(x,T_0,T_1) + \varepsilon u_{12}(x,T_0,T_1), + \cdots,$$
$$u_2(x,t;\varepsilon) = u_{21}(x,T_0,T_1) + \varepsilon u_{22}(x,T_0,T_1) + \cdots, \tag{9, 10}$$

where $T_0 = t$ is the fast time scale and $T_1 = \varepsilon t$ is the slow time scale. Only the primary resonance case is considered and hence, the forcing and damping terms are ordered as

$$\bar{\mu} = \varepsilon\mu, \quad \hat{F}_i = \varepsilon F_i, \tag{11}$$

The time derivatives are expressed in the new time scales

$$(^\bullet) = D_0 + \varepsilon D_1, \ (^{\bullet\bullet}) = D_0^2 + 2\varepsilon D_0 D_1, \ D_n = \partial/\partial T_n. \tag{12}$$

Inserting Eqs. (9)–(12) into Eqs. (6)–(8) and equating coefficients of like powers of ε, one obtains, at order 1,

$$D_0^2 u_{11} + u_{11}^{iv} = 0, \quad D_0^2 u_{21} + u_{21}^{iv} = 0, \tag{13, 14}$$

$$u_{11}(0, T_0, T_1) = u''_{11}(0, T_0, T_1) = u_{21}(1, T_0, T_1) = u''_{21}(1, T_0, T_1) = 0,$$
$$u_{11}(\eta, T_0, T_1) = u_{21}(\eta, T_0, T_1) = 0, \quad u'_{11}(\eta, T_0, T_1) = u'_{21}(\eta, T_0, T_1),$$
$$u''_{11}(\eta, T_0, T_1) = u''_{21}(\eta, T_0, T_1), \tag{15}$$

and at order ε,

$$D_0^2 u_{12} + u_{12}^{iv} = -2D_0 D_1 u_{11} + \frac{1}{2}\left[\int_0^\eta u'_{11}2\,\mathrm{d}x + \int_\eta^1 u'_{21}2\,\mathrm{d}x\right]u''_{11} - 2\mu D_0 u_{11} + F_1 \cos \Omega T_0,$$
$$D_0^2 u_{22} + u_{22}^{iv} = -2D_0 D_1 u_{21} + \frac{1}{2}\left[\int_0^\eta u'_{11}2\,\mathrm{d}x + \int_\eta^1 u'_{21}2\,\mathrm{d}x\right]u''_{21} - 2\mu D_0 u_{21} + F_2 \cos \Omega T_0, \tag{16, 17}$$

$$u_{12}(0, T_0, T_1) = u''_{12}(0, T_0, T_1) = u_{22}(1, T_0, T_1) = u''_{22}(1, T_0, T_1) = 0,$$
$$u_{12}(\eta, T_0, T_1) = u_{22}(\eta, T_0, T_1) = \alpha(T_0, \mathrm{T}_1),$$
$$u'_{12}(\eta, T_0, T_1) = u'_{22}(\eta, T_0, T_1), \quad u''_{12}(\eta, T_0, T_1) = u''_{22}(\eta, T_0, T_1). \tag{18}$$

At order 1, solutions of the form

$$u_{11} = (A(T_1)\mathrm{e}^{\mathrm{i}\omega T_0} + cc)Y_1(x), \quad u_{21} = (A(T_1)\mathrm{e}^{\mathrm{i}\omega T_0} + cc)Y_2(x) \tag{19, 20}$$

is assumed, where cc stands for the complex conjugate of the preceding terms. Substituting Eqs. (19) and (20) into Eqs. (13)–(15), one has

$$Y_1^{iv} - \omega^2 Y_1 = 0, \quad Y_2^{iv} - \omega^2 Y_2 = 0, \tag{21, 22}$$

$$Y_1(0) = Y''_1(0) = Y_2(1) = Y''_2(1) = 0,$$
$$Y_1(\eta) = Y_2(\eta) = 0 \;\; Y'_1(\eta) = Y'_2(\eta) \;\; Y''_1(\eta) = Y''_2(\eta). \tag{23}$$

Solving Eqs. (21)–(23) exactly yields the mode shapes,

$$Y_1(x) = C \sin \beta(1-\eta)\left[\sin \beta x - \frac{\sin \beta\eta}{\sinh \beta\eta}\sinh \beta x\right],$$
$$Y_2(x) = C \sin \beta\eta\left[\sin \beta(1-x) - \frac{\sin \beta(1-\eta)}{\sinh \beta(1-\eta)}\sinh \beta(1-x)\right], \tag{24, 25}$$

and the natural frequencies ω satisfy the transcendental equation

$$\sin \beta\eta \sinh \beta\eta[\cos \beta(1-\eta)\sinh \beta(1-\eta) - \sin \beta(1-\eta)\cosh \beta(1-\eta)]$$
$$+ \sin \beta(1-\eta)\sinh \beta(1-\eta)[\cos \beta\eta \sinh \beta\eta - \sin \beta\eta \cosh \beta\eta] = 0, \tag{26}$$

where

$$\beta = \sqrt{\omega}. \tag{27}$$

Eq. (26) is solved numerically for the first seven modes and results are given in Table 1 for different η values. Due to the symmetry of the problem, results are given up to $\eta = 0.5$. Inspecting Eq. (26), one finds that for some specific β values $\sin \beta\eta$ and $\sin \beta(1-\eta)$ both vanish, and for those degenerate cases the mode shapes take the simpler form

$$Y_1(x) = C \sin \beta x, \quad Y_2(x) = -C \sin \beta(1-x). \tag{28, 29}$$

Table 1
β values for different support locations (η)

η	β_1	β_2	β_3	β_4	β_5	β_6	β_7
0.1	4.2264	7.6313	11.0505	14.4793	17.9123	21.3435	24.7626
0.2	4.6183	8.3915	12.1617	15.7080	17.8725	20.4610	24.1149
0.3	5.1318	9.2769	11.7804	14.2845	18.4048	21.7675	23.5619
0.4	5.7826	8.7679	11.3129	15.7080	17.3296	21.4939	24.4763
0.5	6.2832	7.8532	12.5664	14.1372	18.8496	20.4204	25.1327

Because the homogeneous equations (13)–(15) have a non-trivial solution, the non-homogeneous problem (16)–(18) will have a solution only if a solvability condition [4] is satisfied. To determine this condition, the secular and non-secular terms are separated by assuming a solution of the form

$$u_{12} = \varphi_1(x, T_1)e^{i\omega T_0} + W_1(x, T_0, T_1) + cc,$$
$$u_{22} = \varphi_2(x, T_1)e^{i\omega T_0} + W_2(x, T_0, T_1) + cc. \qquad (30, 31)$$

Substituting this solution into Eqs. (16)–(18), secular and non-secular terms separate (for this order, only the secular ones are of interest):

$$\varphi_1^{iv} - \omega^2\varphi_1 = -2i\omega D_1 A Y_1 + \frac{3}{2}A^2\bar{A}\left(\int_0^\eta Y_1'2\,dx + \int_\eta^1 Y_2'2\,dx\right)Y_1'' - 2\mu i\omega A Y_1 + \frac{F_1}{2}e^{i\sigma T_1},$$
$$\varphi_2^{iv} - \omega^2\varphi_2 = -2i\omega D_1 A Y_2 + \frac{3}{2}A^2\bar{A}\left(\int_0^\eta Y_1'2\,dx + \int_\eta^1 Y_2'2\,dx\right)Y_2'' - 2\mu i\omega A Y_2 + \frac{F_2}{2}e^{i\sigma T_1}. \qquad (32, 33)$$

In obtaining these equations, the order 1 solutions (19) and (20) are substituted into Eqs. (16)–(18). It is also assumed that the external excitation frequency is close to one of the natural frequencies of the system; such that

$$\Omega = \omega + \varepsilon\sigma. \qquad (34)$$

Here σ is a detuning parameter of order 1. After algebraic manipulations, the solvability conditions for Eqs. (32) and (33) are obtained

$$2i\omega(D_1 A + \mu A) + \frac{3}{2}b^2 A^2\bar{A} + kA(Y_2'''(\eta) - Y_1'''(\eta)) - \frac{1}{2}f e^{i\sigma T_1} = 0, \qquad (35)$$

where

$$f = \int_0^\eta F_1 Y_1\,dx + \int_\eta^1 F_2 Y_2\,dx, \qquad (36)$$

and

$$b = \int_0^\eta Y_1'2\,dx + \int_\eta^1 Y_2'2\,dx. \qquad (37)$$

In obtaining Eq. (35) the normalization condition $\int_0^\eta Y_1^2\,dx + \int_\eta^1 Y_2^2\,dx = 1$ is employed. The amplitude of the deflection allowed at the intermediate point is assumed to be of the same form as that of order 1 solution, namely

$$\alpha(T_0, T_1) = kA(T_1)e^{i\omega T_0} + cc, \tag{38}$$

where k is an arbitrary constant of order 1. Eq. (35) determines the modulations in the complex amplitudes. The polar form

$$A = \frac{1}{2}a(T_1)e^{i\theta(T_1)} \tag{39}$$

is to be used to calculate real amplitudes and phases. After separating real and imaginary parts, one obtains

$$\begin{aligned} \omega a' &= -\mu\omega a + \frac{1}{2}f\sin\gamma, \\ \omega a\gamma' &= \omega a\sigma - \frac{3}{16}b^2a^3 - Ka + \frac{1}{2}f\cos\gamma, \end{aligned} \tag{40, 41}$$

where γ and K are defined as

$$\begin{aligned} \gamma &= \sigma T_1 - \theta, \\ K &= \frac{k}{2}\left[Y_2'''(\eta) - Y_1'''(\eta)\right]. \end{aligned} \tag{42}$$

In the steady state case, $a' = \gamma' = 0$ and solving for the detuning parameter yields

$$\sigma = \frac{K}{\omega} + \frac{3}{16\omega}b^2a^2 \pm \sqrt{\frac{f^2}{4\omega^2a^2} - \mu^2}. \tag{43}$$

For free undamped vibrations, non-ideal non-linear natural frequencies are

$$\omega_{ni} = \omega + \varepsilon\left(\frac{3}{16\omega}b^2a^2 + \frac{K}{\omega}\right). \tag{44}$$

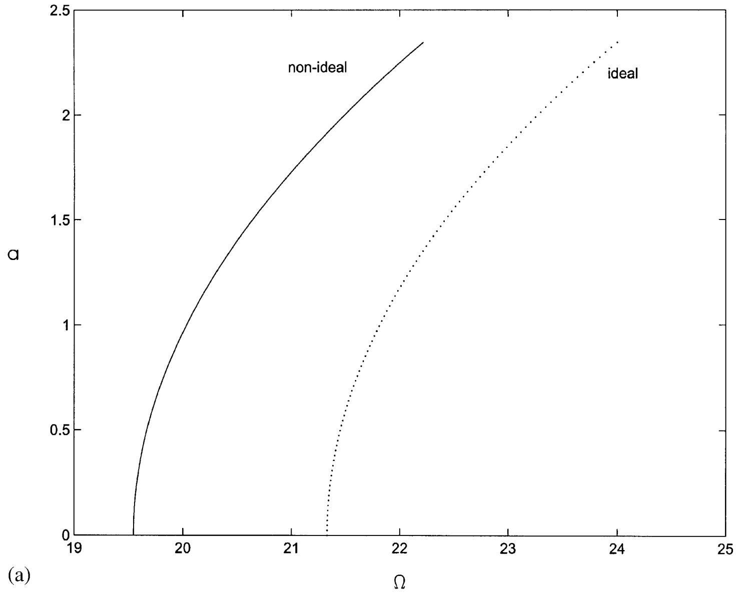
In the above relation, the first term in $O(\varepsilon)$ is due to the non-linearity and the second term is due to the non-ideal boundary condition. If $b = 0$ is taken, non-ideality effects will be isolated. Non-ideal and ideal frequencies are contrasted in Table 2 for the first five frequencies and for different location parameters.

As a general rule, intermediate non-ideal boundary condition causes a decrease in odd numbered frequencies and an increase in even numbered frequencies. However this general rule ceases to be valid for degenerate roots. Degenerate β values are defined previously as the values

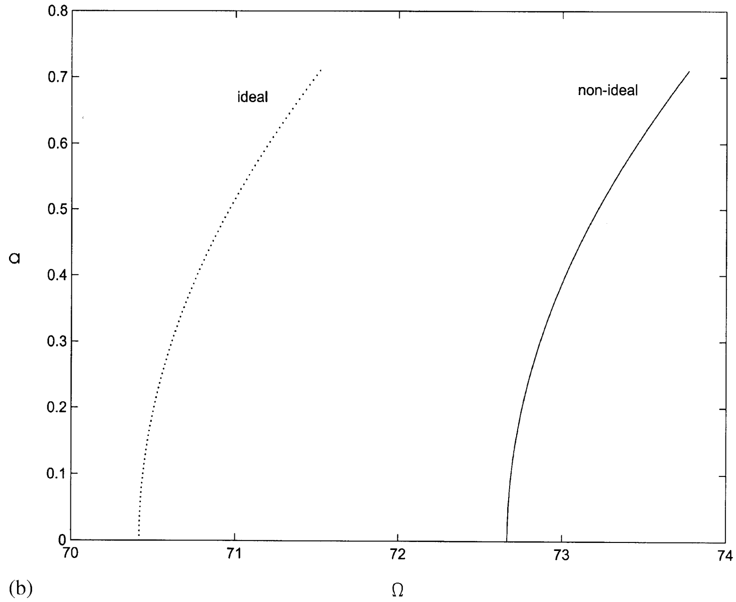
Table 2
Ideal and non-ideal natural frequencies ($\varepsilon k/2 = 0.1$)

η	ω_{1i}	ω_{1ni}	ω_{2i}	ω_{2ni}	ω_{3i}	ω_{3ni}	ω_{4i}	ω_{4ni}	ω_{5i}	ω_{5ni}
0.1	17.8622	15.1822	58.2367	61.4112	122.1138	118.4668	209.6487	213.7117	320.8512	316.4768
0.2	21.3289	19.5473	70.418	72.6689	147.9077	145.6414	246.7401	242.2972	319.4266	312.1935
0.3	26.3352	24.7986	86.0614	87.2882	138.7773	134.8397	204.0458	208.9341	338.7358	335.0162
0.4	33.4385	32.1416	76.8753	79.3723	127.9821	124.2754	246.7401	251.183	300.3159	293.3388
0.5	39.4784	39.4784	61.6728	64.8165	157.9137	157.9137	199.8594	205.5143	355.3058	355.3058

which causes $\sin\beta\eta$ and $\sin\beta(1-\eta)$ both vanish. The degenerate values ($\omega = \beta^2$) in Table 2 are fourth frequencies for $\eta = 0.2$ and 0.4, odd frequencies for $\eta = 0.5$. A centered non-ideal support does not change the odd frequencies since there is a node at this location for these frequencies. For $\eta = 0.4$, the fourth frequency does not violate the general rule, i.e., increase for even modes. However, the general rule is violated for the fourth frequency of $\eta = 0.2$.



(a)



(b)

Fig. 2. Non-linear frequencies versus amplitudes for ideal (dotted) and non-ideal (solid) cases: (a) first mode ($\eta = 0.2$); (b) second mode ($\eta = 0.2$).

In Fig. 2a fundamental non-linear frequencies versus amplitudes are contrasted for the ideal and non-ideal cases for $\eta = 0.2$. The second mode comparison is given in Fig. 2b for the same η value (location parameter). In Figs. 3a and b, the first and second modes of amplitude-dependent vibration frequencies are contrasted for $\eta = 0.4$.

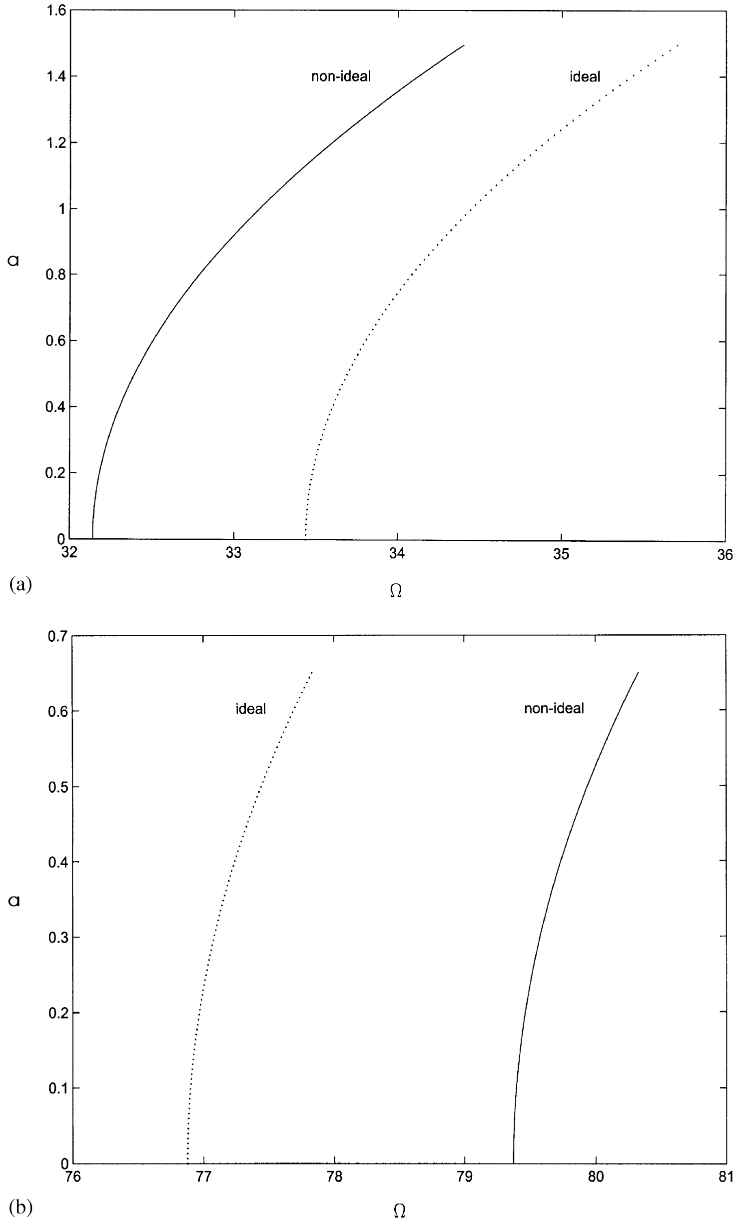
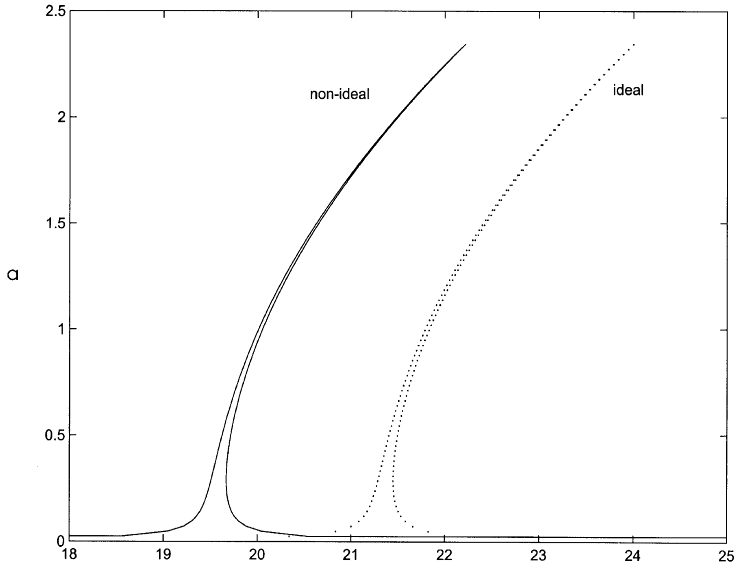
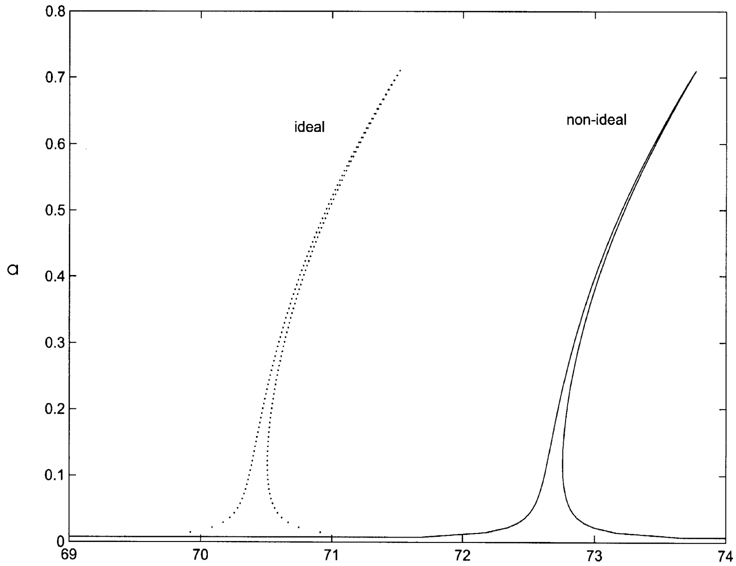


Fig. 3. Non-linear frequencies versus amplitudes for ideal (dotted) and non-ideal (solid) cases: (a) first mode ($\eta = 0.4$); (b) second mode ($\eta = 0.4$).



(a)



(b)

Fig. 4. Frequency-response curves for ideal (dotted) and non-ideal (solid) cases: (a) first mode ($\eta = 0.2$, $\varepsilon\mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$); (b) second mode ($\eta = 0.2$, $\varepsilon\mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$).

One can also obtain amplitude-excitation frequency relation from Eqs. (43) and (34)

$$\Omega = \omega + \varepsilon \frac{3}{16\omega} b^2 a^2 + \varepsilon \frac{K}{\omega} \pm \sqrt{\frac{\varepsilon^2 f^2}{4\omega^2 a^2} - (\varepsilon\mu)^2}. \tag{45}$$

In Fig. 4a, frequency-response graphs for the first modes are compared for the ideal and non-ideal cases for $\eta = 0.2$, $\varepsilon\mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$. In Fig. 4b the second modes are contrasted for the same parameter values. In Figs. 5a and b the first and second modes are contrasted for $\eta = 0.4$, and all other parameters remaining the same. The non-ideal frequencies may increase, decrease or remain unchanged depending on the position parameter η and number

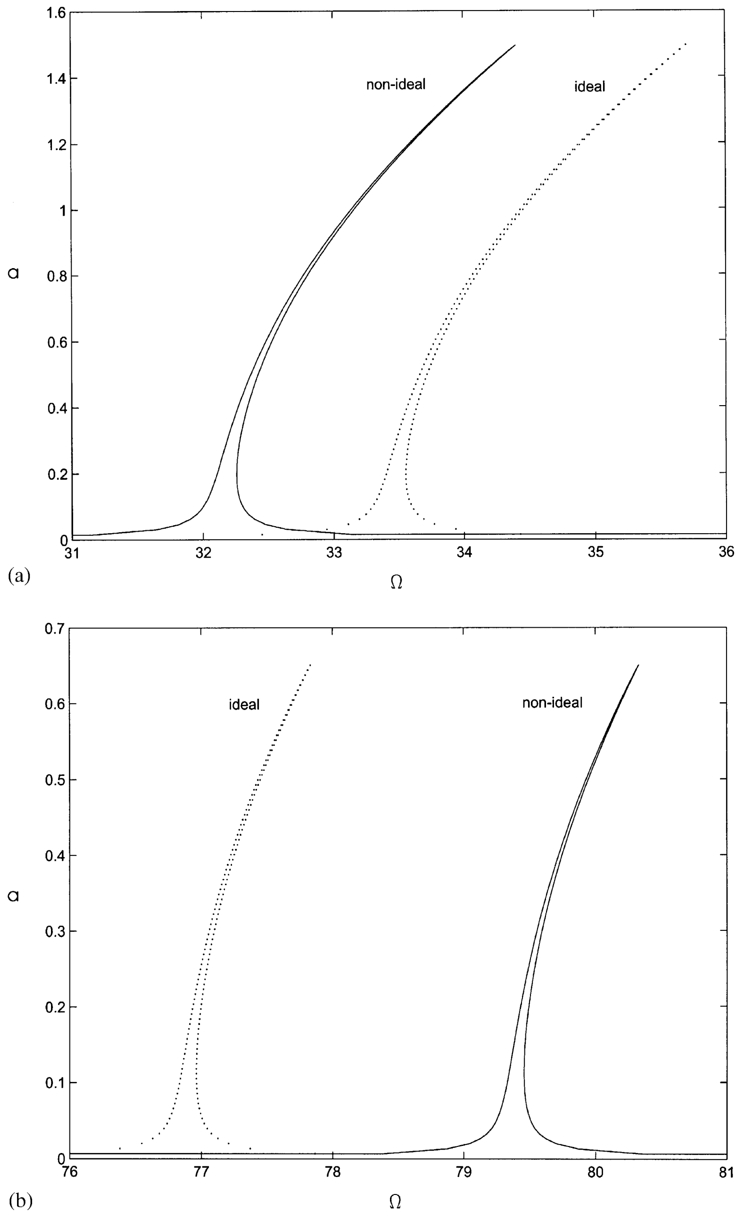


Fig. 5. Frequency-response curves for ideal (dotted) and non-ideal (solid) cases: (a) first mode ($\eta = 0.4$, $\varepsilon\mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$); (b) second mode ($\eta = 0.4$, $\varepsilon\mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$).

of modes. The non-ideality causes a negative, positive or zero drift in the frequency-response curves depending on the location and mode numbers.

The approximate beam deflections to the first order are as follows:

$$w_1 = \sqrt{\varepsilon}\,[a\cos(\Omega t - \gamma)Y_1(x) + O(\varepsilon)],$$
$$w_2 = \sqrt{\varepsilon}\,[a\cos(\Omega t - \gamma)Y_2(x) + O(\varepsilon)], \qquad (46,47)$$

where $Y_1(x)$ and $Y_2(x)$ are given in Eqs. (24,25) or Eqs. (28,29).

3. Concluding remarks

Non-ideal boundary conditions are defined and formulated using perturbation theory. A simply supported beam with a non-ideal simple support at an intermediate point is treated. Non-linear terms are introduced because of the immovable end conditions leading to beam stretching. Approximate analytical solution of the problem is presented using the method of multiple scales. Ideal and non-ideal non-linear natural frequencies are given for different intermediate support locations. Depending on the mode numbers and locations, the frequencies may increase, decrease or remain unchanged. Deviations from the ideal conditions lead to a drift in frequency-response curves which may be positive, negative or zero depending on the mode number and location.

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