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Collision dynamics of two adjacent oscillators

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Abstract

A linear analysis is made of a single collision between two single-degree-of-freedom systems separated by a gap. The contact is modelled by a spring and a viscous damper. The approach is to describe the motion of the pair as being composed of sum and difference displacements. The equation of motion during contact is found and the solution is obtained from the conditions at initial contact. The main parameters are the ratio of strain energy to kinetic energy at initial contact, and the damping of the contact. The contact time and the energy loss are calculated, which gives an expression for the coefficient of restitution for the collision. This coefficient is found to be dependent on the collision velocity, but becomes constant for strong collisions.

1. Introduction

Whenever two mechanical systems collide there is an exchange of momentum and also energy is dissipated in the high stress region of contact. This energy dissipation is the work involved in damped elastic behavior and also plastic deformation and fracture. Although there are many practical examples available, the case of interest here is the collision of a line of adjacent building structures during seismic activity [1,2]. This paper is presented as the first step in the analysis of this problem, but is also has general application.

Collision problems are usually approached by assuming that the contact stiffness is large in comparison to the oscillator stiffness, which justifies the use of a ‘‘coefficient of restitution’’ that is constant for impacts of any strength [1,3–6]. This coefficient is usually derived from the collision of a free body with a general boundary, which assumes that at the moment of initial contact all the energy is kinetic and is available for transfer to the boundary. However, if an oscillator rather

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than a free body is in collision with a boundary, at the moment of initial contact strain energy remains stored within the oscillator stiffness, and is therefore not available for transfer to the boundary. The conventional coefficient of restitution cannot thus be used with complete confidence.

The purpose of this paper is therefore to investigate a single collision between two oscillators separated by a damped contact stiffness, to determine: the energy loss, contact duration and coefficient of restitution as a function of the physical parameters, and the velocities and displacements at initial contact.

Each oscillator consists of a mass and a spring with viscous damping. The contact is also modelled as a spring with a viscous damper as in Ref. [7]. There is a separation between the oscillators when at rest. This choice of damping was made simply because the decay rate of the time response is controlled only by the damping at the resonance frequencies, and is independent of the general frequency characteristic of the damping mechanism. Other possible damping mechanisms more representative of real materials can be composed from a summation of Kelvin–Voigt relaxation elements [8]. These however add to the oscillatory response of a non-oscillatory decay response, the form of which is particular to the exact frequency response between zero and the resonance frequency. The solution to these problems is therefore possible but are not of general application.

The analysis here only considers the actual contact period between two oscillators, and deploys a device where the oscillator motions are not described independently but rather as a pair with two almost uncoupled modes. One mode describes the mean or in-phase motion, which is not greatly affected by the collision. The other mode describes the relative motion and hence the collision dynamics. This basic assumption is only rigorously true for identical oscillators; however, it is shown that it also holds good for a broader application described as "similar oscillators" during the short contact interval following the initial conditions.

The benefit of this representation of "similar oscillators" (such as adjacent buildings) is that only the relative motion is altered by the collision. This implies that the relative motion can be modelled as single degree-of-freedom system, leading to closed form solutions for contact time, momentum exchange, energy dissipation and coefficient of restitution.

The solution depends upon an important non-dimensional parameter called "impact strength", which is the ratio of the oscillator kinetic energy to strain energy at the moment of initial contact. For impact strength greater than unity the solution for relative motion tends to that for a viscously damped contact stiffness applied to a single degree of freedom [6]; the coefficient of restitution and contact time then become independent of impact strength as is generally assumed in the previous work [1,5].

2. Equations of motion

Consider Fig. 1 in which two oscillators a, b, are spaced a distance Δ, and have masses m_a, m_b, viscous damping c_a, c_b and real stiffnesses k'_a, k'_b. The distances u_a, u_b describe the displacements relative to the equilibrium position.

Before contact the oscillators are in free motion, solutions of

$$K_a u_a = 0, \quad K_b u_b = 0, \tag{1a, b}$$

where

$$K_a = \mathrm{D}^2 m_a + \mathrm{D}c_a + k'_a, \quad K_b = \mathrm{D}^2 m_b + \mathrm{D}c_b + k'_b,$$

and D is a differential operator d/dt. The solution to equations 1a and 1b include the two oscillator damped natural frequencies ω_a and ω_b.

The displacements may also be described by the sum and difference displacements u_s, u_d,

$$u_\mathrm{s} = u_a + u_b, \quad u_\mathrm{d} = u_a - u_b \tag{2a, b}$$

and the sum and difference dynamic stiffnesses K_s, K_d, i.e.,

$$K_\mathrm{s} = K_a + K_b, \quad K_\mathrm{d} = K_a - K_b. \tag{3a, b}$$

The sum displacement in Eq. (2a) is the in-phase component of the two oscillator motions which is therefore almost unaffected by the collisions. The difference displacement in Eq. (2b) describes the relative motion outside and within the contact.

Within the contact when $u_\mathrm{d} > \Delta$, a contact force F in proportion to the product of complex contact stiffness, $\mathbf{k} = k' + \mathrm{D}c$, and relative displacement u_r, occurs in accordance with Fig. 2. The stiffness of the contact is k' and the viscous damping constant is c. The relative displacement u_r used during contact is simply related to the general difference term u_d by:

$$\mathbf{F} = \mathbf{k}u_\mathrm{r}, \quad u_\mathrm{r} = u_\mathrm{d} - \Delta. \tag{4a, b}$$

The contact force acts equally on both oscillators, thus

$$K_a u_a = -\mathbf{F}, \quad K_b u_b = \mathbf{F}. \tag{5a, b}$$

Elimination of this force yields the equations for conservation of linear momentum, i.e.,

$$K_a u_a + K_b u_b = 0. \tag{6}$$

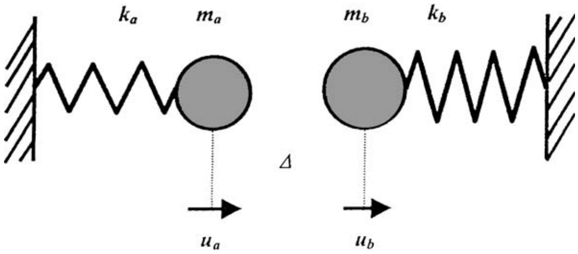

Fig. 1. Two separated oscillators.

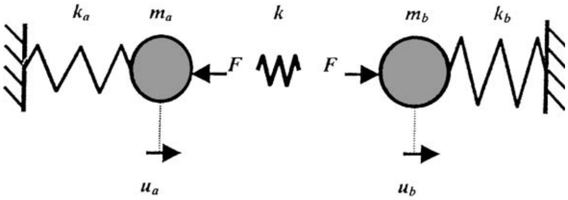

Fig. 2. Contact model for $u_\mathrm{d} > \Delta$.

By substituting from Eqs. (2) and (3) these may written in the sum and difference form of the alternative oscillator pair as

$$K_s u_s + K_d u_d = 0. \tag{7}$$

For the special case of identical oscillators, i.e. $K_a = K_b$ Eqs (3b) and (7) show that the sum and difference motion is entirely independent, i.e.,

$$K_d u_d = 0, \quad K_s u_s = 0.$$

When contact occurs, only the difference motion will be modified, which is the central idea here. To investigate how far the identical oscillator result can be extended the analysis is now continued for the general case. Eqs. (4)–(6) can be combined to give the equation of motion for difference displacement during the contact.

$$(K_a K_b / K_s + \mathbf{k}) u_d = \mathbf{k}\Delta, \tag{8}$$

the solution of which yields the difference displacement, contact period, contact force and energy dissipation. The change in sum displacement u_s can then be calculated from the momentum, Eq. (7).

3. Simplification of the equations of motion

The full solution of Eq. (8) for difference motion during contact is rather awkward as it is for a fourth-order differential equation. It will be shown however that if the oscillators have a similar scale it may be reduced to an equation of the second order for the short time of contact. This simplified equation yields the expression for contact duration and ultimately the coefficient of restitution. The first group of terms may be referred to as the oscillator "interaction dynamic stiffness" K_{d0}, i.e.,

$$K_{d0} = K_a K_b / (K_a + K_b). \tag{9a}$$

Using Eqs. (1a) and (1b) and a harmonic solution $e^{i\omega t}$ it is expanded as

$$K_{d0} = (\mathbf{k}_a - \omega^2 m_a)(\mathbf{k}_b - \omega^2 m_b) / (\mathbf{k}_a + \mathbf{k}_b - \omega^2 (m_a + m_b)). \tag{9b}$$

For a compact form the complex stiffness $\mathbf{k}_a = k'_a + i\omega c_a$, $\mathbf{k}_b = k'_b + i\omega c_b$, are defined. Eq. (9b) is plotted in Fig. 3. At low frequencies in zone (1) the dynamic stiffness K_{d0} asymptotes to the complex "interaction stiffness" $\mathbf{k}_{d0} = \mathbf{k}_a \mathbf{k}_b / (\mathbf{k}_a + \mathbf{k}_b)$:

$$K_{d0} = \mathbf{k}_{d0} \tag{10}$$

At high frequencies in zone (2) the dynamic stiffness asymptotes to the interaction mass $m_{d0} = m_a m_b / (m_a + m_b)$ are

$$K_{d0} = -\omega^2 m_{d0} \tag{11}$$

where the "free interaction frequency" is defined from $\omega_{d0}^2 = \mathbf{k}_{d0} / m_{d0}$. It will be seen later that the interaction mass and stiffness describe the portion of the oscillator mass and stiffness that participates in the collision.

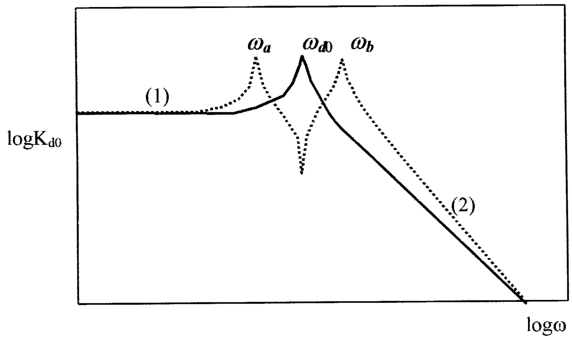

Fig. 3. The dynamic stiffness (……..) and its approximation(——).

This would suggest a dynamic stiffness with an approximate form seen in Fig. 3, corresponding to that of a single degree-of-freedom system

$$K_{\mathrm{d0}} \approx \mathbf{k}_{\mathrm{d0}} - \omega^2 m_{\mathrm{d0}}, \tag{12}$$

To demonstrate that such an approximation is valid for the short contact duration, the free vibration solution of Eq. (8) must be considered. This has roots at $\pm\omega_a$, $\pm\omega_b$ one pair from each equation of motion (1a), (1b) and in the numerator of Eq. (9). The oscillator displacements u_a, u_b are

$$u_a = A_1 \mathrm{e}^{-\mathrm{i}\omega_a t} + A_2 \mathrm{e}^{\mathrm{i}\omega_a t}, \quad u_b = B_1 \mathrm{e}^{-\mathrm{i}\omega_b t} + B_2 \mathrm{e}^{\mathrm{i}\omega_b t}, \tag{13a, b}$$

where A_1, A_2, B_1, B_2 are constants dependent upon the conditions at initial contact at time $t = 0$

$$\dot{u}_a = \dot{U}_a, \ \ \dot{u}_b = \dot{U}_b, \quad u_a = U_a, \ \ u_b = U_b,$$

For real values of initial displacement and velocity equation (13) yields

$$A_2 = A_1^*, \quad B_2 = B_1^*,$$

where * denotes the complex conjugate. As time is measured from initial contact the relative displacement u_{r} is the difference between u_a and u_b in Eqs. (13a,b)

$$u_{\mathrm{r}} = 2\Re(A_1 \mathrm{e}^{-\mathrm{i}\omega_a t} - B_1 \mathrm{e}^{-\mathrm{i}\omega_b t}), \tag{14}$$

where $\Re$ denotes the real part. As the relative motion is zero at initial contact, i.e., $t = 0$, then Eq. (14) gives

$$A_1 = B_1. \tag{15}$$

The relative motion may be further simplified by defining ϖ as the mean of ω_α and ω_b as in Fig. 3, and defining ω_δ as half the difference

$$\varpi = (\omega_a + \omega_b)/2, \quad \omega_\delta = (\omega_a - \omega_b)/2. \tag{16a, b}$$

Substitution of Eqs. (16) and (15) into (14) gives the relative displacement as

$$u_{\mathrm{r}} = 4\Re(A_1 \ \exp(-\mathrm{i}\varpi t))\cos\omega_\delta t. \tag{17}$$

If the impacting pair of oscillators have similar natural frequencies then $\varpi \gg \omega_\delta$, in which case for the contact duration (corresponding to $\pi > \varpi t$), the $\cos\omega_\delta t$ term stays close to unity and so can be neglected. The relative motion then takes the form of the impulse response of a single degree-of-freedom system of natural frequency ϖ. This shows that the approximate form of Eq. (12) is appropriate, provided that $\varpi \approx \omega_{d0}$. To determine the range of values for which this is a valid approximation, the quotient ω_0/ϖ is expanded from Eqs. (11) and (16) to give

$$\omega_{d0}/\varpi = ((1+\mu_0)/(1+\mu_0-(1-\mu_0)\omega_\delta/\varpi))^{1/2}, \tag{18}$$

where $\mu_0 = m_a/m_b$. For identical oscillators $\omega_{d0}/\varpi = 1$, as expected. However even if $\mu_0 = 2$, and $\omega_\delta/\varpi = 1$, as for rather dissimilar oscillators, the quotient $\omega_{d0}/\varpi = 0.86$, which is still sufficiently close to unity to suggest that the approximation is still useful and has a broad range of application.

4. Solution for relative motion during contact

In the previous section some justification was given for expressing the difference displacement u_d as a simplified form of Eq. (8)

$$(K_{d0} + \mathbf{k})u_d = \mathbf{k}\Delta,$$

which on substitution from Eq. (12) and applying a harmonic solution $e^{i\omega t}$, becomes

$$(-\omega^2 m_{d0} + i\omega(c_{d0} + c) + k'_{d0} + k')u_d = \mathbf{k}\Delta. \tag{19}$$

The solution has two parts; the complementary function u_{dc} and particular integral u_{dp}. The particular integral is the steady state response arising from the constant forcing function on the right side

$$u_{dp} = k'\Delta/(k'_{d0} + k'). \tag{20}$$

The complementary function is found by setting the right side of Eq. (19) to zero

$$(-\omega^2 m_{d0} + i\omega(c_{d0} + c) + k'_{d0} + k')u_{dc} = 0 \tag{21}$$

and finding a solution u_{dc} which satisfies the conditions at initial contact $t = 0$. The pair of complex roots from Eq (21) are: $\omega = \boldsymbol{\omega}_r$, $\omega = \boldsymbol{\omega}_r^*$ where the complex form of $\boldsymbol{\omega}_r$ is for light damping, $\zeta_r < 1$, in this analysis

$$\boldsymbol{\omega}_r = \omega_{rr} + i\gamma_{rr}. \tag{22}$$

These give the "interaction frequency" that governs the contact time and the dissipation of energy during contact. This is ω_{d0} in Eq. (11), now modified by the contact stiffness

$$\boldsymbol{\omega}_r = \omega'_{d0}(1 + k'/k'_{d0})\left(\sqrt{1-\zeta_r^2} + i\zeta_r\right), \tag{23}$$

where the viscous damping coefficient is

$$\zeta_r = (c + c_{d0})/2\sqrt{m_{d0}(k' + k'_{d0})}. \tag{24}$$

The general solution is the sum of the complementary function and particular integral,

$$u_d = A_r e^{-i\boldsymbol{\omega}_r^* t} + B_r e^{-i\boldsymbol{\omega}_r t} + u_{dp}. \tag{25}$$

The constants A_r, B_r can now be found from the initial conditions. At $t = 0$, $u_d = \Delta$, giving

$$\Delta_r = A_r + B_r. \tag{26}$$

The modified separation Δ_r is obtained using Eq. (20),

$$\Delta_r = \Delta k'_{d0}/(\mathbf{k}_{d0} + k') \tag{27}$$

Differentiation of Eq. (25) gives the relative velocity

$$\dot{u}_d = -i\boldsymbol{\omega}_r^* A_r e^{-i\boldsymbol{\omega}_r^* t} + i\omega B_r e^{i\boldsymbol{\omega}_r t} \tag{28}$$

The relative velocity at $t = 0$ is defined as

$$\dot{U}_d = \dot{U}_a + \dot{U}_b, \tag{29}$$

which on substitution into Eq. (28) gives

$$\dot{U}_d = -i\boldsymbol{\omega}_r^* A_r + i\boldsymbol{\omega}_r B_r. \tag{30}$$

Rearrangement of Eqs. (25) and (30) give A_r and B_r:

$$A_r,\ B_r = (\Delta_r \boldsymbol{\omega}_r \pm i\dot{U}_d)/(\boldsymbol{\omega}_r + \boldsymbol{\omega}_r^*). \tag{31}$$

These are a complex conjugate pair because the relative velocity in Eq. (28) must be real. Substitution of Eq. (31) into Eq. (25) gives the relative displacement during contact u_r, where u_r is a special case of the difference displacement u_d that applies during contact as defined in Eq. (4)

$$u_r = (\Delta_r \cos\omega_{rr} t + (\Delta_r \delta_r + \dot{U}_d/\omega_{rr}) \sin\omega_{rr} t) e^{-\gamma_{rr} t} - \Delta_r. \tag{32}$$

A "loss ratio" δ_r is defined from Eqs. (22) and (23) as a function of the viscous damping coefficient. As the damping coefficient can take the range between zero and unity, the "loss ratio" in Eq. (33) covers the full range of zero to infinity:

$$\delta_r = \gamma_{rr}/\omega_{rr} = \zeta_r/\sqrt{1 - \zeta_r^2}, \quad \zeta_r < 1. \tag{33}$$

The displacement amplitude is controlled by two input parameters: $\dot{U}_d/\omega_{rr}$ and Δ_r. The first is the maximum displacement that would occur if there was no initial separation between the oscillators, and is related to the kinetic energy at contact T_d. The second is Δ_r, a negative displacement step reducing the depth of contact; it is related to the strain energy at contact S_{0d}. Eq. (27) shows that this term increases both with the oscillator stiffness k_a, k_b, relative to the contact stiffness k, and also the initial separation Δ. The ratio of these two displacements gives a non-dimensional parameter referred to here as the "impact strength", $\hat{\beta}$

$$\hat{\beta} = \dot{U}_d/(\omega_{rr}\Delta_r). \tag{34a}$$

This is related to the "impact ratio" at initial contact β by

$$\beta^2 = T_d/S_{0d}, \tag{34b}$$

where $\hat{\beta}/\beta = \sqrt{k_{d0} + \mathbf{k}/k_{d0}}$.

Eq. (32) can be regrouped using this term and the expansion:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \tag{35}$$

giving the relative displacement during contact as

$$u_r = \Delta_r\left(\frac{\sin(\omega_{rr}t+\phi)e^{-\gamma_{rr}t}}{\sin\phi}-1\right), \tag{36}$$

where

$$\sin\phi = 1/\sqrt{1+(\delta_r+\hat{\beta})^2} \tag{37}$$

and

$$\tan\phi = 1/(\delta_r+\hat{\beta}). \tag{38}$$

Eq. (36) is plotted in Fig. 4(a). The relative displacement is zero at initial contact and has a maximum at a time $t = t_a$ of

$$\hat{u}_r = \Delta_r\left(\sqrt{1+\hat{\beta}^2}-1\right). \tag{39}$$

Before further interrogation of Eq. (36) a relatively brief energy analysis is made of the collision to confirm the outcome and to give some physical insight.

5. Energy relationships and contact duration

Fig. 5 shows the force-difference displacement curve. Between $0 > u_d > \Delta$ there is no contact and the gradient is k_{d0} which is defined in Eq. (12). When $u_d = \Delta$ initial contact is made, thereafter for $u_d > \Delta$ the gradient is that of the combined stiffness, i.e., $\mathbf{k}_{d0} + \mathbf{k}$. At the instant of contact the kinetic energy T_d is

$$T_d = \tfrac{1}{2}\, m_{d0}\dot{U}_d^2. \tag{40}$$

The strain energy $\hat{S}_d$ stored between contact and the peak displacement is the area under the curve $u_d > \Delta$, i.e.,

$$\hat{S}_d = \tfrac{1}{2}\,(\mathbf{k}_{d0}+\mathbf{k})\hat{u}_r^2 + \mathbf{k}_{d0}\hat{u}_r\Delta. \tag{41}$$

This strain energy is shared between the contact stiffness k and the oscillator interaction stiffness k_{d0}. Equating this strain energy to the kinetic energy gives a quadratic in terms of the peak displacement

$$0 = \hat{u}_r^2 + 2\Delta_r\hat{u}_r - (\dot{U}_d/\omega_{rr})^2. \tag{42}$$

The peak relative displacement, taken from the positive root gives the same result as Eq. (39), which was obtained with rather more labour. This equation is however not very convenient and so an approximation is selected which is accurate at the asymptotes $1 \ll \hat{\beta} \gg 1$ and has at worst a 10% error when $\hat{\beta} = 1$, i.e.,

$$\hat{u}_r = (\dot{U}_d/\omega_{rr})\sqrt{\hat{\beta}^2/(4+\hat{\beta}^2)}. \tag{43}$$

The peak strain energy within the contact stiffness alone $\hat{S}_{dc}$ is taken from the square of the peak relative displacement in Eq. (43). This is the energy available for dissipation or damage in

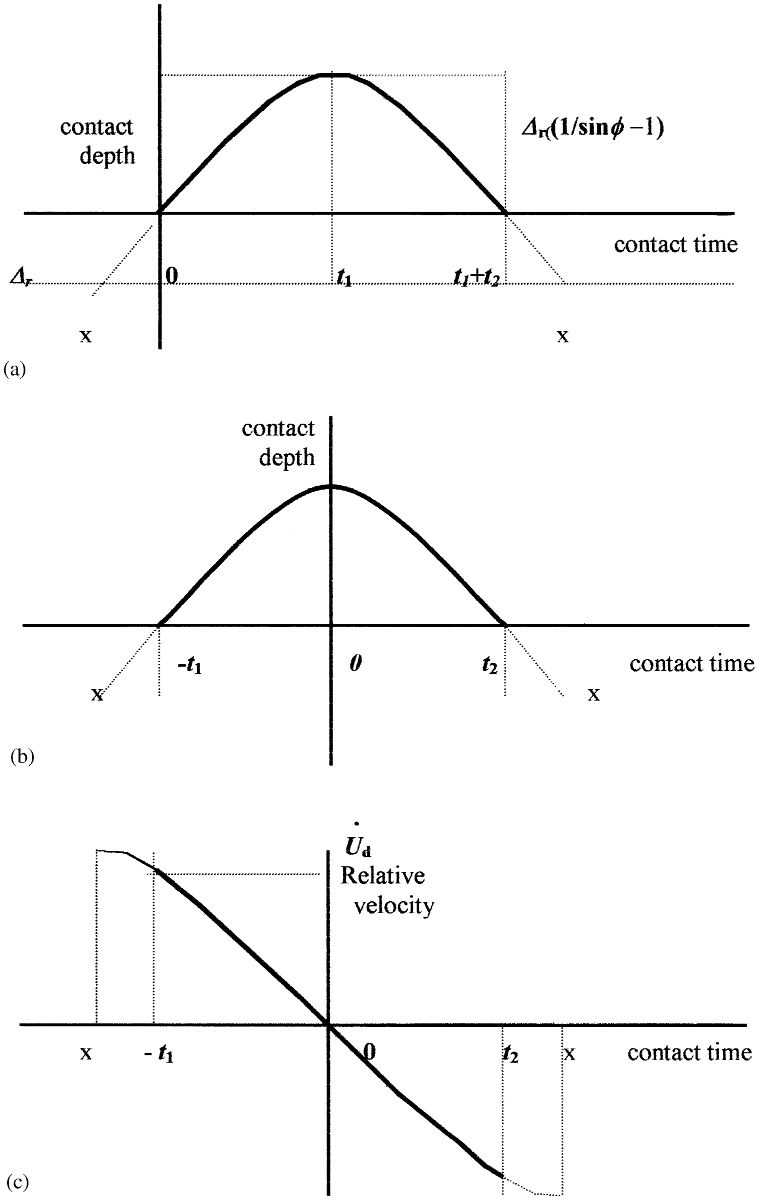

Fig. 4. (a) Contact depth against time; (b) time shifted contact depth; (c) time shifted relative velocity.

the contact

$$\hat{S}_{dc} = \tfrac{1}{2}\,\mathbf{k}(\dot{U}_d/\omega_{rr})^2\hat{\beta}^2/(4+\hat{\beta}^2). \tag{44}$$

If the energy ratio $\hat{\beta} \gg 2$, more energy is stored within the contact stiffness $\mathbf{k}$ than in the oscillator stiffness' $\mathbf{k}_{d0}$. Under this condition the oscillator stiffness can be neglected and the collision is that between free bodies, when the usual constant coefficient of restitution could be used for calculations. Eq. (36) shows that the contact pulse approaches a simple half sine, seen in Fig. (6). If however there is only a light impact such that $\hat{\beta} \ll 2$ then the collision dynamics are more controlled by the oscillator stiffness than the contact stiffness. As indicated from Eqs. (34) and (44), much less energy is available for damage. Fig. 6 and Eq. (36) show a greatly truncated half sine, which is nevertheless still quite similar to a half sine.

If there is no energy dissipation the contact duration can also be estimated from the expressions above. The main assumption is that the kinetic energy given in Eq. (40) can also be written as

$$T_{\mathrm{d}} = \tfrac{1}{2} m_{\mathrm{d}0} \omega_{\mathrm{rc}}^2 \hat{u}_{\mathrm{r}}^2. \tag{45}$$

The contact duration t_{c} is assumed to be a half sine, and gives an “equivalent interaction frequency” ω_{rc} from

$$t_{\mathrm{c}} = \pi/\omega_{\mathrm{rc}}. \tag{46}$$

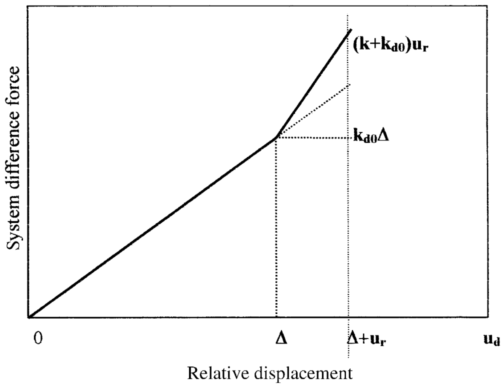

Fig. 5. Difference force against relative velocity.

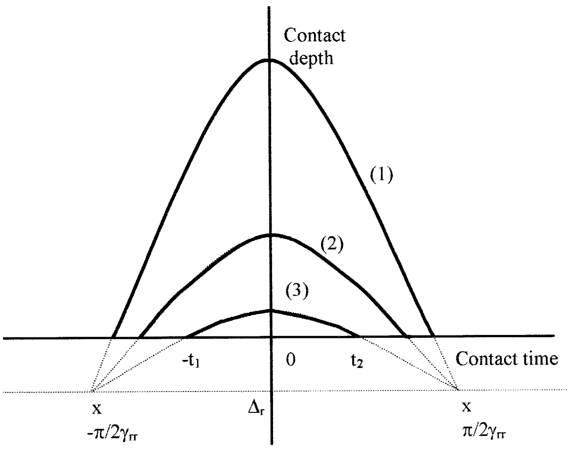

Fig. 6. Contact depth as a function of time and impact strength $\hat{\beta}$: (1) $\hat{\beta} > 1$; (2) $\hat{\beta} - 1$; (3) $\hat{\beta} < 1$.

Although this is only strictly accurate for heavy contact, it can be shown to be quite acceptable even for light contact, as truncation of a half sine does not greatly alter its shape. The contact frequency ω_{rc} can be found from Hamilton's Principle, effectively,

$$\partial(T_d - \hat{S}_d)/\partial\hat{u}_r = 0. \tag{47}$$

T_d and $\hat{S}_d$ are in Eqs. (41) and (45). Using also Eq. (28b) the equivalent interaction frequency is

$$\omega_{rc} = \omega_{rr}\sqrt{1 + \Delta_r/\hat{u}_r}. \tag{48}$$

If a substitution is made from Eq. (43) this becomes

$$\omega_{rc} = \omega_r\sqrt{1 + \sqrt{(4 + \hat{\beta}^2)}/\hat{\beta}^2}. \tag{49}$$

The contact duration t_c is now available from Eqs. (46) and (49), in which ω_{rr} is the real part of Eq. (22). t_c is plotted against impact strength $\hat{\beta}$, as the solid line in Figs. 7(a) and (b). The contact time is important as it controls the coefficient of restitution. For strong impacts when $\hat{\beta} \gg 2$ there is a plateau region where from Eq. (49), $\omega_{rr} \cong \omega_{rc}$. Therefore in this region the coefficient of restitution is constant. For weak impacts when $\hat{\beta} \ll 2$ the contact duration increases with impact strength $\hat{\beta}$ according to

$$t_c = \pi\hat{\beta}/\omega_{rr}\sqrt{2}. \tag{50}$$

6. Energy dissipation and contact duration

In the previous section it was shown that the contact duration is a function of the impact strength, while here the relationship with the loss ratio δ_r is established. In the event of impacts involving damage, large losses are possible and the calculations must therefore accommodate the full range of loss ratio i.e. $0 < \delta_r < \infty$, (see Eq. (33)).

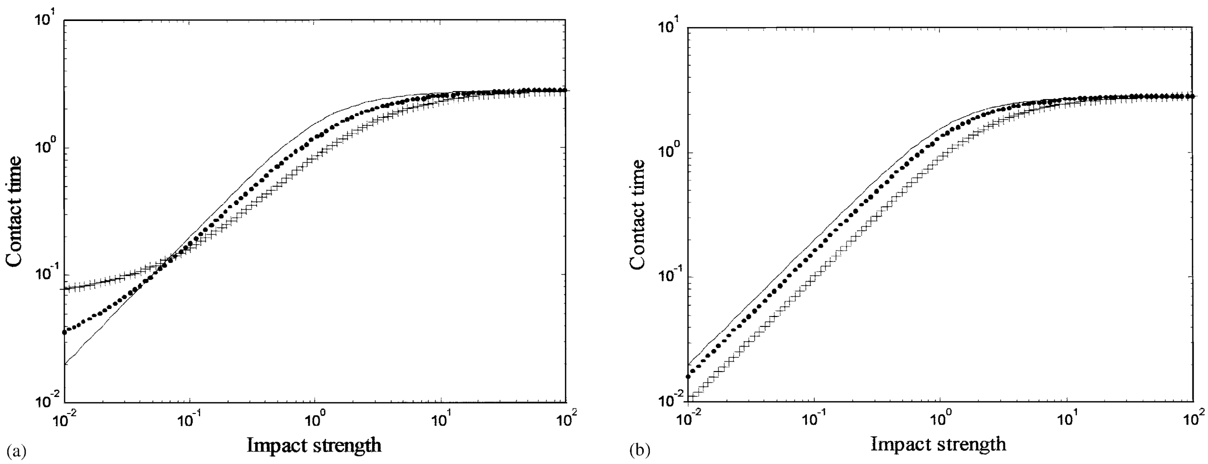

Fig. 7. Approximate contact time (a), and estimated contact time (b), against non-dimensional time: —, loss ratio = 0.01; ….., loss ratio = 0.5, xxxxx, loss ratio = 1.0.

Eq. (36) for the relative displacement during contact is plotted in Fig. 4(a). Initial contact occurs when $t = 0$, and so the other root of this expression yields the general form of the contact duration t_c. As it is difficult to provide a simple algebraic solution directly the following path is employed

Fig. 4(b) shows Eq. (51), a rearrangement of Eq. (36). The contact occurs for time $-t_1 < t < t_2$.

$$u_r = \Delta_r(e^{\gamma_{rr}(t+t_1)} \cos\omega_{rr} - \sin\phi)/\sin\phi. \tag{51}$$

The times t_1 and t_2 are the solution of Eq. (52), the numerator of Eq. (51), at zero relative displacement:

$$0 = e^{r_{rr}(t+t_1)}\cos\omega_{rr}t - \sin\phi. \tag{52}$$

The contact duration is

$$t_c = t_1 + t_2 \tag{53}$$

The first interval t_1 is found by setting $t = -t_1$ in Eq. (52) and making an approximation for the cosine of the form

$$\cos x \approx 1 - x^2/2. \tag{54}$$

This approximation only becomes inaccurate when x is almost $\pm\pi/2$, and so is acceptable within the range of contact. Solution of the quadratic form of Eq. (52) gives two roots

$$t_1 = \pm\sqrt{2(1-\sin\phi)}/\omega_{rr}. \tag{55}$$

The negative root represents the real value of t_1. However if there is no attenuation the contract pulse is symmetrical and the positive root is equal to t_2. The contact duration can thus be expressed as $2t_1$, i.e.,

$$t_c = 2\sqrt{2(1-\sin\phi)}/\omega_{rr} \tag{56}$$

which could be shown to be very close to the estimate made from Eqs. (46) and (49) using the energy approach.

The general form of t_2 may now be found by returning to Eq. (52), setting $t = t_2$ and making the previous approximation for the cosine term, and now also for the exponential using the first three terms of the series expansion, i.e.,

$$e^x \approx 1 + x + x^2/2. \tag{57}$$

The solution of a quadratic equation in x_2, $x_2 = \omega_{rr}t_2$ may with some patience, be found as

$$x_2 = (1 + \delta_r x_1) - \delta_r \sin\phi \pm (x_1 - \delta_r \sin\phi)/(1 + \delta_r^2 \sin\phi), \tag{58}$$

where Eq. (55) was used for the definition of: $x_1 = \omega_{rr}t_1$. The time period t_2 is the positive root of Eq. (58). The contact duration t_c is therefore the sum of t_1 and t_2, i.e.,

$$t_c = 2(t_1 - \delta_r \sin\phi/\omega_{rr})/(1 + \delta_r^2 \sin\phi). \tag{59}$$

This expression is plotted in Fig. 7(a) for four different loss ratios. When the loss ratio is 0.01 and 0.1, the contact period is that of the undamped case given in Eq. (49) or (56). The contact period is seen to decrease when the loss ratio increases to 0.5 or the maximum of 1. This is expected as contact occurs when there is a reaction force between the colliding bodies arising from the stored strain energy; if this energy is dissipated the force is lost and the contact ceases. At first sight it would appear that Eq. (59) has the unfortunate possibility of becoming negative for short contact times. This is not in fact the case because sin ϕ, which also occurs within t_1, also includes the loss ratio seen in Eq. (36).

Eq. (59) is in error when there is very high loss and short contact duration, as represented on Fig. 7(a) for $\hat{\beta}<0.1$ $\delta_r = 0.5,1$. This is because of the approximation used for the exponential in Eq. (57). Inspection of Eq. (59) and Fig. 7(a) suggest an accurate and simple contact time which is valid for all cases is

$$t_c = 2t_0/(1 + \delta_r^2 \sin\phi_0), \tag{60}$$

where the undamped parameters from Eqs. (36) and (37) are used, namely,

$$\sin\phi_0 = 1/\sqrt{1 + \hat{\beta}^2} \tag{61}$$

and t_0 is half the undamped contact period, i.e.,

$$t_0 = (2(1 - \sin\phi_0))^{1/2}/\omega_{rr}. \tag{62}$$

given in Eq. (49) in an approximate form. Eq. (60) is displayed in Fig. 7(b), where it can be seen to correspond to the accurate section of Fig. 7(a)

To confirm the contact time calculations the relative displacement, from Eq. (36), is plotted in Figs. 8(a), 9(a), 10(a) as a function of loss ratio δ_r and the impact strength $\hat{\beta}$ the accompanying relative velocities given from the differential of Eq. (36) are

$$\dot{u}_r = \Delta_r \boldsymbol{\omega}_r e^{-\gamma_{rr} t} \cos(\omega_{rr} t + \phi + \delta_r')/\sin\phi, \tag{63}$$

where

$$\delta_r' = \arctan\delta_r$$

are plotted in Figs. 8(b), 9(b) and 10(b). These are normalized by setting $\Delta_r = 1$, $\omega_{rr} = 1$. Four loss ratios are used: 0.01, 0.1, 0.5, 1.0, covering the full range of possibilities. In Fig. 8 the impact

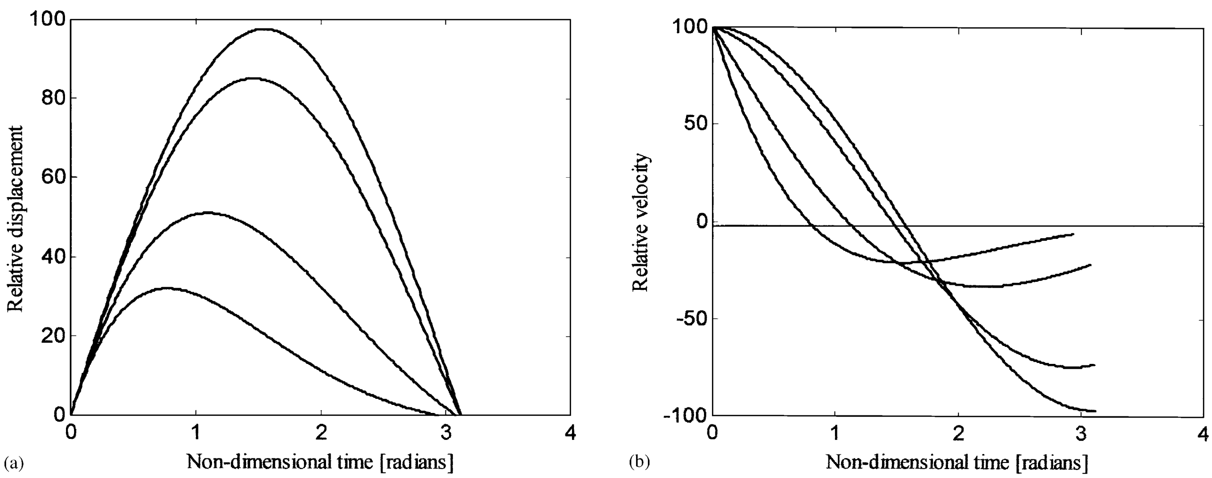

Fig. 8. Relative displacement (a) and relative velocity (b) against non-dimensional time: impact strength = 100; loss ratios: 0.01, 0.1, 0.5, 1.0, in order from the top.

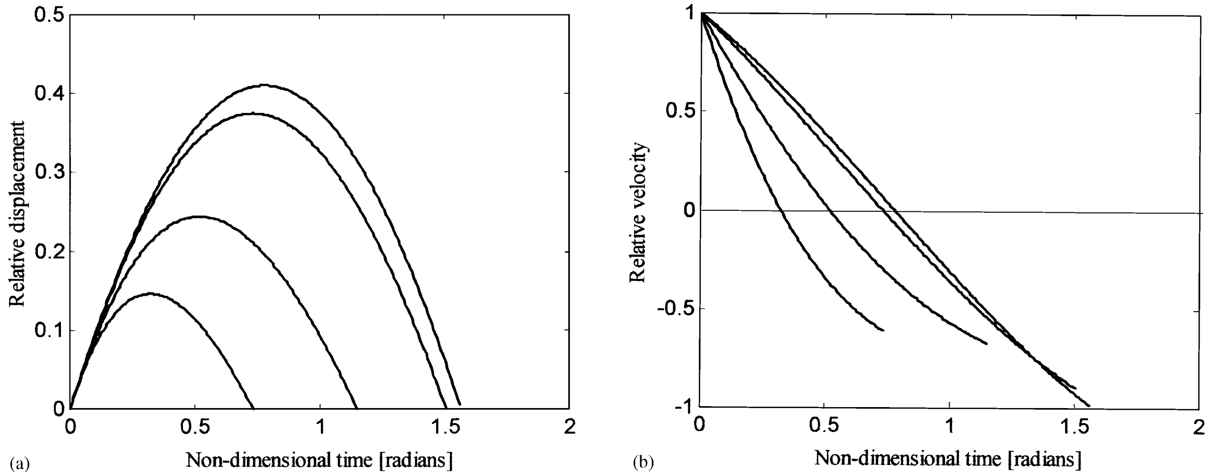

Fig. 9. Relative displacement (a) and relative velocity (b) against non-dimensional time: impact strength = 1.0; loss ratios: 0.01, 0.1, 0.5, 1.0, in order from the top.

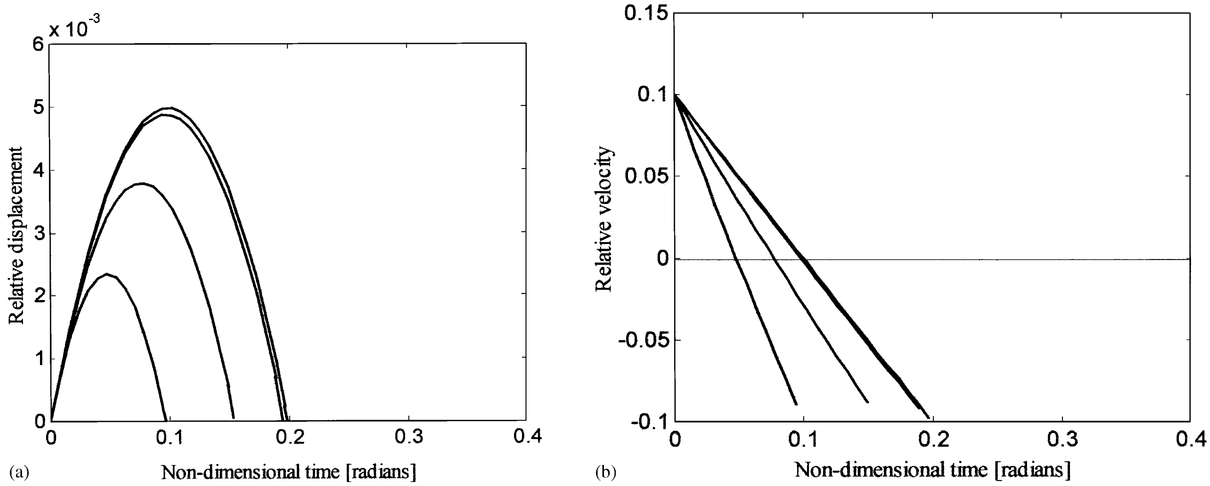

Fig. 10. Relative displacement (a) and relative velocity (b) against non-dimensional time: impact strength = 0.1; loss ratios: 0.01, 0.1, 0.5, 1.0, in order from the top.

strength is 100. For low loss the relative displacement is a half sine wave and the relative velocity is a half cosine. When the damping is increased the contact time is almost unaffected, as seen in Fig. 7, there is however considerable attenuation to the peak relative displacement and to the exit relative velocity.

Fig. 9 shows the contact pulse when the impact ratio is unity, indicating that only one half of the kinetic energy at contact is transferred into strain energy within the contact. For low damping the pulse duration is half that of the high impact ratio example in Fig. 8, while for heavy damping the contact time is a quarter of that of Fig. 8. Inspection of the exit velocity shows that the reduction in contact duration leads to considerably less energy dissipation.

Fig. 10 gives the contact pulses for slight contact when the impact ratio is one-tenth. The contact duration is very brief and accordingly there is little attenuation, the relative velocities only include the zero crossing section of the cosine form. The contact times are all correctly given by Eq. (60) in Fig. 7(b), confirming its utility.

The work done during contact may be simply estimated from the difference in the initial and final kinetic energy, calculated from the initial and final velocities at times 0 and t_c as

$$\hat{W}_d = T_d(1-\varepsilon^2). \tag{64}$$

The coefficient of restitution ε is defined in the usual way:

$$\dot{u}_r(t_c) = -\varepsilon\dot{u}_r(0).$$

or from Eq. (36)

$$\varepsilon = \exp(-\delta_r\omega_{rr}t_c).$$

T_d is the energy of the collision described in Eq. (40). The contact time t_c increases with impact strength $\hat{\beta}$ as given in Eqs. (60) and (61) and seen in Fig. 7(a). The coefficient of restitution is therefore a function of the impact strength $\hat{\beta}$. For impact strength greater than unity the coefficient of restitution will be independent of collision velocity (as is generally assumed).

7. The relative and mean motion during contact

The approach used in this analysis is to regard the dynamics in terms of sum and difference motion. The collision and energy dissipation is associated with the relative or difference motion, causing no direct change to the sum motion. There are however secondary changes to the sum motion u_s in response to the changes in the relative motion to conserve linear momentum. This interaction as described in Eq. (7) has two parts; a forced solution u_{s1} representing the change sum motion due to the change in relative motion, and a free vibration solution u_{s2} which satisfies the initial conditions for sum motion at the instant of contact.

The forced solution u_{s1} is found by substituting the relative motion u_r, from Eq. (51), into Eq. (7)

$$K_s u_s = -\Delta_r e^{-\gamma_{rr}t_1}\Re\{(k_d - \omega_r^2 m_d)e^{i\boldsymbol{\omega}_r t}\}/\sin\phi + \Delta_r\Re\{k_d\}. \tag{65}$$

The solution takes the form

$$u_{s1} = \mathbf{A}e^{i\boldsymbol{\omega}_r t} + \mathbf{A}^* e^{i\boldsymbol{\omega}_r^* t} + C, \tag{66}$$

where $\mathbf{A}$ is a complex constant and C is a real constant. Substitution into Eq. (65) yields

$$u_{s1} = \frac{-\Delta_r e^{-\gamma_{rr}t_1}}{\sin\phi}\Re\left\{\left(\frac{k_d - \omega_r^2 m_d}{k_s - \omega_r^2 m_s}\right)e^{i\boldsymbol{\omega}_r t}\right\} + \Delta_r\Re\left\{\frac{k_d}{k_s}\right\}. \tag{67}$$

As expected from the symmetry of the momentum equation, the sum motion u_{s1} is similar to the relative motion u_r, given in Eq. (51). The ratio u_{s1}/u_r decreases with the quotient in the centre of Eq. (67), which is K_d/K_s at frequency $\boldsymbol{\omega}_r$. If the two oscillators are identical there is no forced sum motion as K_d is zero. This means that if the colliding structures are physically similar as would be the case for adjacent buildings, the sum and difference motions can be regarded independently.

A further simplification may be made by observing that the interaction frequency $\boldsymbol{\omega}_r$ in Eq. (22) is always greater than the free interaction frequency ω_{d0} in Eq. (12) i.e.

$$\boldsymbol{\omega}_r/\omega_{d0} = \sqrt{(\mathbf{k} + \mathbf{k}_{d0})/\mathbf{k}_{d0}}. \tag{68}$$

The sum frequency ω_s can be defined from Eq. (1) as

$$\omega_s^2 = \mathbf{k}_a + \mathbf{k}_b/(m_a + m_b). \tag{69}$$

so the ratio of uncoupled difference frequency to sum frequency is

$$\omega_{d0}/\omega_s = \omega_a\omega_b/\omega_s^2. \tag{70}$$

This ratio becomes unity if oscillators a and b are similar, i.e., of the same elastic modulus, density and shape, but of a different size, then $k_a = pk_b$, $m_b = pm_{a,}$; where p is the scaling factor. Under these conditions $\boldsymbol{\omega}_r > \boldsymbol{\omega}_s$ and Eqs. (51) and (67) give the ratio of sum motion to relative motion as $u_s/u_r = R$, where R is

$$R \approx (-m_d + \omega_r^2 k_d)/m_s. \tag{71}$$

For a hard contact, i.e., $\boldsymbol{\omega}_r \gg \boldsymbol{\omega}_s$, $R = -m_d/m_s$ as for the collision of two free bodies. Fig. (11) illustrates this change to the sum displacement and sum velocity within the contact period because of the relative motion, which is also shown. The peak relative displacement is given from Eq. (51) as

$$\hat{u}_r = e^{-\gamma_{rr}t_1}\sqrt{\Delta_r^2 + \dot{U}_d^2/\omega_{rr}^2} - \Delta_r. \tag{72}$$

The quantity u_{s1} discussed above only represents the change in u_s during the contact period, as there is also the free vibration component u_{s2}, which satisfies the left- side of Eq. (7), i.e.,

$$K_s u_{s2} = 0, \tag{73}$$

such that the total response within the contact period is

$$u_s = u_{s1} + u_{s2}. \tag{74}$$

The free vibration solution takes the usual form, used for example in Eq. (27) as

$$u_{s2} = \mathbf{A}_{s2}e^{i\boldsymbol{\omega}_s t} + \mathbf{A}_{s2}^* e^{-i\boldsymbol{\omega}_s^* t}. \tag{75}$$

This is valid during contact, i.e., $-t_1 < t < t_1$, and must satisfy the initial conditions at $t = -t_1$:

$$u_s = U_s, \;\; u_{s1} = 0, \;\; \dot{u}_s = \dot{U}_s, \;\; \dot{u}_{s1} = R\dot{U}_d. \tag{76}$$

These substitutions yield:

$$\mathbf{A}_{s2} = e^{i\boldsymbol{\omega}_s t_1}(U_s - i((\dot{U}_s - R\dot{U}_d)/\boldsymbol{\omega}_s)/2. \tag{77}$$

To summarize the simplest and most useful case which is that of a hard contact, i.e., $\omega_r \gg \omega_s$, the dynamics become those of two colliding masses and

$$u_{s1}/u_r = -m_d/m_s$$

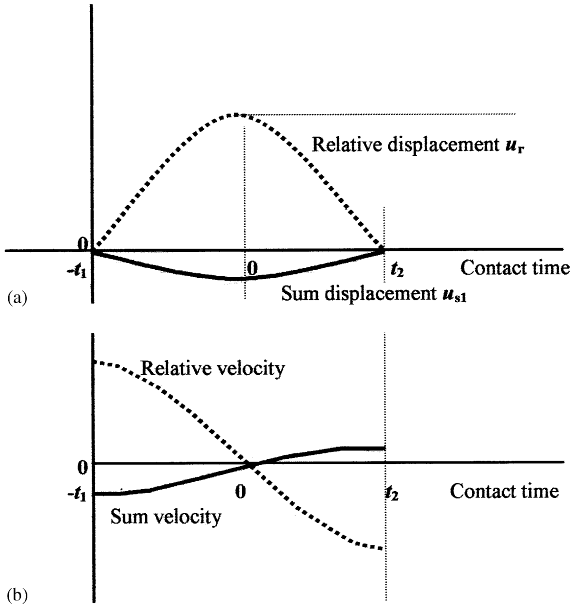

Fig. 11. Relative displacement and sum displacement due to contact (a), relative velocity and sum velocity due to contact (b).

at initial contact:

$$\dot{u}_s = \dot{U}_s, \quad \dot{u}_r = \dot{U}_d, \tag{78}$$

at final contact:

$$\dot{u}_s = \dot{U}_s + \varepsilon R \dot{U}_d, \quad \dot{u}_r = -\varepsilon \dot{U}_d. \tag{79}$$

These terms may then be used to obtain the energy loss and exchanged in the collision, and they may also be substituted back into Eq. (2) for the motions of the individual oscillators. The only difference between expressions (78) and (79) with those normally used for the impact of two free-bodies is that the coefficient of restitution ε is a function of the contact stiffness, damping and the impact strength.

8. Conclusions

The contact dynamics between two colliding oscillators has been calculated using the sum and difference motions. The contact mainly influences the difference or relative motion as is usually assumed when a coefficient of restitution is used. There are changes in the sum motion in response to the change in relative motion. These are calculated from the momentum equation in the usual way. For identical oscillators the relative and sum motion are shown to be independent; however, it is also shown that within the short contact duration these motions are almost independent provided that the two oscillators have ratio of resonance frequencies that is less than two. In this range a new coefficient of restitution can be defined in terms of all the mechanical properties of the oscillators.

The contact time was found to increase with the contact stiffness and relative velocity and decrease with the damping ratio and the oscillator separation. A strong impact gives a half sine displacement pulse, while a weak impact gives a truncated half sine pulse. The analysis was confirmed using an alternative approach employing an energy balance.

The main outcome is that a coefficient of restitution describing the energy loss in the collision was calculated, which increases with the contact stiffness, damping, and the relative velocity. However, it decreases with the oscillator spacing. Above a certain threshold the contact time and coefficient of restitution become invariant with the magnitude of the relative velocity at impact. This threshold occurs when at the moment of contact, for relative motion, the strain energy in the oscillators is equal to the kinetic energy. For free bodies there is therefore no such threshold and the coefficient of restitution is constant, as is usually assumed.

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