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# Estimation of power transmission to a flexible receiver from a stiff source using a power mode approach

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**Abstract**

A power mode approach to estimating the vibrational power transmitted to a receiver structure by multiple sources is reviewed. This approach is then extended to estimating the power transmitted to a flexible receiver from a stiff source through discrete couplings. There may be both translational and rotational coupling degrees of freedoms, and/or force and moment excitations. Approximations are developed for the upper and lower bounds and the frequency average of the transmitted power. These depend only on the point mobilities of the source and receiver, and thus require much less data than an exact description. The approximations are most accurate when the mobility or stiffness mismatch of the coupled system is large enough, e.g., the source has, on average, low mobility compared to that of the receiver. Numerical examples are presented concerning a multi-point coupled beam/plate model.


## 1. Introduction

The analysis of vibration transmission in complex built-up structures is relevant to many practical problems, such as occur in vehicles, aircraft structures and mechanical equipment, etc. Often these structures are assembled from many substructures, possibly with quite different vibration properties, joined together at their interfaces. The mismatch between the local dynamic properties of the substructures may then present a number of challenges to predicting the vibration response of the system. Some of these are characteristic of the so-called "mid-frequency" region, for which generally accepted methodologies have not yet been fully developed.

One common arrangement consists of a stiff source and a flexible receiver. Here the "stiff" source is usually well-defined (i.e., deterministic) with long-wavelengths and/or low modal density,

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while the "flexible" receiver has relatively short-wavelengths and/or high modal density, and its dynamic properties may have significant uncertainties. For such coupling cases it is difficult, or even impossible, to predict the system response with a full, exact description, although theoretically such exact solutions do exist, e.g., through a full finite element (FE) analysis or by the conventional frequency-response-function (FRF)-based substructuring method. Moreover, the dynamic response becomes increasingly sensitive to geometrical imperfections, so that even a very detailed deterministic mathematical model based on the nominal system properties may not yield a reliable response prediction. Therefore, it is generally more useful to approximate the main properties of the system response rather than attempt to predict precisely its details.

Various specialist methods [1–12] have been developed to approximate the dynamic response of such complex built-up systems, especially the power transmitted to the receiver, which has been increasingly accepted as an effective parameter to estimate the mean square response of the receiver and hence the structure-borne sound emission, for example. However, more research is still required due to certain limitations of these existing methods.

In a previous study [13], a power mode approach was developed for estimating the power transmission to a flexible receiver from multiple sources. Ideal force and/or moment excitations were assumed. Here, this approach is extended to the estimation of the power transmitted to a flexible receiver from a stiff source through discrete point couplings. The coupling degrees of freedom (d.o.f.) may involve both translational and rotational motions, and there may be simultaneous force and moment excitations. The difference between this paper and Ref. [13] is that here the influences of the dynamics of the source structure on the transmitted power are considered.

In the next section the power mode technique is briefly reviewed for a structure with multiple sources. In Section 3, this technique is extended to a multi-point coupled stiff source/flexible receiver system. In the first instance, the coupling d.o.f. at the interface points are assumed to be of the same type, e.g., the translational motion due to force coupling. Approximations are made for the upper and lower bounds and the frequency average of the transmitted power, which require less information than an exact description. Then a matrix scaling technique is introduced to extend these approximations for more general cases where the coupling d.o.f. of the system may involve different types, e.g., simultaneous translational and rotational motions. Finally, numerical examples are presented concerning a multi-point coupled beam/plate model with force and moment excitations.

## 2. Power transmission to a structure by multiple sources: review

In this section some results from Ref. [13] are reviewed. Later, these are developed for the case of the power transmission between a stiff source and a flexible receiver.

### 2.1. Power mode theory

Suppose an array of $N$ time harmonic forces at a frequency $\omega$ are applied to a region of a receiving structure whose properties are uniform and homogenous. The time-averaged power transmitted to the receiver can be expressed as

$$P = \tfrac{1}{2}\mathrm{Re}\left\{\mathbf{F}^{\mathrm{H}}\bar{\mathbf{M}}\mathbf{F}\right\} = \tfrac{1}{2}\mathbf{F}^{\mathrm{H}}\mathbf{M}\mathbf{F}, \tag{1}$$

where $\mathbf{F}$ is the vector of amplitudes of the forces, $\bar{\mathbf{M}}$ is the complex mobility matrix of the structure (assumed symmetric), $\mathbf{M} = \mathrm{Re}\left[\bar{\mathbf{M}}\right]$ is the real part of $\bar{\mathbf{M}}$, and the superscript H denotes the conjugate transpose. By matrix theories [14,15], $\mathbf{M}$ can be decomposed into the form

$$\mathbf{M} = \mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^{\mathrm{T}}, \tag{2}$$

where $\mathbf{\Lambda}$ is a real and non-negative diagonal matrix of the eigenvalues $\lambda_n$ of $\mathbf{M}$, $\mathbf{\Psi}$ is the orthogonal matrix composed of the corresponding eigenvectors (in columns), so that $\mathbf{\Psi}\mathbf{\Psi}^{\mathrm{T}} = \mathbf{\Psi}^{\mathrm{T}}\mathbf{\Psi} = \mathbf{I}$, and the superscript T denotes the transpose. The eigenvalues are ordered such that $\lambda_1 > \lambda_2 > \cdots > \lambda_N$. The mean value and standard deviation of these eigenvalues are given by [13]

$$\bar{\lambda} = \frac{\sum_{n=1}^{N} M_{nn}}{N}, \tag{3}$$

$$\sigma = \sqrt{\frac{|M|_2}{N} - \bar{\lambda}^2}, \tag{4}$$

where

$$|M|_2 = \sum_{m=1}^{N}\sum_{n=1}^{N} M_{mn}^2 \tag{5}$$

is the second order norm of $\mathbf{M}$.

Let the force vector $\mathbf{F}$ now be weighted by $\mathbf{\Psi}$ so to give a new set of power mode forces whose amplitudes are given by

$$\mathbf{Q} = \mathbf{\Psi}^{\mathrm{T}}\mathbf{F}. \tag{6}$$

Combining Eqs. (1), (2) and (6) gives

$$P = \frac{1}{2}\sum_{n=1}^{N}|Q_n|^2\lambda_n. \tag{7}$$

Eq. (7) shows that the vibrational power transmitted to the receiver by $N$ forces can be regarded as the power transmitted by $N$ independent power mode contributions, each one of them being related to only one force distribution (eigenvector) and one eigenvalue [13].

Mobility matrix eigenproperties were first used to find simple approximations of the transmitted power in Refs. [7,9,10]. Further investigations were made in Ref. [13] to estimate the upper and lower bounds of the transmitted power, as well as its mean value. These are given below.

### 2.2. Approximations to the power transmission

Strict upper and lower bounds of the transmitted power can be given from Eqs. (6) and (7), by

$$P_{up} = \frac{1}{2}\left(\sum_{n=1}^{N}|F_n|^2\right)\lambda_1, \tag{8}$$

$$P_{low} = \frac{1}{2}\left(\sum_{n=1}^{N}|F_n|^2\right)\lambda_N, \tag{9}$$

where $\lambda_1$ and $\lambda_N$ are the maximum and minimum eigenvalues of $\mathbf{M}$ [7,9]. It is seen that the bounds need less information than an exact description in Eq. (1). However, the range between these strict bounds may be so large as to be of little value. Hence there is interest in approximating these bounds to narrow the range between them.

For a short-wavelength receiver structure, the correlations between the individual excitations can be regarded as being relatively small, at least when frequency averaged [13]. If it is also assumed that the local driving point properties of the receiver have the same order of magnitude, the eigenvalues $\lambda_n$ typically are then of the same order of magnitude. As a result, both $\lambda_1$ and $\lambda_N$ will often lie within, say, one standard deviation of the mean, so that they can be simply approximated as

$$\lambda_1 \approx (\bar{\lambda} + \sigma), \tag{10}$$

$$\lambda_N \approx (\bar{\lambda} - \sigma), \tag{11}$$

where the mean value $\bar{\lambda}$ and standard deviation $\sigma$ are given by Eqs. (3) and (4). The upper and lower bounds for the power transmitted to a short-wavelength receiver, therefore, can be simply approximated by

$$P'_{up} \approx \frac{1}{2}\left(\sum_{n=1}^{N}|F_n|^2\right)(\bar{\lambda} + \sigma), \tag{12}$$

$$P'_{low} \approx \frac{1}{2}\left(\sum_{n=1}^{N}|F_n|^2\right)(\bar{\lambda} - \sigma). \tag{13}$$

For a long-wavelength receiver structure, however, individual excitations may be strongly correlated. For example, when the response is dominated by a single resonant mode, then, at the resonant frequency, the driving point mobility tends to be comparable to the transfer mobility, i.e.,

$$|M_{nn}| \approx |M_{mn}|. \tag{14}$$

As a result, $\lambda_1$ tends to be much larger than the smallest eigenvalue $\lambda_N$. Then the lower bound in Eq. (9) will be too conservative to be of practical value. Under such circumstances, it is more useful to replace the lower bound by an approximation for the power associated with the first power mode. In Ref. [13], this is approximated by

$$P_1 \approx \frac{1}{2N}\left|\sum_{n=1}^{N}F_n\right|^2(\bar{\lambda} + \sigma). \tag{15}$$

It is also useful to estimate the mean value of the power over a range of frequencies. An estimate of this frequency-averaged power can be found by taking the average over all the power modes, as [13]

$$\bar{P} = \frac{N}{2}\left(\frac{1}{N}\sum_{n=1}^{N}|F_n|^2\right)\left(\frac{1}{N}\sum_{n=1}^{N}M_{nn}\right). \tag{16}$$

Eq. (16) is in a very simple form and was also given in Ref. [8].

So far, the power transmitted to a receiver structure by an array of point forces has been described in terms of upper and lower bounds and a mean value.

### 2.3. Combined force and moment excitations

It is known that in many cases of practical interest, vibration sources apply moments as well as forces, and the power transmitted by moment excitations is generally greater at higher frequencies [11,12]. Therefore it is necessary to also consider moment excitations. For such cases, however, the elements of $\mathbf{F}$ (and $\mathbf{M}$) have different units, and the approximations developed in the above subsections can no longer be applied. In Refs. [10,13] a scaling technique was used to deal with this problem of dimensional incompatibility. The main principle of this scaling technique is to scale the mobility matrix $\mathbf{M}$ by a specified diagonal matrix to give a new "dimensionless" matrix, and then to weight the physical force vector using the same diagonal matrix to give a new set of forces each of which has the same units. This is briefly described below.

The scaling matrix $\mathbf{D}_C$ is defined as a real and diagonal matrix, with the $n$th diagonal element given by

$$D_{C,nn} = \frac{1}{\sqrt{M_{nn}}}, \tag{17}$$

where $M_{nn} = \mathrm{Re}\{\bar{M}_{nn}\}$. Let $\mathbf{M}$ and $\mathbf{F}$ be scaled by $\mathbf{D}_C$ as

$$\mathbf{M}_C = \mathbf{D}_C\mathbf{M}\mathbf{D}_C, \tag{18}$$

$$\mathbf{F}_C = \mathbf{D}_C^{-1}\mathbf{F}. \tag{19}$$

It is seen that $\mathbf{M}_C$ is now a dimensionless, real, symmetric and non-negative matrix, and the elements of $\mathbf{F}_C$ have the same units. Combining Eqs. (18) and (19) with (1) gives

$$P = \tfrac{1}{2}\mathbf{F}_C^{\mathrm{H}}\mathbf{M}_C\mathbf{F}_C. \tag{20}$$

Then approximations for the upper and lower bounds and the mean value of the transmitted power can be given by Eqs. (12), (13) (or (15)) and (16), with $\mathbf{F}$ being replaced by $\mathbf{F}_C$, and $\mathbf{M}$ by $\mathbf{M}_C$.

Instead of Eq. (17), an alternative scaling matrix may be defined where

$$D_{C,nn}^{\infty} = \frac{1}{\sqrt{M_{nn}^{\infty}}}, \tag{21}$$

where $M_{nn}^{\infty}$ is the real part of the characteristic point mobility of the receiver structure, i.e., the point mobility if the receiver structure is extended to infinity. The differences between Eqs. (17) and (21) are that the former gives better estimates of the power while the latter requires no detailed knowledge of the properties of the receiver structure, e.g., the boundary conditions.

A similar scaling technique can be applied if the characteristic point mobilities for different input points are substantially different, for example if there is simultaneous excitation of in-plane and out-of-plane vibrations of a plate.

The power mode approach described above assumes ideal source excitations, so that it has only limited practical applications. In the following section a similar approach will be used to estimate the power transmitted to a flexible receiver from a stiff source through discrete couplings.

## 3. Power transmission between a stiff source and a flexible receiver

Suppose a source structure is coupled to a receiver structure through $N$ discrete points. The power transmitted from the source to the receiver can be written as

$$P = \tfrac{1}{2}\mathbf{F}_I^{\mathrm{H}}\mathbf{M}_R\mathbf{F}_I, \tag{22}$$

where $\mathbf{M}_R$ is the real part of the mobility matrix $\bar{\mathbf{M}}_R$ of the receiver at the interface points before coupling and $\mathbf{F}_I$ is the interface force vector caused by the interaction between the source and the receiver. By using the conventional FRF-based substructuring method [16,17], the interface force vector is given by

$$\mathbf{F}_I = \left(\bar{\mathbf{M}}_S + \bar{\mathbf{M}}_R\right)^{-1}\mathbf{V}_{sf}, \tag{23}$$

where $\bar{\mathbf{M}}_S$ is the complex mobility matrix of the source at the interface points before coupling, and $\mathbf{V}_{sf}$ is the free velocity vector of the source substructure at the interface points. Here free velocity is used to define the source strength, which has the advantage of allowing simple comparisons between different sources [8].

As a result, a general expression for the power transmission within the system can be written as

$$P = \frac{1}{2}\mathrm{Re}\left\{\mathbf{V}_{sf}^{\mathrm{H}}\left[\left(\bar{\mathbf{M}}_S + \bar{\mathbf{M}}_R\right)^{-1}\right]^{\mathrm{H}}\bar{\mathbf{M}}_R\left(\bar{\mathbf{M}}_S + \bar{\mathbf{M}}_R\right)^{-1}\mathbf{V}_{sf}\right\}. \tag{24}$$

Obviously Eq. (24) will be inconvenient if the number of interface d.o.f. is very large, and/or the required FRF data are not known to sufficient accuracy, due to some uncertainty of the receiver properties, for example. Approximations for the upper and lower bounds and the frequency average power are thus of interest, especially if these require less information than an exact description. In this section, the power mode approach is used in two stages: first, the coupling forces are assumed to involve only translational motion at the coupling d.o.f., and secondly, there may be both translational and rotational coupling d.o.f.

### 3.1. Translational coupling motion only

In this case it is assumed that the coupling d.o.f. of the source/receiver system all act in the same direction, e.g., normal to the surface of a plate-like receiver. This can be further defined to be where all the coupling forces and d.o.f. are dynamically similar, e.g., all out-of-plane, bending vibrations of a flexible plate.

*3.1.1. Approximations to the upper and lower bounds of the transmitted power*

Similar to Eq. (9), a strict lower bound for the transmitted power of Eq. (22) can be expressed as

$$P_{low} = \frac{1}{2}\left(\sum_{n=1}^{N}\left|F_{I,n}\right|^2\right)\lambda_{min}^{R}, \tag{25}$$

where $\lambda_{min}^{R}$ is the minimum eigenvalue (power mode mobility) of $\mathbf{M}_R$. From Eq. (23), it follows that

$$\sum_{n=1}^{N}\left|F_{I,n}\right|^2 = \mathbf{V}_{sf}^{\mathrm{H}}\bar{\mathbf{M}}_{RS}^{-1}\mathbf{V}_{sf}, \tag{26}$$

where

$$\bar{\mathbf{M}}_{RS} = (\bar{\mathbf{M}}_R + \bar{\mathbf{M}}_S)(\bar{\mathbf{M}}_R + \bar{\mathbf{M}}_S)^{\mathrm{H}} \tag{27}$$

is given by the combination of mobility matrices of the source and the receiver. (Note that $\bar{\mathbf{M}}_{RS}$ here does not represent a mobility matrix.) It is seen that $\bar{\mathbf{M}}_{RS}$ is a Hermitian matrix, and thus can be decomposed into a diagonal form

$$\bar{\mathbf{M}}_{RS} = \mathbf{\Phi}\mathbf{\Lambda}_{RS}\mathbf{\Phi}^{\mathrm{H}}. \tag{28}$$

Since Eq. (26) is a positive semi-definite quadratic form, $\mathbf{\Lambda}_{RS}$ is a real and non-negative diagonal matrix, and $\mathbf{\Phi}$ is a unitary matrix ($\mathbf{\Phi}\mathbf{\Phi}^{-1} = \mathbf{\Phi}^{-1}\mathbf{\Phi} = \mathbf{I}$). Then a strict lower bound for $\sum_{n=1}^{N}\left|F_{I,n}\right|^2$ is

$$\sum_{n=1}^{N}\left|F_{I,n}\right|^2 \geqslant \left(\sum_{n=1}^{N}\left|V_{sf,n}\right|^2\right)\frac{1}{\lambda_{max}^{RS}}, \tag{29}$$

where $\lambda_{max}^{RS}$ is the maximum eigenvalue of matrix $\bar{\mathbf{M}}_{RS}$, and satisfies [18]

$$\sqrt{\lambda_{max}^{RS}} \leqslant \max_n\left|\bar{M}_{R,nn} + \bar{M}_{S,nn}\right| + \sqrt{\sum\nolimits_{n\neq m}\sum\left|\bar{M}_{R,nm} + \bar{M}_{S,nm}\right|^2}. \tag{30}$$

Combining Eqs. (25), (29) and (30), a strict lower bound of the transmitted power is then given by

$$P_{low} = \frac{\frac{1}{2}\left(\sum_{n=1}^{N}\left|V_{sf,n}\right|^2\right)\lambda_{min}^{R}}{\left(\max_n\left|\bar{M}_{R,nn} + \bar{M}_{S,nn}\right| + \sqrt{\sum_{n\neq m}\sum\left|\bar{M}_{R,nm} + \bar{M}_{S,nm}\right|^2}\right)^2}. \tag{31}$$

Eq. (31) implies that the power transmission is small either if the free-velocity distribution of the source structure is proportional to the eigenvector corresponding to the largest eigenvalue of $\bar{\mathbf{M}}_{RS}$, or if the interface force distribution is proportional to the eigenvector corresponding to the smallest eigenvalue of $\mathbf{M}_R$.

It is difficult to find an expression for the upper bound of the transmitted power, similar to Eq. (5), using the above eigendecomposition approach. However, a convenient approximation to the maximum transmitted power can be estimated by simply assuming the mobility matrix $\bar{\mathbf{M}}_S$ in

Eq. (24) is zero. As a result, the upper bound of the transmitted power can be written as

$$P'_{up} \approx \tfrac{1}{2}\mathrm{Re}\left\{\mathbf{V}_{sf}^{\mathrm{H}}\bar{\mathbf{M}}_R^{-1}\mathbf{V}_{sf}\right\}. \tag{32}$$

Although it is not a strict maximum, it tends to be a good, somewhat conservative approximation which is rarely exceeded in practice. Physically Eq. (32) means that the maximum power transmission occurs when the source can almost be treated as a set of free velocities.

A flexible receiver structure usually implies a relatively short wavelength and/or heavy damping and also a relatively high modal density. In such cases, as mentioned above, the correlations between the individual driving points are often relatively small, at least when frequency averaged, so that the local driving point properties of the receiver at different coupling points can be regarded as being uncorrelated. Under such circumstances, one can write the approximate relations

$$\lambda_{min}^R \approx \min_n\{M_{R,nn}\}, \tag{33}$$

$$M_{R,mn} \approx 0, \ m \neq n. \tag{34}$$

Hence, the upper and lower bounds of the transmitted power can be approximated as

$$P'_{up} \approx \frac{1}{2}\sum_{n=1}^{N}\frac{|V_{sf,n}|^2}{M_{R,nn}}, \tag{35}$$

$$P'_{low} \approx \frac{1}{2}\left(\sum_{n=1}^{N}|V_{sf,n}|^2\right)\frac{\min_n\ \{M_{R,nn}\}}{\left(\max_n\ |\bar{M}_{R,nn}| + \left(1+\sqrt{N(N-1)}\right)\max_n |\bar{M}_{S,nn}|\right)^2}. \tag{36}$$

It is seen from Eqs. (35) and (36) that these approximations for the upper and lower bounds depend only on the diagonal elements of the mobility matrices of the source and receiver. Although these approximations are less conservative than the corresponding exact results, they are much easier to predict since the amount of data required is substantially reduced.

From Eqs. (35) and (36), it is seen that the width of the range of power is closely related to the mobility mismatch between the source and the receiver. The lower the mobility of the source compared to that of the receiver, or the stiffer the source compared to the receiver, the narrower is the range between these limits. If it is assumed that the local driving point mobilities of the receiver points are approximately equal, or at least of comparable magnitudes, the approximations to both the upper and lower bounds of the transmitted power can be quite close to the "exact" value, provided that the mobility mismatch between the source and the receiver is big enough. If the receiver structure is much more flexible than the source so as to meet the condition

$$\max_n|\bar{M}_{R,nn}| >> \left(1+\sqrt{N(N-1)}\right)\max_n|\bar{M}_{S,nn}| \tag{37}$$

the approximations to both the upper and lower limits of the power can then be very close to the exact value. Under such circumstances, the power can actually be treated as that transmitted by a set of free velocities $\mathbf{V}_{sf}$, i.e.,

$$P \approx \frac{1}{2}\sum_{n=1}^{N}|V_{sf,n}|^2\frac{1}{M_{R,nn}}. \tag{38}$$

Eq. (37) actually gives the condition under which the flexibility of the source structure itself may be neglected, so that the power transmitted to the receiver can be regarded as that by a set of ideal velocities, i.e., a set of point-velocities uncorrelated with each other. Therefore, provided the mobility-mismatch of the system meets the condition of Eq. (37), the approximations developed in Section 2 may be quite useful for estimating the power transmitted to a receiver structure which is not flexible enough to meet the conditions of Eqs. (33) and (34).

#### 3.1.2. Approximation to the frequency average transmitted power

In the above subsection, the upper bound is estimated by assuming that the mobilities of the source structure at the interface points are zero, while the approximation for the lower bound was made when both the mobility terms of the source and receiver at the interface are included. The upper bound given in Eq. (35) tends to be more conservative than the lower bound given in Eq. (36), especially when the local driving properties of the receiver at each coupling point are similar. This implies that the latter approximation is likely to be closer to the exact value than the former. It is also known that the frequency average of the real part of the point mobility of a structure approximates that of the characteristic point mobility, i.e., the point mobility of the equivalent infinite structure [19]. Therefore, one can write an approximation to the frequency average of the transmitted power as

$$\bar{P} \approx \frac{1}{2}\left(\sum_{n=1}^{N} |V_{sf,n}|^2\right) \frac{Re\left\{\bar{M}^{\infty}_{R,nn}\right\}}{\left(|\bar{M}^{\infty}_{R,nn}| + \left(1+\sqrt{N(N-1)}\right)\, \max_n\, |\bar{M}_{S,nn}|\right)^2}, \tag{39}$$

where $\bar{M}^{\infty}_{R,nn}$ is the characteristic point mobility of the receiver structure. This relation is particularly valid for non-tonal excitation or for tonal excitation if the modal overlap of the receiver is large enough, i.e., if the excitation excites resonant response in at least a few receiver modes.

Generally, Eq. (39) is accurate if the source/receiver system has a large enough mobility mismatch. Otherwise, Eq. (39) tends to underestimate the true frequency average.

### 3.2. Translational and rotational coupling

In many cases of practical interest, the power transmitted to the receiver may arise partly from the translational and partly from the rotational motions of the coupling d.o.f., e.g., due to simultaneous force and moment excitations. The former is usually the largest, but the latter can be substantial at higher frequencies [11,12]. Therefore, it is important to consider combined translational and rotational motions of the coupling d.o.f.

Under such circumstances, however, $\mathbf{V}_{sf}$, $\bar{\mathbf{M}}_S$ and $\bar{\mathbf{M}}_R$ are composed of elements with different units. The approximations made above can no longer be applied. Nevertheless, this problem of dimensional incompatibility can be overcome by using a matrix scaling technique similar to that of Section 2.3. This is described in the following subsections. Similar scaling approaches can be extended for more general cases where the coupling d.o.f. are different types, e.g., the simultaneous in-plane and out-of-plane vibrations of a structure.

### 3.2.1. Approximations to the upper and lower bounds of the transmitted power

Let the mobility matrices $\bar{\mathbf{M}}_S$ and $\bar{\mathbf{M}}_R$ be scaled by a real diagonal matrix $\mathbf{D}'_C$ whose $n$th diagonal element is defined as

$$D'_{C,nn} = \frac{1}{\sqrt{|\bar{M}_{R,nn}|}} \tag{40}$$

so as to give two dimensionless matrices $\bar{\mathbf{M}}_S^C$ and $\bar{\mathbf{M}}_R^C$ as

$$\bar{\mathbf{M}}_S^C = \mathbf{D}'_C \bar{\mathbf{M}}_S \mathbf{D}'_C, \tag{41}$$

$$\bar{\mathbf{M}}_R^C = \mathbf{D}'_C \bar{\mathbf{M}}_R \mathbf{D}'_C. \tag{42}$$

It is seen from Eq. (41) that

$$|\bar{M}_{R,nn}^C| = 1. \tag{43}$$

Let $\mathbf{F}_I$ and $\mathbf{V}_{sf}$ be weighted by $\mathbf{D}'_C$ to give a new set of forces and a new set of free velocities, whose elements all have the same units, as

$$\mathbf{F}_I^C = \mathbf{D}'^{-1}_C \mathbf{F}_I, \tag{44}$$

$$\mathbf{V}_{sf}^C = \mathbf{D}'_C \mathbf{V}_{sf}. \tag{45}$$

The power transmission in Eq. (24) can then be re-written as

$$P = \tfrac{1}{2}\mathrm{Re}\left\{\mathbf{V}_{sf}^{C\mathrm{H}}\left[\left(\bar{\mathbf{M}}_S^C + \bar{\mathbf{M}}_R^C\right)\right]^{-1\mathrm{H}}\bar{\mathbf{M}}_R^C\left(\bar{\mathbf{M}}_S^C + \bar{\mathbf{M}}_R^C\right)^{-1}\mathbf{V}_{sf}^C\right\}. \tag{46}$$

Consequently, the approximations for the upper and lower bounds of the transmitted power can be made in analogy to Eqs. (35) and (36), with $\mathbf{V}_{sf}$, $\bar{\mathbf{M}}_S$ and $\bar{\mathbf{M}}_R$ being replaced by $\mathbf{V}_{sf}^C$, $\bar{\mathbf{M}}_S^C$ and $\bar{\mathbf{M}}_R^C$.

By replacing the terms of $M_{R,nn}^C$ by its maximum possible value of unity (given by Eq. (43)), the approximations for the upper and lower bounds of the power can be further simplified as

$$P'_{up} \approx \frac{1}{2}\sum_{n=1}^{N}\left|V_{sf,n}^C\right|^2, \tag{47}$$

$$P'_{low} \approx \frac{1}{2}\left(\sum_{n=1}^{N}\left|V_{sf,n}^C\right|^2\right)\frac{1}{\left(1 + \left(1 + \sqrt{N(N-1)}\right)\ \max_n \left|\bar{M}_{S,nn}^C\right|\right)^2}. \tag{48}$$

So far, the bounds of the power transmission for a stiff source/flexible receiver system with both translational and rotational motions of coupling d.o.f. can be estimated by Eqs. (47) and (48).

### 3.2.2. Approximation to the frequency average transmitted power

A similar scaling approach can be used to approximate the frequency average of the transmitted power. Here the scaling matrix $\mathbf{D}'^{\infty}_C$ is used where

$$D'_{C,nn}\infty = \frac{1}{\sqrt{|\bar{M}_{R,nn}^{\infty}|}},\ D'_{C,nm}\infty = 0, n \neq m \tag{49}$$

and where $\bar{M}^{\infty}_{R,nn}$ is the characteristic point mobility of the receiver. Then the approximation for the frequency average of the transmitted power can be written in a form similar to Eq. (39) as

$$\bar{P} \approx \frac{1}{2}\left(\sum_{n=1}^{N}\left|V^{C_\infty}_{sf,n}\right|^2\right)\frac{1}{\left(1+\left(1+\sqrt{N(N-1)}\right)\max_n\left|\bar{M}^{C_\infty}_{S,nn}\right|\right)^2}, \tag{50}$$

where $V^{C_\infty}_{sf,n}$ are a new set of scaled free velocities of the source structure, given by

$$\mathbf{V}^{C_\infty}_{sf} = \mathbf{D}'^{\infty}_{C}\mathbf{V}_{sf} \tag{51}$$

and $\bar{M}^{C_\infty}_{S,nn}$ is the $(n,n)$th element of the scaled matrix

$$\bar{\mathbf{M}}^{C_\infty}_{S} = \mathbf{D}'^{\infty}_{C}\bar{\mathbf{M}}_{S}\mathbf{D}'^{\infty}_{C}. \tag{52}$$

The approximations of Eqs. (47), (48) and (50) are thus related to only the scaled point mobility terms of the source and the receiver.

## 4. Numerical examples

Numerical examples are presented here to demonstrate the approach developed in Section 3. Since many of the basic features of structures of practical concern can be reduced to relatively simple configurations of beams and plates, the numerical model considered here is a beam/plate system coupled by four evenly spaced points, as shown in Fig. 1. The beam is chosen to be relatively stiff and with a low modal density compared to that of the plate. Both the beam and the plate are simply supported for simplicity. External time harmonic force and moment excitations act at a distance $\xi$ from one end of the beam. The dimensions of the system and the coupling positions are listed in Table 1. The material properties of the system are those of perspex, given in Table 2. Three different plate thicknesses are used to vary the stiffness of the plate receiver. The stiffness (mobility) mismatch of the system here is indicated by wavenumber ratios $k_p/k_b$ =2.5, 3.5 and 5.6, corresponding to plate thicknesses of 0.010, 0.005 and 0.002 m, respectively, with $k_p$ and $k_b$ being the wavenumbers of the plate and the beam. Approximations to the transmitted power from the beam to the plate are made under the following two circumstances: first, only translational coupling d.o.f. are involved; secondly, both translational and rotational coupling d.o.f. are considered. The approximate results are compared to exact results found using the conventional FRF-based substructuring technique. A running frequency average, i.e., smoothing

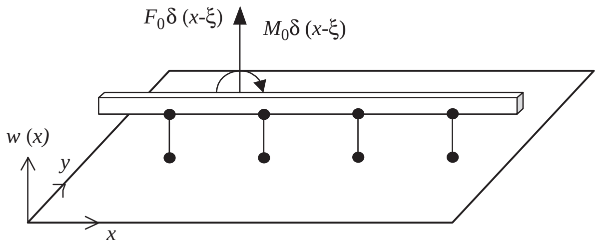


Fig. 1. Multi-point coupled beam/plate system.

Table 1
System dimensions and coupling positions

| | Dimension sizes (m) | Coupling positions (m) | Excitation position (m) |
|---|---|---|---|
| Beam | Length = 2<br>Width = 0.059<br>Height = 0.068 | $x_1 = 0.40$, $x_2 = 0.80$<br>$x_3 = 1.20$, $x_4 = 1.60$ | $\xi = 0.73$ |
| Plate | Length = 2<br>Width = 0.9<br>Thickness = 0.010, 0.005, 0.002 | $(x_1, y_1) = (0.40, 0.45)$<br>$(x_2, y_2) = (0.80, 0.45)$<br>$(x_3, y_3) = (1.20, 0.45)$<br>$(x_4, y_4) = (1.60, 0.45)$ | |

Table 2
Material properties for the beam/plate system

| Young's modulus (GN/m$^2$) | The Poisson ratio | Loss factor | Density (kg/m$^3$) |
|---|---|---|---|
| 4.4 | 0.38 | 0.05 | 1152 |

technique has been used for all figures given in this section to determine the broad features of the transmitted power. The plate modal densities are 0.15, 0.30 and 0.74 mode/Hz when the plate thickness is 0.010, 0.005 and 0.002 m, respectively. The bandwidth used in the smoothing is 10 Hz so that each band consists of a few (plate) vibration modes.

### 4.1. *Transmitted power with translational coupling d.o.f. only*

When the system is assumed to have only translational coupling d.o.f., the beam and the plate generally rotate through different angles, and the coupling moments are zero. Under such circumstances, the transmitted power only has contributions from the translational coupling d.o.f., so that the estimates for the power transmitted to the plate can be found using the expressions given in Section 3.1. Figs. 2–4 compare the exact and the approximate results for the transmitted power, when a time harmonic force of magnitude 1 is applied to the beam, where $k_p/k_b = 2.5$, 3.5 and 5.6, respectively. It is seen clearly that the accuracy of the approximations increases as the mobility mismatch between the beam and the plate increases, as expected. When the plate is much more flexible than the beam, the transmitted power can be simply approximated by that transmitted by a set of free velocities, as shown in Fig. 4.

The above upper and lower bound calculations need the point-mobilities of the plate at all the interface d.o.f. to be known exactly. In principle, this requires detailed knowledge of the modal properties of the plate. In many cases, however, this is impractical, or it may even be impossible to find these values accurately. In such cases, the plate receiver may be approximated by regarding it as extending uniformly to infinity in a manner analogous to that described in Section 3.2.2. The upper bound can then be approximated by replacing $M_{R,nn}$ in Eq. (35) by $Re\left\{\bar{M}^{\infty}_{R,nn}\right\}$, while the lower bound expression of Eq. (36) becomes Eq. (39). Fig. 5 shows such approximations for the

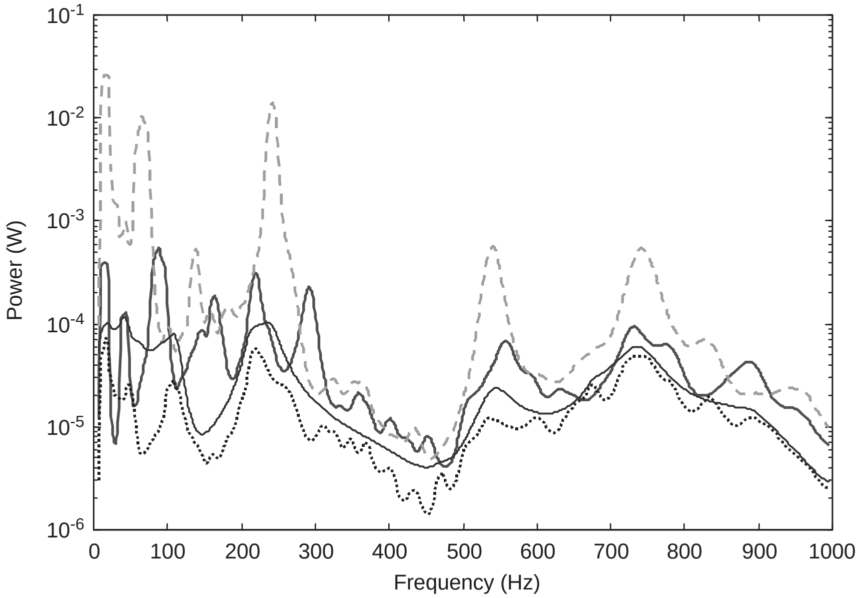


Fig. 2. Power transmitted to the plate when only translational coupling d.o.f. are assumed, $k_p/k_b = 2.5$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (35)), lower bound (. . . . . . , Eq. (36)) and frequency average (——, Eq. (39)).

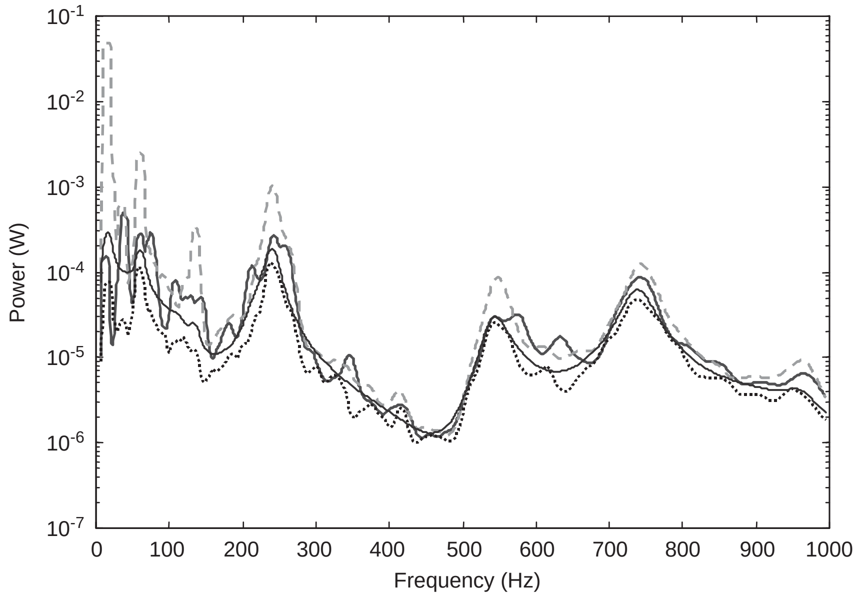


Fig. 3. Power transmitted to the plate when only translational coupling d.o.f. are assumed, $k_p/k_b = 3.5$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (35)), lower bound (. . . . . . , Eq. (36)) and frequency average (——, Eq. (39)).

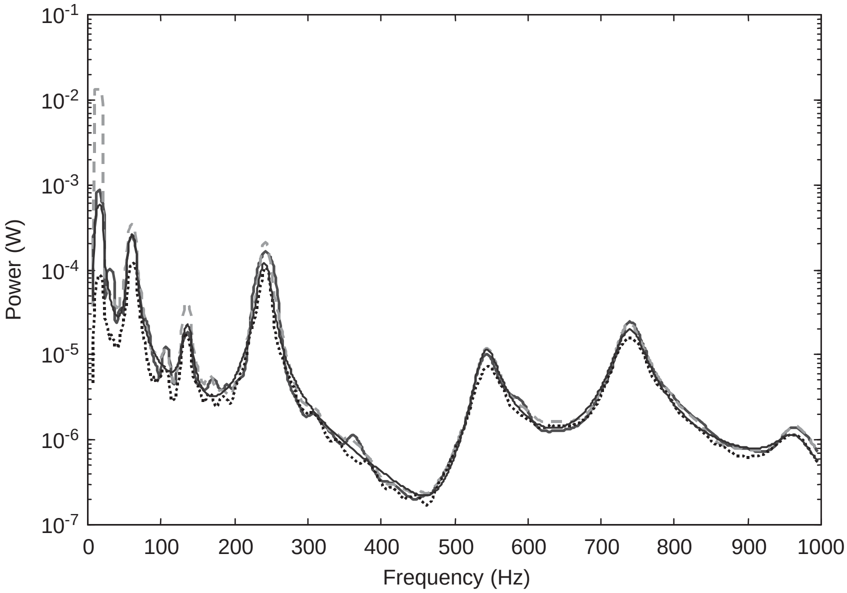


Fig. 4. Power transmitted to the plate when only translational coupling d.o.f. are assumed, $k_p/k_b = 5.6$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (35)), lower bound (. . . . . ., Eq. (36)) and frequency average (——, Eq. (39)).

case where $k_p/k_b = 2.5$. By comparison with Fig. 2, it can be seen that this further approximation gives reasonable results even for this case of modest wavenumber mismatch. When the plate receiver is relatively very flexible compared to the beam, however, one needs to use only Eq. (39) to get good estimates for the transmitted power, as shown in Figs. 3 and 4.

### 4.2. Transmitted power with both translational and rotational coupling d.o.f.

When the system has both translational and rotational coupling d.o.f., the translation and the rotation $\partial w/\partial y$ of the beam and the plate at the interface d.o.f. are equal. (There is assumed to be no coupling between the torsion in the beam and the rotation $\partial w/\partial x$ in the plate.) Under such a circumstance, the transmitted power has contributions from both translational and rotational coupling d.o.f., and thus can be approximated by the expressions described in Section 3.2. Figs. 6–8 compare the exact transmitted power to the approximations, when a time harmonic force and moment of magnitudes 1 and 0.5, respectively, are applied to the beam at the point $\xi = 0.73$, where $k_p/k_b = 2.5$, 3.5 and 5.6, respectively. Once again the approximations become closer to the exact values as the wavenumber ratio increases.

The plate receiver may also be simply approximated as being an infinite structure when it is difficult to determine the exact values of the relevant point mobilities. Consequently, the upper bound of the transmitted power can be estimated by replacing $V^{C}_{sf,n}$ in Eq. (47) by $V^{C_\infty}_{sf,n}$, while the lower bound is now given by Eq. (48). When the plate receiver is relatively very flexible compared to the beam, as shown in Fig. 8, only Eq. (50) is needed to approximate the transmitted power

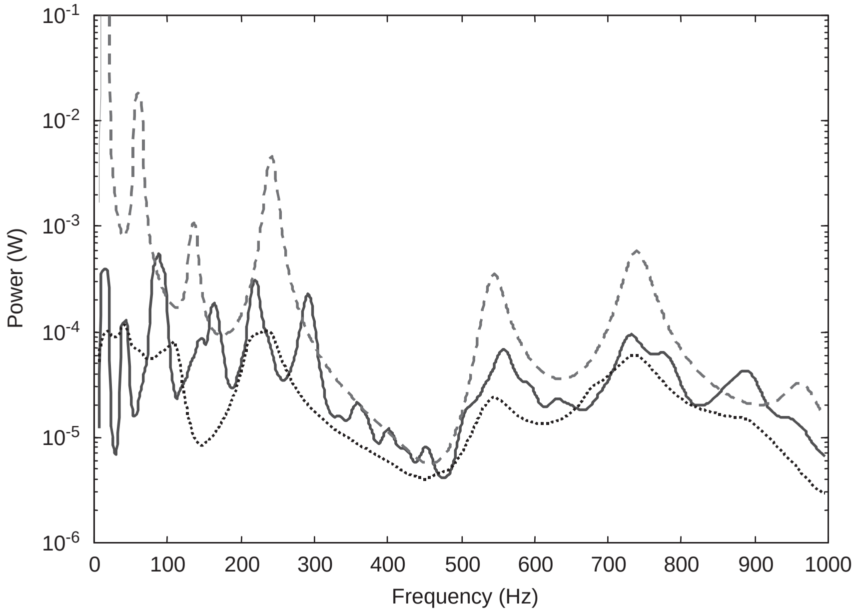


Fig. 5. Power transmitted to the plate when only translational coupling d.o.f. are assumed and the plate is approximated as being infinite, $k_p/k_b = 2.5$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (35)) and lower bound (. . . . . ., Eq. (39)).

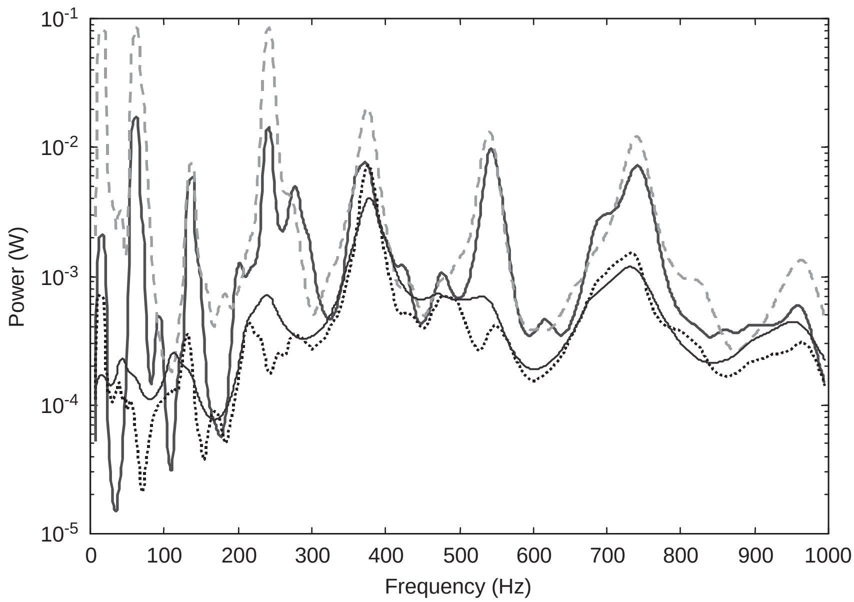


Fig. 6. Power transmitted to the plate when both translational and rotational coupling d.o.f. are considered, $k_p/k_b = 2.5$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (47)), lower bound (. . . . . ., Eq. (48)) and frequency average (——, Eq. (50)).

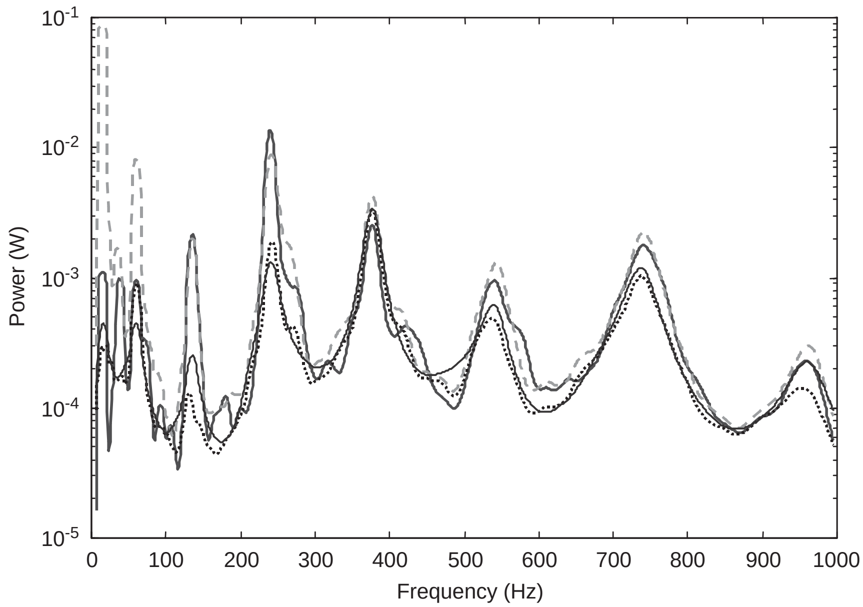


Fig. 7. Power transmitted to the plate when both translational and rotational coupling d.o.f. are considered, $k_p/k_b = 3.5$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (47)), lower bound (......, Eq. (48)) and frequency average (——, Eq. (50)).

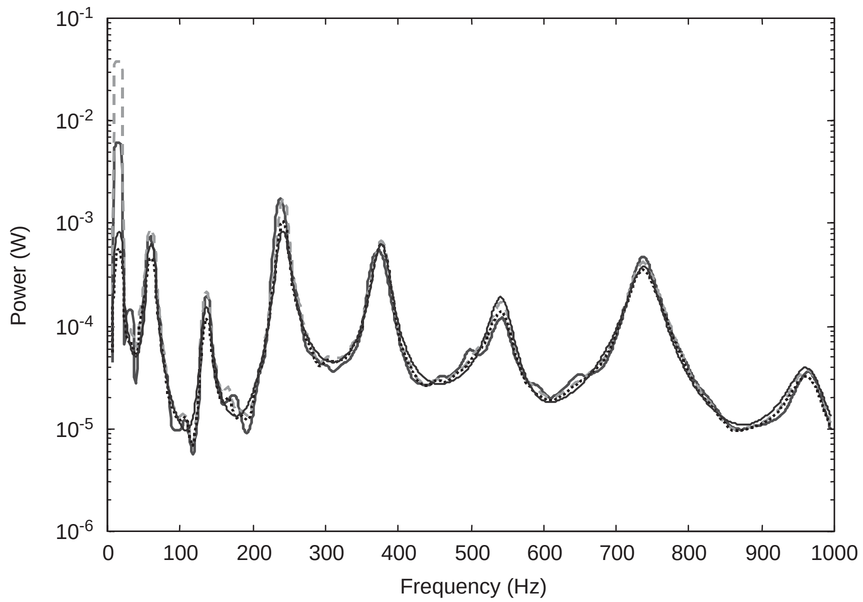


Fig. 8. Power transmitted to the plate when both translational and rotational coupling d.o.f. are considered, $k_p/k_b = 5.6$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (47)), lower bound (......, Eq. (48)) and frequency average (——, Eq. (50)).

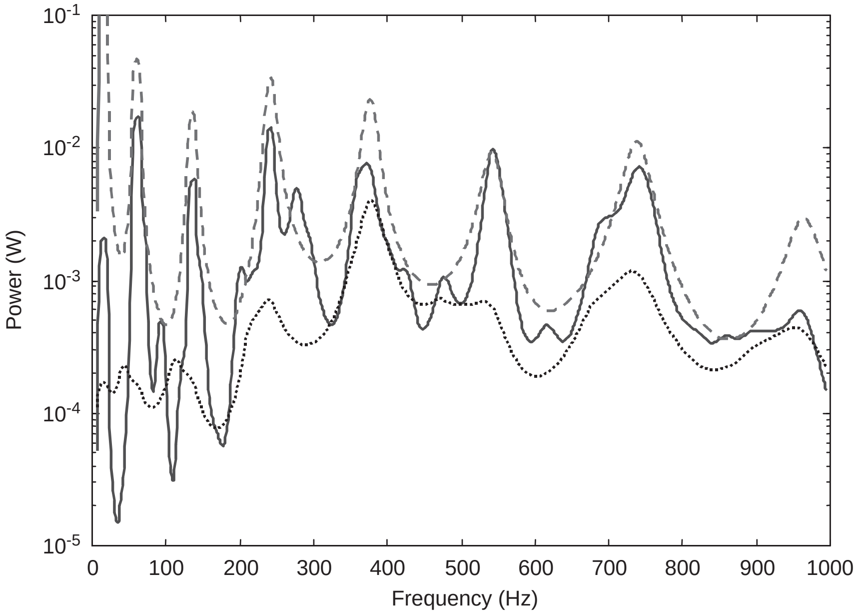


Fig. 9. Power transmitted to the plate when both translational and rotational coupling d.o.f. are considered and the plate is approximated as being infinite, $k_p/k_b = 2.5$: exact (▬▬, Eq. (24)) and approximations to the upper bound (– – –, Eq. (47)) and lower bound (……, Eq. (48)).

accurately. If that is not the case, however, Eq. (47) (where $V^C_{sf,n} = V^{C_\infty}_{sf,n}$) can be used together with Eq. (48) to roughly approximate the broad features of the transmitted power. This is shown in Fig. 9, where $k_p/k_b = 2.5$.

## 5. Concluding remarks

In this paper a power mode approach to estimating the power transmitted to a receiver structure from multiple sources was extended to the case of power transmission between a stiff source and a flexible receiver through discrete couplings. In the first instance, the coupling d.o.f. at the interface points were assumed to be of a dynamically similar type, e.g., the translational motion due to force coupling. Then more general source/receiver systems were considered where the coupling d.o.f. could be of different types, e.g., the simultaneous translational and rotational motions. A matrix scaling technique was introduced. Approximations were developed for the upper and lower bounds and the frequency average of the transmitted power.

These approximations were found depending only on the point mobilities of the source and receiver, and thus the amount of data required can be reduced substantially compared to an exact description.

This power mode approach is particularly useful when the mobility mismatch between the source and the receiver is big enough.

## References

[1] R.S. Langley, P. Bremner, A hybrid method for the vibration analysis of complex structural-acoustic systems, Journal of Acoustical Society of America 105 (1999) 1657–1671.
[2] P.J. Shorter, B.K. Gardner, P.G. Bremner, A hybrid method for full spectrum noise and vibration prediction, Fifth World Congress on Computational Mechanics, Vienna, Austria, July 7–12, 2002.
[3] A.D. Pierce, V.W. Sparrow, D.A. Russell, Fundamental structural-acoustic idealisations for structures with fuzzy internals, Transactions of American Society of Mechanical Engineers Journal of Vibration and Acoustics 117 (1995) 339–348.
[4] M. Strasberg, D. Feit, Vibration damping of large structures induced by attached small resonant structures, Journal of the Acoustical Society of America 99 (1996) 335–344.
[5] R.J. Pinnington, R.AF ulford, M. Terry, The use of polar mobility for predicting the coupled response of machine mounting system, Proceedings of inter-noise 96 (1996) 1587–1592.
[6] B.R. Mace, P.J. Shorter, A local modal/perturbational method of estimating frequency response statistics of built-up structures with uncertain properties, Journal of Sound and Vibration 242 (2001) 793–811.
[7] J. Su, A.T. Moorhouse, B.M. Gibbs, Towards a practical characterisation for structure-borne sound sources based on mobility techniques, Journal of Sound and Vibration 185 (1995) 737–741.
[8] A.T. Moorhouse, B.M. Gibbs, Prediction of the structure-borne noise emission of machines: development of a methodology, Journal of Sound and Vibration 167 (1993) 223–237.
[9] A.T. Moorhouse, B.M. Gibbs, Simplified characterisation of multiple point excited structures using mobility matrix eigenvalues and eigenvectors, Acustica united with Acta Acustica 5 (84) (1998) 843–853.
[10] A.T. Moorhouse, A mobility matrix simulation for force and moment excitation of structures, Proceedings of inter-noise 99 (1999) 789–794.
[11] B.A.T. Petersson, Structural acoustic power transmission by point moment and force excitation, Part I: Beam- and frame-like structures, Journal of Sound and Vibration 160 (1993) 43–66.
[12] B.A.T. Petersson, Structural acoustic power transmission by point moment and force excitation, Part II: Plate-like structures, Journal of Sound and Vibration 160 (1993) 67–91.
[13] L. Ji, B.R. Mace, R.J. Pinnington, A power mode approach to estimating vibrational power transmission by multiple sources, Journal of Sound and Vibration, 2002, to appear.
[14] H.R. Schwarz, H. Rutishauser, E. Stiefel, Numerical Analysis of Symmetric Matrices, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1973.
[15] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.
[16] R.R. Craig Jr., A review of time-domain and frequency-domain component-mode synthesis methods, International Journal of Analytical and Experimental Modal Analysis 2 (1987) 59–72.
[17] R.R. Craig Jr., Substructure methods in vibration, Transactions of American Society of Mechanical Engineers Journal of Vibrations and Acoustics 117 (1995) 207–213.
[18] E. Bodewig, Matrix Calculus, North-Holland Publishing Company, Amsterdam, 1959.
[19] L. Cremer, M. Heckl, E.E. Ungar, Structure-Borne Sound, 2 Edition, Springer, Berlin, 1988.