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Letter to the Editor

Free vibration of an infinite magneto-electro-elastic cylinder

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1. Introduction

A recent study by Pan and Heyliger [1] focused on the vibration of magneto-electro-elastic simply supported plates. Their vibration results [1] for coupled magnet-electro-elastic materials were based upon combining the effects of two materials. A form of the piezoelectric material barium titanate ($BaTiO_3$) was combined as a layered material with the magnetostrictive cobalt iron oxide ($CoFe_2O_4$). Both materials are transversely isotropic of the 6 mm crystal class. The coupled magneto-electro-elastic analysis resulted from combining layers of the two materials.

In this study the fundamental problem described in Ref. [1] is extended to cylindrical co-ordinates. The analysis is rendered one-dimensional by assuming certain axisymmetric solutions that satisfy the governing equations. The three-dimensional character of the solution is preserved by assuming a solution that would characterize the cylinder as infinite. In some sense the results presented here are an extension of earlier work by Buchanan and Peddieson [2] on infinite piezoelectric cylinders. The piezoelectric cylinder has been studied analytically by Paul [3], later Paul and Raju [4] computed frequencies for solid cylinders and Paul and Venkatesan [5] extended the analysis to include hollow cylinders. The papers by Paul and co-workers can serve as benchmark solutions to validate the analysis given here.

The formulation presented here can be applied to a fully coupled magneto-electro-elastic material. A fiber reinforced material wherein the matrix is $CoFe_2O_4$ and the fibers are $BaTiO_3$ has been studied by Huang and Kuo [6] and they proposed material properties for the combined materials based upon the aspect ratio of ellipsoidal inclusions of $BaTiO_3$ in a matrix of $CoFe_2O_4$ with volume ratio of 0.5 for the two materials. Subsequently, Huang et al. [7] proposed an analysis that predicts a fiber volume fraction of 0.46 for developing an optimum magnitude for the electromagnetic coupling constant. Aboudi [8] used the general method of cells to predict the various electro-magneto-elastic material constants for a fully coupled composite material and related the results to the fiber volume fraction. The optimum fiber content predicted by Aboudi [8] of approximately 0.44 was in excellent agreement with that predicted in Ref. [7]. The material

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properties computed by Aboudi [8], and based upon his optimum fiber content are used in this paper to study the coupled vibrational behavior of magneto-electro-elastic cylinders.

2. Governing equations

The complete equations governing the behavior of a piezoelectric cylinder have been recorded by Paul and Raju [4] in terms of displacements and electric potential and the extension to coupled magnet-electro-elasticity in cylindrical coordinates is straightforward. The governing equations that relate the magnetic field to the magnetic potential are identical, in form, to those that relate the electric field to the electric potential. It follows that all equations that govern electric displacement are similar to those that govern magnetic induction.

The significant equations, for a finite element analysis, are the strain–displacement, electric field–electric potential and magnetic field–magnetic potential equations along with the constitutive equations. The strain–displacement equations are as follows:

$$S_{rr} = S_1 = \frac{\partial u}{\partial r}, \quad S_{\theta\theta} = S_2 = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right),$$
$$S_{zz} = S_3 = \frac{\partial w}{\partial z}, \quad S_{\theta z} = S_4 = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta},$$
$$S_{rz} = S_5 = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \quad S_{r\theta} = S_6 = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \tag{1}$$

where u, v and w are the mechanical displacements corresponding to the cylindrical co-ordinate directions r, θ, and z. The strains, S_i, are written in matrix notation using a single subscript and Eq. (1) identifies the relation between elasticity notation and matrix notation. The electric field vector E_i is related to the electric potential φ as

$$E_r = E_1 = -\frac{\partial \varphi}{\partial r}, \quad E_\theta = E_2 = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta}, \quad E_z = E_3 = -\frac{\partial \varphi}{\partial z}. \tag{2}$$

Similarly, the magnetic field H_i is related to the magnetic potential ψ as

$$H_r = H_1 = -\frac{\partial \psi}{\partial r}, \quad H_\theta = H_2 = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad H_z = H_3 = -\frac{\partial \psi}{\partial z}. \tag{3}$$

The constitutive equations, following [1], relate stress T_k, electric displacement D_j and magnetic induction B_j to strain, electric field and magnetic field as follows:

$$T_k = C_{jk}S_k - e_{kj}E_k - q_{kj}H_k, \tag{4}$$

$$D_j = e_{jk}S_k + \varepsilon_{jk}E_k + m_{jk}H_k, \tag{5}$$

$$B_j = q_{jk}S_k + m_{jk}E_k + \mu_{jk}H_k, \tag{6}$$

where C_{jk}, ε_{jk} and μ_{jk} are the elastic, dielectric and magnetic permeability coefficients, respectively; e_{kj}, q_{kj} and m_{jk} are the piezoelectric, piezomagnetic and magnetoelectric material coefficients.

The equations of motion, for the record, can be written in a general format that can be specialized to cylindrical co-ordinates:

$$\operatorname{div}\, \bar{\bar{T}} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}, \quad \operatorname{div}\, \vec{D} = 0, \quad \operatorname{div}\, \vec{B} = 0, \tag{7}$$

where $\bar{\bar{T}}$ is the stress tensor, ρ is the density and body forces, electric charge and current densities have been neglected. Eq. (7) are given in expanded format in Appendix A for cylindrical co-ordinates.

A completely coupled material matrix, assuming a hexagonal crystal class, corresponds to Eqs. (4)–(6) and following Ref. [6], is written as

$$\left\{\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ \hline D_1 \\ D_2 \\ D_3 \\ \hline B_1 \\ B_2 \\ B_3 \end{array}\right\} = \left[\begin{array}{cccccc|ccc|ccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} & 0 & 0 & q_{31} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} & 0 & 0 & q_{31} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & e_{33} & 0 & 0 & q_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & e_{15} & 0 & 0 & q_{15} & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & e_{15} & 0 & 0 & q_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & e_{15} & 0 & \varepsilon_{11} & 0 & 0 & m_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \varepsilon_{11} & 0 & 0 & m_{11} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33} & 0 & 0 & m_{33} \\ \hline 0 & 0 & 0 & 0 & q_{15} & 0 & m_{11} & 0 & 0 & \mu_{11} & 0 & 0 \\ 0 & 0 & 0 & q_{15} & 0 & 0 & 0 & m_{11} & 0 & 0 & \mu_{11} & 0 \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 & 0 & 0 & m_{33} & 0 & 0 & \mu_{33} \end{array}\right] \left\{\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ \hline E_1 \\ E_2 \\ E_3 \\ \hline H_1 \\ H_2 \\ H_3 \end{array}\right\} \tag{8}$$

with

$$C_{66} = (C_{11} - C_{12})/2.$$

3. Finite element formulation

A solution, similar to that used in Ref. [2], that satisfies the assumption of an infinite cylinder and also satisfies Eq. (7) is as follows:

$$\begin{aligned} u(r,\theta,z,t) &= U(r)\cos m\theta \cos(kz - \omega t), \\ v(r,\theta,z,t) &= V(r)\sin m\theta \cos(kz - \omega t), \\ w(r,\theta,z,t) &= W(r)\cos m\theta \sin(kz - \omega t), \\ \varphi(r,\theta,z,t) &= \Phi(r)\cos m\theta \sin(kz - \omega t), \\ \psi(r,\theta,z,t) &= \Psi(r)\cos m\theta \sin(kz - \omega t), \end{aligned} \tag{9}$$

where m is an integer and is the circumferential wave number, k is the longitudinal wave number and ω is the circular frequency. The analysis has effectively been reduced to a single co-ordinate but retains a three-dimensional dependence for the solution depending upon the choice of m and k.

The finite element model can be developed using the Rayleigh–Ritz variational formulation or the Galerkin weighted residual method. The finite element equations will not be derived here but the derivation would follow the discussion given by Buchanan and Peddieson [2] or Buchanan [9]. Assume the mechanical displacements, electric potential and magnetic potential can be represented using suitable shape functions, such as

$$U_i = [N_u]\{U\}, \quad \Phi = [N_\varphi]\{\Phi\}, \quad \Psi = [N_\psi]\{\Psi\}. \tag{10}$$

In application the same shape functions are used for mechanical displacements, electrical potential and magnetic potential, but for derivation they are kept separate. In principal, different shape functions could be used to represent the different variables. A formulation that corresponds to a completely coupled system could be written in terms of the following stiffness matrices:

$$\begin{aligned}
[[K_{uu}] - \omega^2[M]]\{U\} + [K_{u\varphi}]\{\Phi\} + [K_{u\psi}]\{\Psi\} &= 0,\\
[K_{u\varphi}]^{\mathrm{T}}\{U\} + [K_{\varphi\varphi}]\{\Phi\} + [K_{\varphi\psi}]\{\Psi\} &= 0,\\
[K_{u\psi}]^{\mathrm{T}}\{U\} + [K_{\varphi\psi}]^{\mathrm{T}}\{\Phi\} + [K_{\psi\psi}]\{\Psi\} &= 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
[K_{uu}] &= \int_v [B_u]^T[C][B_u]\,\mathrm{d}V,\\
[K_{u\varphi}] &= \int_v [B_u]^T[e][B_\varphi]\,\mathrm{d}V,\\
[K_{u\psi}] &= \int_v [B_u]^T[q][B_\psi]\,\mathrm{d}V,\\
[K_{\varphi\varphi}] &= \int_v [B_\varphi]^T[\varepsilon][B_\varphi]\,\mathrm{d}V,\\
[K_{\psi\psi}] &= \int_v [B_\psi]^T[\mu][B_\psi]\,\mathrm{d}V,\\
[K_{\varphi\psi}] &= \int_v [B_\varphi]^T[m][B_\psi]\,\mathrm{d}V,\\
[M] &= \int_V [N]^T[\rho][N]\,\mathrm{d}V
\end{aligned} \tag{12}$$

and

$$\mathrm{d}V = 2\pi r\,\mathrm{d}r. \tag{13}$$

The so-called B matrix is the assumed shape function matrix pre-multiplied by an operator matrix whereby the operator matrix is dictated by the equation that is to be modelled. A three-node quadratic shape function is assumed and combined with an operator matrix based upon Eq. (1) to form $[B_u]$ as

$$[B_u] = [L_u][N_u] = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ \frac{1}{r} & \frac{1}{r}\frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 \end{bmatrix} \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 \end{bmatrix}. \quad (14)$$

Substituting Eq. (9) into Eq. (14) gives

$$[B_u] = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & 0 & \cdots\cdots \\ \frac{N_1}{r} & \frac{mN_1}{r} & 0 & \cdots\cdots \\ 0 & 0 & kN_1 & \cdots\cdots \\ 0 & -kN_1 & -m\frac{N_1}{r} & \cdots\cdots \\ -kN_1 & 0 & \frac{\partial N_1}{\partial r} & \cdots\cdots \\ -m\frac{N_1}{r} & \frac{\partial N_1}{\partial r} - \frac{N_1}{r} & 0 & \cdots\cdots \end{bmatrix}, \quad (15)$$

where there are six additional columns for N_2 and N_3. The matrix $[B_\varphi]$ is based upon Eq. (2) and is

$$[B_\varphi] = [L_\varphi][N_\varphi] = \begin{bmatrix} -\frac{\partial}{\partial r} \\ -\frac{1}{r}\frac{\partial}{\partial \theta} \\ -\frac{\partial}{\partial z} \end{bmatrix} [N_1 \quad N_2 \quad N_3] = \begin{bmatrix} -\frac{\partial N_1}{\partial r} & -\frac{\partial N_2}{\partial r} & -\frac{\partial N_3}{\partial r} \\ m\frac{N_1}{r} & m\frac{N_2}{r} & m\frac{N_3}{r} \\ -kN_1 & -kN_2 & -kN_3 \end{bmatrix}. \quad (16)$$

Similarly, Eq. (3) leads to

$$[B_\psi] = [L_\psi][N_\psi] = \begin{bmatrix} -\frac{\partial}{\partial r} \\ -\frac{1}{r}\frac{\partial}{\partial \theta} \\ -\frac{\partial}{\partial z} \end{bmatrix} [N_1 \quad N_2 \quad N_3] = \begin{bmatrix} -\frac{\partial N_1}{\partial r} & -\frac{\partial N_2}{\partial r} & -\frac{\partial N_3}{\partial r} \\ m\frac{N_1}{r} & m\frac{N_2}{r} & m\frac{N_3}{r} \\ -kN_1 & -kN_2 & -kN_3 \end{bmatrix}. \quad (17)$$

The local stiffness matrix of Eq. (11) will be a 15×15 matrix of which only the mechanical displacement have a corresponding mass term. The electrical potential and magnetic potential terms are eliminated using standard condensation techniques. The resulting stiffness matrix that will be solved for eigenvalues is defined as

$$[K_{eq}]\{U\} - \omega^2[M]\{U\} = 0, \tag{18}$$

where

$$[K_{eq}] = [K_{IV}] - [K_{III}][K_{II}]^{-1}[K_I]. \tag{19}$$

The component matrices for Eq. (19) are

$$\begin{aligned}
[K_I] &= [K_{u\psi}]^{\mathrm{T}} - [K_{\varphi\psi}]^{\mathrm{T}}[K_{\varphi\varphi}]^{-1}[K_{u\varphi}]^{\mathrm{T}},\\
[K_{II}] &= [K_{\psi\psi}] - [K_{\varphi\psi}]^{\mathrm{T}}[K_{\varphi\varphi}]^{-1}[K_{\varphi\psi}],\\
[K_{III}] &= [K_{u\psi}] - [K_{u\varphi}][K_{\varphi\varphi}]^{-1}[K_{\varphi\psi}],\\
[K_{IV}] &= [K_{uu}] - [K_{u\varphi}][K_{\varphi\varphi}]^{-1}[K_{u\varphi}]^{\mathrm{T}}.
\end{aligned} \tag{20}$$

The eigenvectors that correspond to the distribution of $\{\Phi\}$ and $\{\Psi\}$ can be computed as

$$\{\Phi\} = -[K_{VI}]^{-1}[K_V]\{U\} \tag{21}$$

and

$$\{\Psi\} = -[K_{II}]^{-1}[K_I]\{U\}, \tag{22}$$

where

$$\begin{aligned}
[K_V] &= [K_{u\varphi}]^{\mathrm{T}} - [K_{\varphi\psi}][K_{\psi\psi}]^{-1}[K_{u\psi}]^{\mathrm{T}},\\
[K_{VI}] &= [K_{\varphi\varphi}] - [K_{\varphi\psi}][K_{\psi\psi}]^{-1}[K_{\varphi\psi}]^{\mathrm{T}}.
\end{aligned} \tag{23}$$

4. Analysis and results

Free vibration frequencies will be compared for cylinders of $BaTiO_3$ as a single material, $CoFe_2O_4$ as a single material and the magneto-electro-elastic material defined by Aboudi [8]. A consistent definition for non-dimensional terms is necessary and they are assumed as follows. The piezoelectric terms are those defined by Refs. [2,4] and the additional non-dimensional terms were derived based upon the governing equations. Some terms do not appear in the final formulation of equations used in this report, but would be required to non-dimensionalize the complete set of coupled equations as given in Appendix A.

$$\begin{aligned}
&\bar{C}_{ij} = C_{ij}/C_{44}, \quad \bar{r} = r/a, \quad \bar{e}_{ij} = e_{ij}/e_{33}, \quad \bar{q}_{ij} = q_{ij}/q_{33},\\
&\bar{m}_{ij} = m_{ij}C_{44}/(e_{33}q_{33}), \quad \bar{\varepsilon}_{ij} = \varepsilon_{ij}C_{44}/e_{33}^2, \quad \bar{\mu}_{ij} = \mu_{ij}C_{44}/q_{33}^2,\\
&\bar{t} = t(C_{44}/\rho a^2)^{1/2}, \quad \bar{\Phi} = \Phi e_{33}/C_{44}a, \quad \bar{\Psi} = \Psi q_{33}/C_{44}a,\\
&\bar{U} = U/a, \quad \bar{V} = V/a, \quad \bar{W} = W/a, \quad \Omega = \omega a(\rho/C_{44})^{1/2}.
\end{aligned} \tag{24}$$

The material constants are given in several papers and Aboudi [8] is the source for the material parameters used here. The actual material constants along with their non-dimensional

Table 1
Material properties for piezoelectric barium titanate, magnetostrictive cobalt iron oxide and a electromagnetic composite

Material coefficient	Actual			Non-dimensional		
	$BaTiO_3$	$CoFe_2O_4$	[8][a]	$BaTiO_3$	$CoFe_2O_4$	[8][a]
C_{11} $(10)^9 N/m^2$	166	286.0	218	3.86047	6.31346	4.36
C_{33}	162	269.5	215	3.76744	5.94923	4.30
C_{12}	77	173.0	120	1.79069	3.81898	2.40
C_{13}	78	170.0	120	1.81395	3.75275	2.40
C_{44}	43	45.3	50	1.0	1.0	1.0
C_{66}	44.5	56.5	49	1.03488	1.24724	0.98
e_{15} C/m^2	11.6	0	0	0.62366	0	0.0
e_{31}	−4.4	0	−2.5	−0.23656	0	−0.33333
e_{33}	18.6	0	7.5	1.0	0	1.0
ε_{11} $(10^{-9})C/V\,m$	11.2	0.08	0.4	1.39206	[b]	0.35556
ε_{33}	12.6	0.093	5.8	1.56667	[b]	5.15556
q_{15} $N/A\,m$	0	550.0	200	0	0.78605	0.57971
q_{31}	0	580.3	265	0	0.82936	0.76812
q_{33}	0	699.7	345	0	1.0	1.0
μ_{11} $(10^{-6})N\,s^2/C^2$	5.0	−590.0	−200	[b]	−344.663	−84.0159
μ_{33}	10.0	157.0	95	[b]	91.715	39.9076
m_{11} $(10^{-9})N\,s/V\,C$	0	0	0.0074	0	0	0.000143
m_{33}	0	0	2.82	0	0	0.054493

[a] Material properties scaled from the graphical results of Ref. [8].
[b] Non-dimensional constant cannot be computed.

counterparts are given in Table 1. The electromagnetic coupled material, that is, the material with electromagnetic constants is based upon the graphical results of Aboudi [8]. Non-dimensional equivalents cannot be computed for μ_{ij} for $BaTiO_3$ or ε_{ij} for $CoFe_2O_4$, however any term (usually unity) can be used on the diagonal of Eq. (8) for μ_{ij} when q_{ij} and m_{ij} are zero and similarly for ε_{ij}.

The anticipated accuracy of the analysis is demonstrated in Table 2. The results for a solid cylinder of hexagonal PZT-4 using the formulation of Eq. (18) is compared with the solution given by Paul and Raju [4]. The material constants are given in numerous Refs. [2,10]. The accuracy of the analysis, as demonstrated in Table 2, is considered to be acceptable.

Frequency of free vibration for the materials of interest are given in Tables 3–5. Sufficient results are tabulated in order that a complete description of the vibration behavior of each material is established. The circumferential wave number m is an integer and varies from 0 to 6. The longitudinal wave number varies from 0 to 4.

Frequencies for free vibration of an infinite piezoelectric solid cylinder of $BaTiO_3$ are given in Table 3. The case $m = 0$ and $k = 0$ gives pure uncoupled radial, torsional and longitudinal modes of vibration. Results are tabulated for various values of the longitudinal wave number k and for $m = 0$ the torsional modes remain uncoupled. Similarly, longitudinal modes remain uncoupled for $k = 0$. Note that for small values of k, greater than zero, the frequency decreases slightly for $m = 0$ and 2, but increases with increasing k for $m = 1$. For higher modes the frequency continues to increase with increasing k, but the increase is slight.

Table 2
Comparison of frequencies Ω with Paul and Raju [4] for an infinite solid cylinder of PZT-4

Mode	$k = 0.1$					
	$m = 0$		$m = 1$		$m = 2$	
		Ref. [4]		Ref. [4]		Ref. [4]
1	4.6553 l	4.6566	1.8993 l	1.8992	2.5664 r,t	2.5675
2	5.1364 r,l	5.1351	3.1592 r,t	3.1595	3.1869 l	3.1870
3	5.6150 t	5.6157	6.4131 l	6.4137	4.9322 r,t	4.9324
4	8.5426 l	8.5427	7.2250 r,t	7.2255	8.0348 l	8.0351
5	9.2020 t	9.2031	8.6812 r,t	8.6812	8.7667 r,t	8.7672
6	12.3862 l	12.3858	10.3487 l	10.3475	11.5787 r,t	11.5790
7	12.6774 r,l	12.6771	10.9107 r,t	10.9117	12.0578 l	12.0573
8	12.7028 t	—	14.2200 l	14.2195	12.6654 r,t	—
9	16.1747 t	—	14.3831 r,t	—	15.9757 l	15.9756
10	16.2241 l	16.2235	16.1993 r,t	—	16.0084 r,t	—

Table 3
Frequencies Ω for an infinite solid cylinder of piezoelectric barium titanate ($BaTiO_3$), r—radial mode, l—longitudinal mode, t—torsional mode

m	Mode	$k = 0.0$	$k = 0.5$	$k = 1.0$	$k = 2.0$	$k = 3.0$	$k = 4.0$
0	1	4.216 r	3.996	3.860	4.232	5.508	6.579
	2	4.334 l	4.656	5.111	5.594 t	6.025 t	6.797 t
	3	5.224 t	5.248 t	5.319 t	6.182	7.283	8.312
	4	7.935 l	7.942	7.968	8.151	8.719	9.451 t
	5	8.563 t	8.577 t	8.621 t	8.793 t	9.073 t	9.853
	6	10.650 r	10.688	10.793	11.092	11.396	11.786
	7	11.507 l	11.529	11.602	11.983	12.196 t	12.479 t
	8	11.821 t	11.831 t	11.863 t	11.989 t	12.727	13.692
	9	15.052 t	15.060 t	15.080	15.132	15.269	15.544
	10	15.070 l	15.072	15.085 t	15.184 t	15.348 t	15.574 t
1	1	1.875 l	2.064	2.476	3.246	3.942	4.708
	2	2.886	2.946	3.138	4.013	5.018	5.763
	3	5.983 l	5.952	5.906	5.997	6.583	7.619
	4	6.610	6.657	6.763	7.044	7.421	7.937
	5	7.387	7.459	7.666	8.346	9.091	9.752
	6	9.626 l	9.640	9.684	9.908	10.423	10.876
	7	10.141	10.154	10.193	10.347	10.647	11.521
	8	13.220 l	13.197	13.170	13.187	13.322	13.608
	9	13.316	13.345	13.397	13.530	13.721	13.978
	10	13.707	13.756	13.897	14.397	15.104	15.914
2	1	2.389	2.365	2.363	2.643	3.259	4.048
	2	3.139 l	3.252	3.512	4.136	4.774	5.460
	3	4.500	4.555	4.721	5.375	6.246	7.016

Table 3 (*continued*)

m	Mode	$k=0.0$	$k=0.5$	$k=1.0$	$k=2.0$	$k=3.0$	$k=4.0$
	4	7.504 1	7.501	7.500	7.597	7.967	8.693
	5	8.070	8.095	8.165	8.400	8.743	9.218
	6	9.889	9.933	10.057	10.462	10.901	11.339
	7	11.225 1	11.241	11.294	11.529	11.889	12.240
	8	11.682	11.695	11.737	11.930	12.415	13.280
	9	14.858	14.850	14.852	14.904	15.042	15.306
	10	14.859 1	14.879	14.913	15.023	15.195	15.434
3	1	3.680	3.667	3.658	3.796	4.186	4.774
	2	4.345 1	4.414	4.590	5.083	5.653	6.278
	3	6.162	6.208	6.342	6.848	7.549	8.252
	4	8.958 1	8.962	8.978	9.086	9.378	9.912
	5	9.461	9.480	9.536	9.741	10.057	10.509
	6	12.091	12.112	12.170	12.358	12.597	12.908
	7	12.763 1	12.780	12.831	13.021	13.280	13.582
	8	13.316	13.337	13.405	13.712	14.293	15.081
4	1	4.806	4.803	4.806	4.908	5.188	5.640
	2	5.525 1	5.570	5.693	6.085	6.583	7.151
	3	7.812	7.847	7.952	8.346	8.906	9.509
	4	10.368 1	10.375	10.398	10.512	10.736	11.179
	5	10.839	10.855	10.903	11.087	11.387	11.825
	6	13.884	13.892	13.915	14.011	14.176	14.428
	7	14.258 1	14.271	14.308	14.446	14.651	14.915
	8	15.205	15.234	15.321	15.665	16.210	16.878
5	1	5.864	5.867	5.880	5.974	6.199	6.565
	2	6.690 1	6.721	6.812	7.126	7.558	8.071
	3	9.410	9.438	9.520	9.827	10.274	10.780
	4	11.474 1	11.755	11.780	11.894	12.115	12.454
	5	12.223	12.238	12.281	12.453	12.743	13.169
	6	15.379	15.404	15.426	15.514	15.669	16.245
6	1	6.887	6.893	6.912	7.006	7.201	7.512
	2	7.845 1	7.869	7.939	8.196	8.570	9.030
	3	10.942	10.963	11.027	11.269	11.628	12.051
	4	13.102 1	13.110	13.136	13.244	13.439	13.724
	5	13.624	13.638	13.679	13.844	14.127	14.536
	6	16.812	16.820	16.843	16.939	17.101	17.336

Results are given in Table 4 for a material with $CoFe_2O_4$ hexagonal material properties. The frequencies of Table 4 represent the first tabulated results for an infinite solid cylinder with magnetostrictive material properties. Observations similar to those for the behavior of $BaTiO_3$ are apparent.

The electromagnetic material proposed by Aboudi [8] is analyzed in the format of Eq. (8) using the non-dimensional constants of Table 1. Results are reported in Table 5. The lowest frequency

Table 4
Frequencies Ω for an infinite solid cylinder of magnetostrictive cobalt iron oxide ($CoFe_2O_4$), r—radial mode, l—longitudinal mode, t—torsional mode

m	Mode	$k=0.0$	$k=0.5$	$k=1.0$	$k=2.0$	$k=3.0$	$k=4.0$
0	1	3.828 l	3.849	3.945	4.645	5.884	6.992 t
	2	5.583 r	5.701	5.822 t	6.074 t	6.473 t	7.023
	3	5.735 t	5.757 t	6.024	6.957	7.852	8.854
	4	7.009 l	7.053	7.188	7.853	9.229	10.216 t
	5	9.400 t	9.414 t	9.453 t	9.611 t	9.868 t	10.854
	6	10.164 l	10.185	10.246	10.494	10.934	11.710
	7	12.977 t	12.987 t	13.016 t	13.130 t	13.319 t	13.580 t
	8	13.312 l	13.316	13.338	13.487	13.783	14.219
	9	13.687 r	13.746	13.912	14.498	15.372	16.431
	10	16.456 l	16.470	16.510	16.645 t	16.794 t	17.001 t
1	1	1.841 l	2.055	2.512	3.387	4.107	4.869
	2	3.249	3.306	3.484	4.262	5.200	5.984
	3	5.327 l	5.364	5.477	5.959	6.835	7.905
	4	7.408	7.427	7.485	7.707	8.058	8.535
	5	8.529 l	8.540	8.583	8.819	9.291	10.008
	6	9.353	9.430	9.649	10.416	11.350	11.807
	7	11.155	11.167	11.203	11.354	11.703	12.505
	8	11.696 l	11.716	11.777	12.033	12.534	13.566
	9	14.700	14.709	14.735	14.838	15.010	15.247
	10	14.850 l	14.864	14.904	15.067	15.345	15.748
2	1	2.625	2.587	2.586	2.892	3.507	4.281
	2	3.054 l	3.200	3.512	4.253	4.976	5.695
	3	5.077	5.123	5.261	5.794	6.527	7.257
	4	6.701 l	6.735	6.836	7.250	7.955	8.872
	5	8.987	9.004	9.052	9.246	9.564	10.001
	6	9.961 l	9.980	10.036	10.269	10.671	11.262
	7	12.293	12.322	12.406	12.676	12.992	13.330
	8	13.135	13.121	13.138	13.286	13.583	14.022
	9	13.159 l	13.221	13.349	13.826	14.641	15.731
	10	16.333 l	16.346	16.386	16.496	16.657	16.887
3	1	4.052	4.001	3.982	4.156	4.574	5.163
	2	4.200 l	4.325	4.559	5.160	5.838	6.539
	3	6.931	6.964	7.062	7.430	7.957	8.552
	4	8.009 l	8.040	8.133	8.509	9.129	9.928
	5	10.534	10.548	10.592	10.767	11.057	11.458
	6	11.336 l	11.354	11.409	11.628	11.996	12.523
	7	14.124	14.136	14.172	14.313	14.537	14.831
	8	14.573 l	14.586	14.625	14.779	15.041	15.418
4	1	5.302	5.236	5.222	5.356	5.680	6.152
	2	5.316 l	5.438	5.620	6.114	6.724	7.395
	3	8.746	8.766	8.826	9.059	9.421	9.883
	4	9.275 l	9.306	9.400	9.766	10.336	11.045

Table 4 (*continued*)

m	Mode	$k=0.0$	$k=0.5$	$k=1.0$	$k=2.0$	$k=3.0$	$k=4.0$
	5	12.076	12.089	12.130	12.290	12.556	12.919
	6	12.671 l	12.688	12.740	12.945	13.289	13.776
	7	15.703	15.713	15.742	15.859	16.054	16.320
	8	15.950	15.963	16.002	16.154	16.404	16.756
5	1	6.413 l	6.389	6.389	6.511	6.783	7.180
	2	6.477	6.549	6.688	7.097	7.639	8.268
	3	10.486	10.458	10.460	10.579	10.837	11.209
	4	10.512 l	10.582	10.706	11.077	11.593	12.215
	5	13.628	13.640	13.677	13.822	14.060	14.382
	6	13.975 l	13.992	14.041	14.237	14.567	15.030
6	1	7.498 l	7.498	7.513	7.635	7.875	8.222
	2	7.615	7.656	7.760	8.100	8.581	9.164
	3	11.726 l	11.737	11.771	11.912	12.155	12.491
	4	12.141	12.165	12.237	12.506	12.918	13.438
	5	15.196	15.203	15.228	15.341	15.541	15.820
	6	15.255 l	15.275	15.331	15.536	15.864	16.310

Table 5
Frequencies Ω for an infinite solid cylinder of electromagnetic composite made of ($BaTiO_3$) and ($CoFe_2O_4$), r—radial mode, l—longitudinal mode, t—torsional mode

m	Mode	$k=0.0$	$k=0.5$	$k=1.0$	$k=2.0$	$k=3.0$	$k=4.0$
0	1	3.824 l	3.771	3.746	4.284	5.505	6.469 t
	2	4.580 r	4.750	5.128	5.463 t	5.903 t	6.615
	3	5.084 t	5.109 t	5.181 t	6.132	7.163	8.249
	4	7.002 l	7.023	7.094	7.454	8.244	9.243 t
	5	8.333 t	8.348 t	8.392 t	8.569 t	8.856 t	9.792
	6	10.153 l	10.157	10.174	10.288	10.554	11.107
	7	11.353 r	11.407	11.546 t	11.676 t	11.888 t	12.179 t
	8	11.503 t	11.514 t	11.560	12.090	12.802	13.494
	9	13.297 l	13.307	13.340	13.489	13.816	14.487
	10	14.647 t	14.656 t	14.681 t	14.783 t	14.951 t	15.184 t
1	1	1.841 l	2.023	2.424	3.198	3.907	4.683
	2	2.855	2.916	3.111	3.947	4.870	5.617
	3	5.322 l	5.332	5.380	5.694	6.444	7.438
	4	6.535	6.560	6.629	6.882	7.267	7.802
	5	7.779	7.830	7.962	8.332	8.787	9.421
	6	8.520 l	8.551	8.657	9.177	10.041	10.615
	7	9.879	9.892	9.932	10.093	10.404	11.285
	8	11.683 l	11.691	11.719	11.848	12.113	12.581
	9	13.023	13.033	13.063	13.180	13.373	13.638
	10	14.515	14.540	14.602	14.762	14.965	15.251

Table 5 (*continued*)

m	Mode	$k=0.0$	$k=0.5$	$k=1.0$	$k=2.0$	$k=3.0$	$k=4.0$
2	1	2.326	2.304	2.305	2.605	3.241	4.042
	2	3.053 1	3.158	3.409	4.040	4.699	5.402
	3	4.457	4.511	4.672	5.275	6.049	6.767
	4	6.694 1	6.709	6.763	7.033	7.589	8.389
	5	7.931	7.950	8.006	8.223	8.572	9.064
	6	9.950 1	9.942	9.938	10.027	10.287	10.740
	7	10.393	10.456	10.617	11.087	11.544	11.935
	8	11.450	11.468	11.521	11.771	12.329	13.096
	9	13.145	13.154	13.184	13.320	13.599	14.122
	10	14.493	14.503	14.530	14.639	14.819	15.073
3	1	3.588	3.576	3.568	3.718	4.128	4.734
	2	4.199 1	4.264	4.433	4.930	5.524	6.175
	3	6.091	6.133	6.256	6.704	7.307	7.929
	4	8.001 1	8.017	8.068	8.310	8.772	9.430
	5	9.294	9.310	9.360	9.554	9.876	10.334
	6	11.324 1	11.329	11.348	11.450	11.675	12.052
	7	12.345	12.367	12.428	12.634	12.910	13.241
	8	13.440	13.469	13.556	13.882	14.336	14.786
4	1	4.692	4.689	4.692	4.799	5.095	5.566
	2	5.314 1	5.356	5.475	5.870	6.391	6.989
	3	7.700	7.731	7.822	8.153	8.613	9.125
	4	9.266 1	9.281	9.330	9.550	9.952	10.513
	5	10.650	10.666	10.711	10.892	11.194	11.622
	6	12.658	12.665	12.687	12.792	13.002	13.337
	7	13.821	13.835	13.873	14.019	14.245	14.542
5	1	5.729	5.732	5.744	5.841	6.077	6.459
	2	6.411 1	6.441	6.528	6.844	7.297	7.841
	3	9.247	9.270	9.336	9.579	9.928	10.342
	4	10.5011	10.516	10.563	10.765	11.123	11.612
	5	12.016	12.031	12.074	12.245	12.530	12.929
	6	13.961 1	13.968	13.991	14.094	14.291	14.598
6	1	6.733	6.738	6.756	6.851	7.054	7.377
	2	7.495 1	7.518	7.584	7.842	8.235	8.726
	3	10.723	10.739	10.787	10.964	11.229	11.563
	4	11.714 1	11.729	11.773	11.960	12.283	12.717
	5	13.397	13.411	13.452	13.614	13.882	14.249

corresponds to $m = 1$ for $k = 0$ and 0.5 and as k increases the lowest frequency occurs when $m = 2$. Similar behavior is observed for $BaTiO_3$. It is observed that the lowest frequencies for $CoFe_2O_4$ occur for $m = 1$ for higher values of k. In general, the behavior of the three materials is similar.

Eq. (15) shows the effect of assuming both m and k equal zero. In that case S_{rr} and $S_{\theta\theta}$ govern the behavior of U. V is governed by $S_{r\theta}$ alone and W is given by S_{rz} and all motions are uncoupled. The potentials are functions of the radial co-ordinate as shown by Eq. (16) and (17). If $m = 0$ and $k \neq 0$ it is observed that U and W are coupled through S_{rz}. Similarly, if $m \neq 0$ and $k = 0$, U and V are coupled through $S_{\theta\theta}$ and $S_{r\theta}$.

5. Conclusions

Free vibrations of infinite magneto-electro-elastic cylinders have been studied using a finite element formulation. The analysis of a completely coupled electromagnetic material that has been proposed in the literature is included. Results are presented in tabular format for combinations of circumferential wave number and longitudinal wave number. The governing equations in cylindrical co-ordinates are recorded for future reference. A complete set of consistent non-dimensional parameters have been proposed and used for the magneto-electro-elastic equations in cylindrical co-ordinates.

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Appendix A. Governing equations in terms of displacements

The governing equations in cylindrical coordinates in terms of mechanical displacements u, v, w, electrical potential φ, and magnetic potential Ψ for material properties given by Eq. (8) are as follows:

$$\begin{aligned} &C_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+C_{66}\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+C_{44}\frac{\partial^2 u}{\partial z^2}+(C_{66}+C_{12})\frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta} \\ &-(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial v}{\partial\theta}+(C_{44}+C_{13})\frac{\partial^2 w}{\partial r\partial z}+(e_{31}+e_{15})\frac{\partial^2\varphi}{\partial r\partial z} \\ &+(q_{31}+q_{15})\frac{\partial^2\psi}{\partial r\partial z}=\rho\frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{A.1}$$

$$\begin{aligned} &(C_{66}+C_{12})\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}+(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial u}{\partial\theta}+C_{66}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)+C_{44}\frac{\partial^2 v}{\partial z^2}+C_{11}\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} \\ &+(C_{44}+C_{13})\frac{1}{r}\frac{\partial^2 w}{\partial\theta\partial z}+(e_{31}+e_{15})\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta\partial z}+(q_{31}+q_{15})\frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}=\rho\frac{\partial^2 v}{\partial t^2}, \end{aligned} \tag{A.2}$$

$$(C_{44}+C_{13})\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial z}\right)+C_{44}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)+C_{33}\frac{\partial^2 w}{\partial z^2}+e_{33}\frac{\partial^2\varphi}{\partial z^2}$$
$$+q_{33}\frac{\partial^2\psi}{\partial z^2}+e_{15}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)+q_{15}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)=\rho\frac{\partial^2 w}{\partial t^2}, \tag{A.3}$$

$$e_{15}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)+(e_{31}+e_{15})\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial z}\right)+e_{33}\frac{\partial^2 w}{\partial z^2}-\varepsilon_{33}\frac{\partial^2\varphi}{\partial z^2}$$
$$-m_{33}\frac{\partial^2\psi}{\partial z^2}-\varepsilon_{11}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)-m_{11}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)=0, \tag{A.4}$$

$$q_{15}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)+(q_{31}+q_{15})\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial z}\right)+q_{33}\frac{\partial^2 w}{\partial z^2}-\mu_{33}\frac{\partial^2\psi}{\partial z^2}$$
$$-m_{33}\frac{\partial^2\varphi}{\partial z^2}-\mu_{11}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)-m_{11}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)=0. \tag{A.5}$$

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